Mathematical Methods and Techniques in Signal Processing - I

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Lecture – 04
Modes in a linear system

Let us discuss Modes in a Linear System. This is an important idea particularly when we analyze linear time invariant systems and we would like to see what is the natural behavior of the system.

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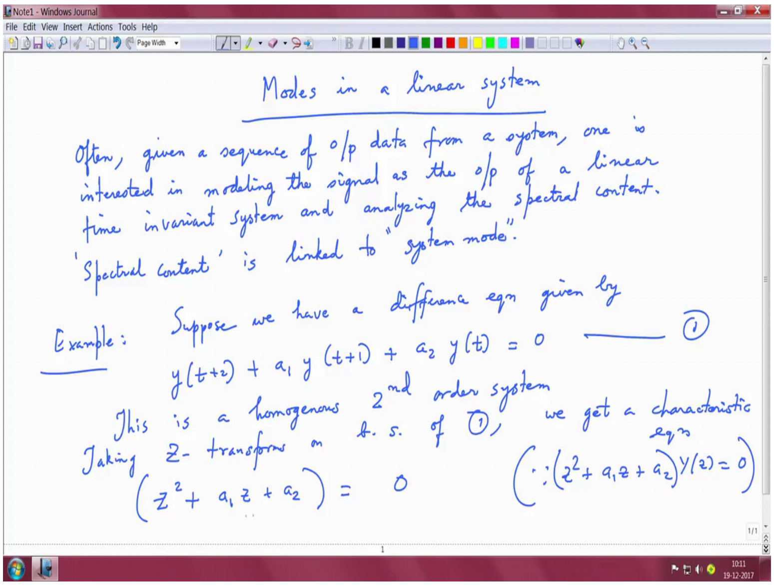

So, often when we do measurements given a sequence of output data from a system one is interested in modeling the signal as the output of a linear time invariant system and analyzing the spectral content or spectral properties. So, throughout this discussion we will assume that the input signal or the forcing signal is essentially deterministic and one can extend these 2 random signals as necessary. And the spectral content is linked to system modes. So, the system modes basically determine where there are peaks in the spectrum right, I mean these you can think of them as eigen modes or characteristic modes.

Now, let us analyze this from the standpoint of view of difference equations and things become very clear for us. But before we begin a little bit of motivation, why these things are very useful. So, imagine an application where you have a an oil refinery and you have to figure out perhaps where at what point in the ground, at what depth in the ground within the earth you get oil or some minerals etcetera. So, people might send impulses essentially to the earth they may get the reflectivities from various layers in the earth and based upon the signals that are captured and certain properties of the earth layers in terms of the reflection coefficients etcetera etcetera, one can come up with some estimate of where the oil can be found or where certain minerals can be found etcetera. So, this is basically done through sending an impulse or a shockwave through the ground right.

Another good example for modeling this is the vocal tract model. So, if you have an impulse that is given to one and of this filter basically if you can think about your vocal tract that can be discretized as several, discretized and thought about as having sections of these acoustic filters which are coupled to each other right and giving the impulse at one end going through this set of filters you can get actually speech. So, if you were to model these carefully then you can kind of think about such modes, I mean what are the natural modes in which the system will resonate towards right. And let us see this from the linear systems angle.

So, let us take an example suppose we have a difference equation given by y of t plus 2 plus a 1 y of t plus 1 plus a 2 y of t equals 0. So, you have to realize the connection between mathematical abstraction and in practice right. I mean practical systems do not tell you that this is the governing equation it is for the modeling people to figure out what is the right model that can fit observe data. So, in that sense we have to develop the theory and advance the modeling ideas forward so that we can match reality as closely as possible. So, hypothetically let us assume. In one case we have this difference equation, this is a homogeneous second order system ok.

Now, if we take the z transforms on both sides of equation 1, we get a characteristic equation given by z square plus a 1 z plus a 2 equal to 0. Now, if you take the z transforms you will have this times z squared plus a 1 z plus a 2 times y of z is 0, but y or z is not equal to 0 therefore, this has to be 0 right, z square plus a 1 z plus a 2 has to be 0.

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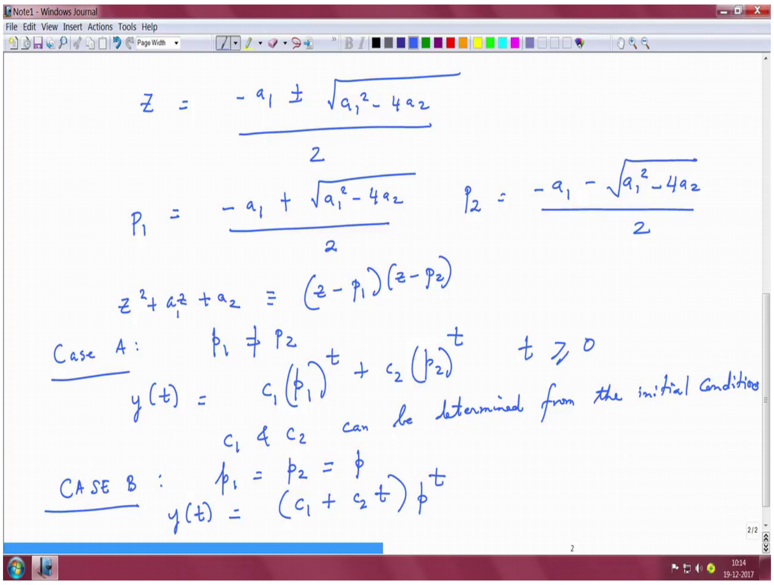

Now, the z can be solved it is a quadratic. So, therefore, it is minus a 1 plus minus square root a 1 square minus 4 a 2 upon 2. So, you have 2 solutions p 1 which is minus a 1 plus square root of a 1 square minus 4 a 2 upon 2 and you have p 2 which is minus a 1 minus square root of a 1 square minus 4 a 2 upon 2. That means, you can write z square plus a z a 1 z plus a 2 as z minus p 1 times z minus p 2.

Now, we have two cases, case a when p 1 is not equal to p 2 we can write y of t is some constant c 1 times p 1 power t plus c 2 times p 2 power t for times greater than or equal to 0 and c 1 and c 2 can be determined from the initial conditions right. I mean you know what y 0 is, what y 1 is, just plug in and then you can have two equations for solving the unknowns c 1 and c 2.

Now, case b, p 1 equals p 2; that means, you have a repeated root then y of t is c 1 plus c 2 times t times p of t where p is the identical pole which is p 1 equals p 2. So, this is basically from our theory of ordinary differential equation. So, differential equations you can think about the similar analog to difference equations and from which you can build up this theory.

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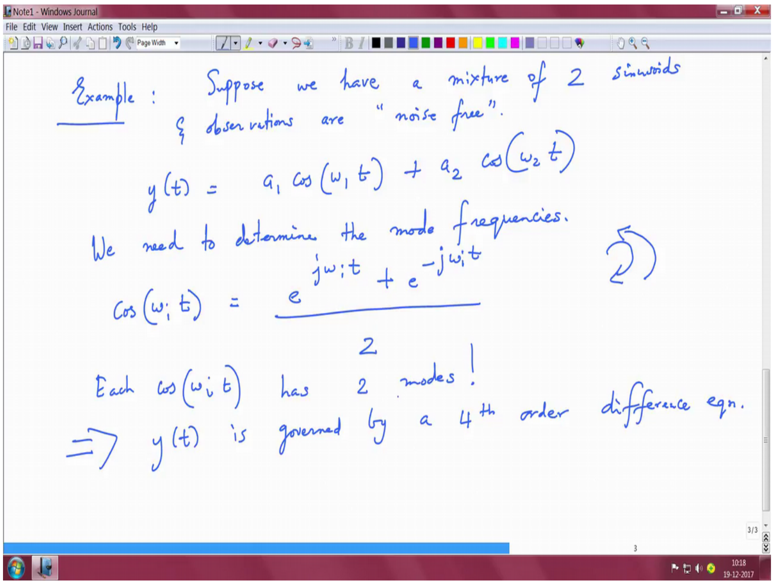

Now, let us consider an example here. Suppose we have a mixture of 2 sinusoids and observations are noise free, we have a mixture of 2 sinusoids and observations of are noise free. We have y of t equals a 1 cosine omega 1 t plus a 2 cosine omega t, omega 2 times t. So, a cause or a sin is the same except that there is a phase shift of pi by 2. So, I think without making a first here I just write cosine I could put sine as well is not a big deal.

Now, we need to determine the mode frequencies. You need to determine the mode frequencies and we need to figure out how many observations we need. So, at a first glance it looks like possibly there are only 2 modes right, you have 2 frequencies, 2 modes that is a wild guess one can take, but it is not. The reason is as follows if you carefully look at what cos omega i t, i equals 1 comma 2. So, let us say cos omega i t this basically e power j omega i t plus e power minus j omega i times t divided by 2.

We have 2 phasers e power j omega t and e power minus j omega t right, I mean one is going in this direction the other is going in this direction and this is giving you the standing wave here right. So, each cos omega i t has 2 modes therefore, you have two such frequencies here omega 1 and omega 2. This implies y of t is governed by a fourth order difference equation right, 2 modes for omega 1, 2 modes similarly for omega 2, 2 plus 2 is 4, therefore, we have 4 modes.

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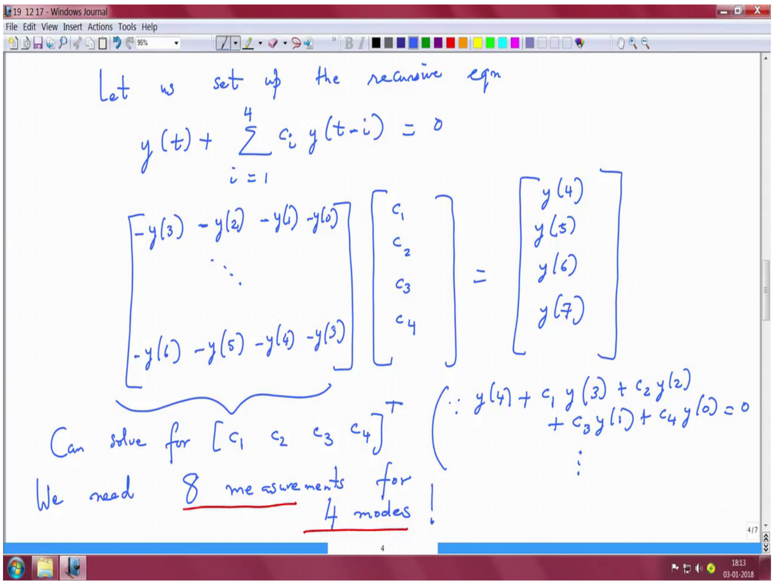

So, with this let us set up the recursive equation right. We can say y of t plus summation I going from 1 to 4 c i y of t minus I equals 0.

Now, let us expand this in matrix form. So, we have 4 unknowns which is which are c 1, c 2, c 3 and c 4 right. So, if I take y 4, y 4 is minus c 1 times y 3, this is a minus y 3 minus c 1 times y 3 minus c 2 times y 2 minus c 1 times minus c 1 times y 3 minus c 2 times y 2 minus c 3 times y 1 minus c 4 times y 0 and so on. So, if you just sort of populate the entries like this the last would be, if you have measurements by 4, y 5, y 6, y 7 and I get this in matrix form as follows right. I mean if you just rewrite this equation slightly you will get this it is not too difficult I just show you one term one sample calculation. So, y 4 plus expand this out c 1 times y, I mean I equal 0 this is 4 minus 1 which is 3 plus c 2 times y 2 this c 3 times y 1 right plus c 4 times y 0 this is equal to 0 so on and so forth I mean I am just expanding this equation and putting this in matrix form.

So, assuming this atoms, a set of equations are full rank right I mean this is a 4 by 4 matrix assuming this is full rank, then you can solve for the tuple c 1, c 2, c 3 and c 4 towards the solution. And if you wonder how many measurements we needed, we need eight measurements for determining 4 modes this is very important. In general we have n modes in the system we need 2 n measurements right. So, what is you take home from this lesson. Erase this, this has to be a y 3. They take home from this lesson is often we are given measurements this is what you get and observations can be noisy here. I have taken a very simple case because this is a very beginning of the course I have just taken a deterministic signal which is non random right. So, given a deterministic system we want to figure out the system modes, to compute the system modes from the measurements you have to formulate a difference equation and this gives you an example how we can formulate a difference equation and then arrive at computing the system modes ok. From the difference equation then we solve for the roots and then we can get the system modes.

I will give you some homework exercises that will help you understand this module better.

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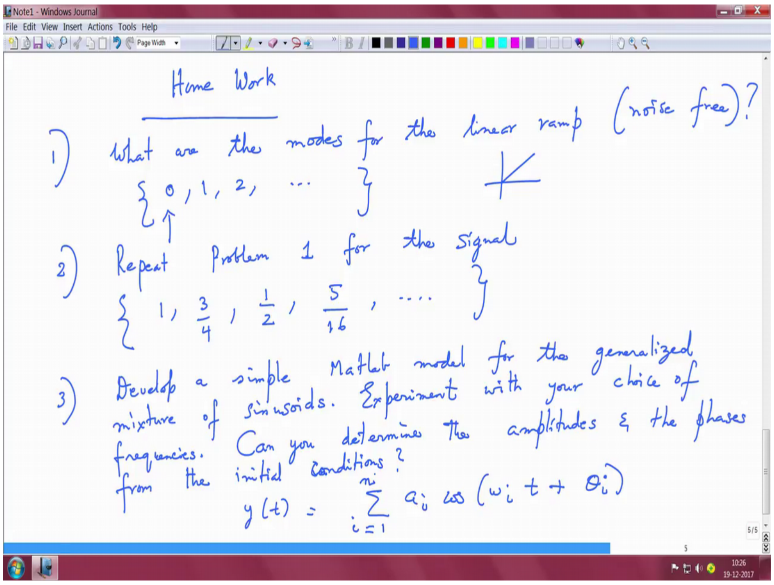

First exercise, what are the modes for the linear ramp again I assume noise free. I give you a sequence which is 0 1 2 dot dot dot right this is a linear ramp. Now, I am asking you what are the system modes for this signal. Next, repeat problem one for the signal or the sequence 1, 3 by 4th, 3 quarters, half, 5 upon 16 dot dot dot, I give you 4 measurements can you compute the signal modes.

Last is, develop a simple MATLAB model for the generalized mixture of sinusoids. Experiment with your choice of frequencies. Can you determine the amplitudes and the phases from the initial conditions, right? Basically I give you a mixture which is like this a i cosine omega i t plus theta i, i equals to 1 to some n. Develop a simple MATLAB model for analyzing the system modes of the generalized mixture of sinusoids with the amplitudes and phases. So, these are some exercise problems which can help you analyze a material, taught in the class and then ponder slightly on this.

We will stop here.