**Mathematical Methods and Techniques in Signal Processing - I**
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**Lecture - 39**
**Noble Identities**

So, let us delve a little bit deeper into multi rate operations. 1 of the questions that we get in our mind is can we swap the down samplers and the up samplers. It might be useful for some reason first slashed sampling rate and then expand or may be may be expand first and then down sample right and what conditions can we swap these operations. So, that is an interesting an idea itself that leads us to these interconnecting systems.

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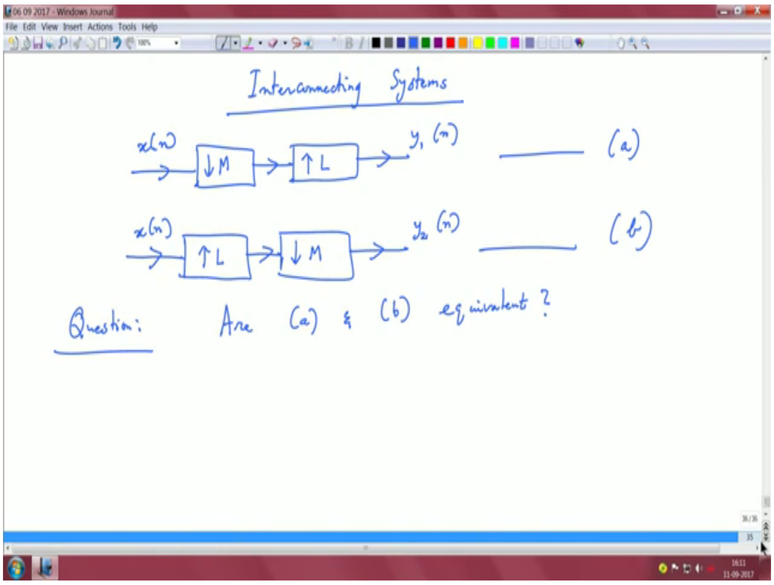


So, we have a signal x of n, we down sampling this first by a factor of M and then up sample this by a factor of L and we get say suppose a signal Y 1 of m let us say this is case a. The second case we take the same input, but we did other way round up sample by a factor of L followed by down sampling by a factor of M and we get the signal Y 2 of n and this is say b.

The question is are a and b equivalent? Of course, a trivial operation would be up sample by L and down sample by L I mean they are not doing anything there and that is a very trivial operation.

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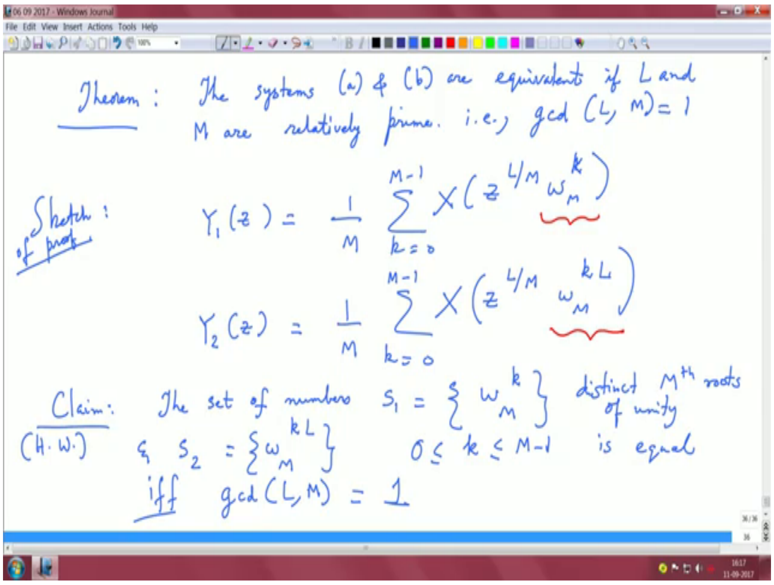

So, there is a theorem which links the scenario and theorem is stated as follows. The systems a and b are equivalent if L and M are relatively prime that is gcd of L comma M equals 1.

So, I will basically sketch the proof because it has small detail which I want you to work as a homework exercises ok. Now Y 1 of Z is the frequency domain response and this is Z transform of the of the signal Y 1 of n which is L upon M sum of k equals 0 to M minus 1 x of Z power L upon m omega omega m power k.

Now, how did we get this basically you have a factor Z power L. So, first you look at the interim signal which is at this step, because your x of n and your down sampling it by m. So, you have a factor of 1 upon 1 upon m,+ you have m copies and your looking at the translates of all these copies right and then you have this Z power 1 over M factor and omega power k of the translation and then this is basically for the stretch 1 upon m right and then that interim signal when it is going through an L fold expander, you have a factor L that appears here. So, this is basically straightforward.

And then similarly we do y for Y 2 of n and we get Y 2 of Z is 1 upon m sum of k equals 0 to M minus 1 at the very first step we have x of Z power L to start with. So, therefore, we have x of Z power L upon M and this is omega M power k times L.

Now, there is a little bit of detail here. So, Y 1 and Y 2 seem almost the same except that if you observe at this modulation factors, one set is omega m power k the other set is omega m power k times L that is the only difference right. So, now, if we ensure that both of them are equivalent of the sets are the same then I think we are proving that this to a systems are identical.

So, I will put this as a claim which is part of your homework, the set of numbers and these are complex quantities S 1 comprising of omega M power k, these are basically distinct M th roots of unity and this set S 2 which is omega m power k times L right for 0 less than or equal to k less than or equal to M minus 1 is equal if and only if gcd of L comma M equals 1. If the gcd of L comma M equals 1 then this set of numbers are equivalent. We can do a simple test I mean you can choose L equals 2.

M equals 3 and then look at all the roots here, and at least get an get an get an intuitive feel if the sets ate the same right and once you get this feel I can just start with trying to expand this relationship for gcd of L comma M equals 1 more carefully in in precise math form and then prove this as a formulae is not too difficult, I have just given this is a homework exercise so that you can work this out.

So, now we know that we could interchange the decimation process and expansion process provided the gcd of L comma m equals 1 right. So, we will delve into the next concept which is on noble identities.

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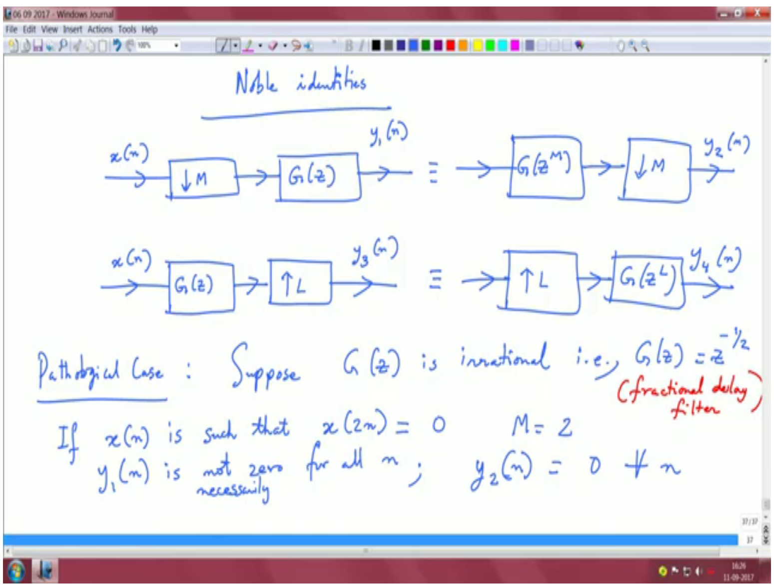

So, one can ponder about what is noble about these identities, but we will see they have some interesting consequences. So, the idea behind noble identities is, can be link down sampling followed by filtering to perhaps filtering with down sample like the way we did swapping of operations, for down sampling and up sampling possible possibilities when there equivalent up to swaps. The question sort of can be extended if we can do down sampling followed by filtering as possibly filtering first and then down sampling.

Similarly, if we are filtering and then up sampling, in what sense is it equivalent to up sampling first and then filtering so on so forth right. So, these are 2 questions and fortunately we have answers to these and that is part of noble identities here. So, let us consider the first case we are down sampling by M followed by filtering and let us say this interim signal is y 1 of n the question is if this is equivalent to filtering first to some filter, followed by down sampling. And similarly we have another case here where we filter first and then up sampling and we are wondering if this is equivalent to up sampling followed by some filter not necessarily the original filter that started with.

So, before we get an idea to formally prove this we have to see if this results make sense right and I will call this a pathological case, suppose G of Z is irrational, that is G of Z is of the form Z power minus half and I think load is important for you to interpret this is basically a fractional fractional delay filter.

Now, let us consider the first case itself those down sampling operation, if x of n is such that x of 2 n is 0, and that is every alternate sample is 0, right. Y 1 is not I mean let us assume that m equals 2 here right M equals 2 and let us assume that x of n such that x of 2 n is 0.

Let us look at the first case if you are down sampling by 2, right you are throwing away all the zeros anyway and whatever you have is basically filtered through G of Z. So, y 1 is not 0 for all n and its a necessarily you know at some point you may get zeros, but it is not that for all n y 1 of n is 0, right y 1 of n is not 0 necessarily for all n right some cases it can be 0 and it is not necessary that all of for all n it has to be 0.

But let us look at the case for y 2 n right. Y 2 n now you have a Z power minus 1 here; that means, we are taking the signal x of n we are filtering it through a unit delay and when we are doing the filtering through a unit delay and then down sampling by 2 you are getting everything to be 0 right. So, y 2 of n is 0 for every n. So, just observe the 2 cases here.

So, which basically means that you know this 2 operations are not the same if we have to deal with filter switch, which are irrational and interpret these irrational power you know error that is you know irrational in the irrationality this as a function has basically fractional delay filters right. And this is the pathological case, but outside of this pathological case the in the systems are equivalent and we will show this by evaluating the Z transform of the signal at the output of each of the systems ok.

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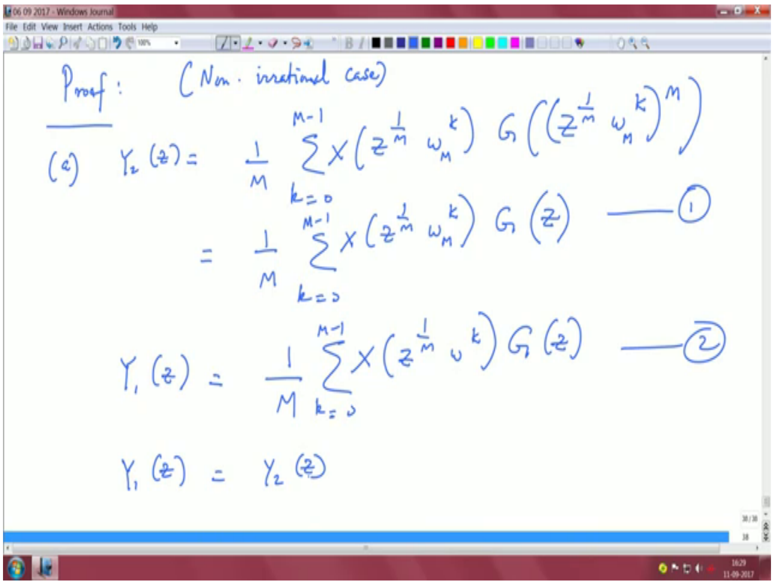


Now, we can do the proof this is basically non irrational case. Let us consider case a now Y 2 of Z is basically. So, first we have you filtering that g of Z power n and then this is followed by decimation. So, Y 2 of Z is basically 1 of 1 upon M is sum M copies of x of Z power 1 upon m omega power k times G of Z power 1 upon m omega power k whole upon m right and this is detail that we have to get, because that down sampler is over this filter G of Z power m. So, there is a modulation here and Z power m ok.

Now, this can be simplified as 1 upon M k equals 0 to M minus 1 this term would be remaining the way it is and this is basically G of Z, because omega though you can put a factor and if you want to hear, but throughout implied and omega m power k times and this is basically it is unity, right. Therefore, this is this can be choosy and let us look at Y 1 of Z,Y 1 of Z is basically a first look at the output of the decimation. So, the system transfer function is basically straightforward in the output response is pretty straightforward you have a scale factor of 1 upon M you have to add those m copies that are basically stretched and translated, right.

And now this times G of Z is basically the overall response over frequency response. So, from these 2 equations say call this 1 or call this 2 we getting for Y 1 of Z is Y 2 of Z right. So, basically I can replace a down sampler followed by a filter through possibly a filter and a down sampler and the filter is basically link to the original decimation filter as it is it has to be G of Z power M right.

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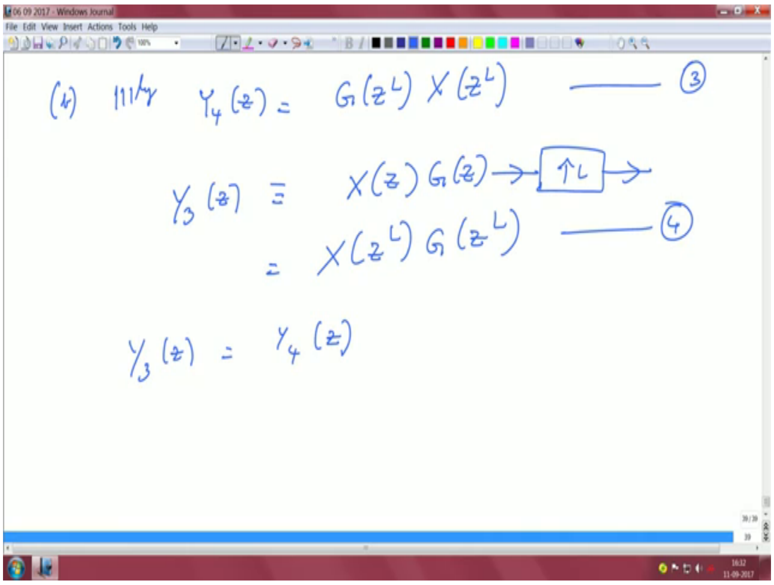

So, now let us do this for the second case. So, we have Y 4 of Z is basically G of Z power L times X of Z power L why because we have x of n go through an up sampler and that system can be replaced as a x of Z power L, and the x of Z power L times G of Z power L is what you have for Y 4 and Y 3 of Z is basically now we have x of Z times g of Z it is it is equivalent to passing this through this up sampler and which is basically x of Z power L g of Z power L right.

X of n filtering to G of Z is basically x of Z times G of Z and that if you up sample passes through an up sampler, and you can you can this equivalent to this this this relationship. So, now, from 3 and 4 Y 3 of Z is Y 4 of Z and we have showed that these identities are correct or that true ok.

So, the pathologically case would have arrows if you started I mean if you just started off with writing it normally just I mean in in the prove, then we looked at the verification steps right we just did not assume any irrationality here as we are just doing the algebra in the routine way right, but the moment you ponder about the fractional powers, then I think that becomes very important then you will see that and there is a case if G of Z cleverly happens to be a fractional delay filter then they are not equivalent and then the whole this will basically does not work out it is a very important step. So, they work with filters that are not fractional delays.

So, with this we end this module.