Mathematical Methods and Techniques in Signal Processing - I
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Lecture - 38
DFT as filter bank

Welcome all to this lecture. So, in the last class we basically discussed analysis filters and we concluded that, we can take a full band signal we can decompose that through an analysis bank, comprising of analysis filters and then we pass this through the synthesis bank and hopefully we get the reconstructed signal. Now there are there are a lot of details here when we have to delve into specifics as to how we can build these analysis filters, synthesis filters etcetera and that would be the study during our course of the module on multi rate signal processing ok.

So, first let us start with the discrete Fourier transform and analyze this as a simplest filter bank.

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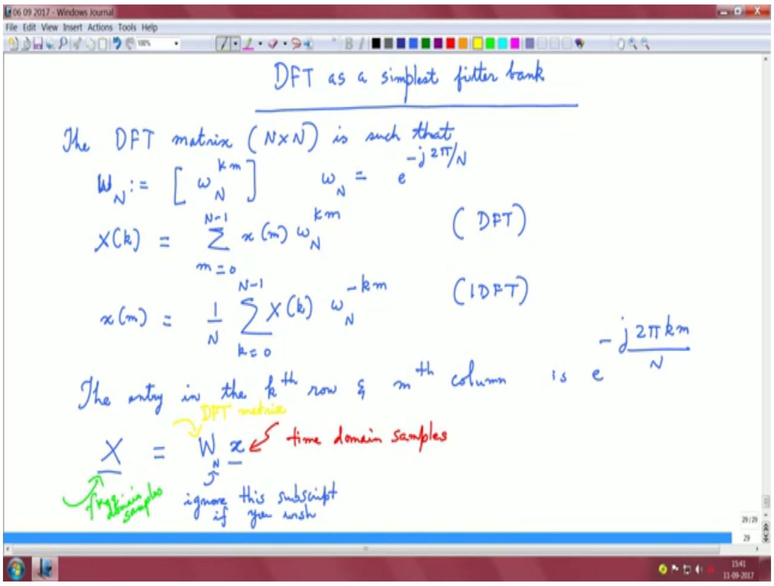

The DFT matrix is an N by N matrix such that this matrix given by W N, N for the length of the DFT that you are interested in computing it is given by omega omega N power k times m I can put that such that equal to indicating that this is a matrix comprising of entries, which are omega N power k times m where omega N and I would say this is this is a capital W and omega N is e power this small omega N which is e power minus j 2 pi upon N.

Now, the time domain samples can be transformed into the frequency domain using the discrete Fourier transform and I will recall the discrete Fourier transform, x of k is basically summation m equals 0 to N minus 1 x of m omega N power k m this is a small omega and this is a small omega power k m and this is a capital this is a matrix, with comprising of these elements, which are small omega N power k m.

Now, we have the inverse Fourier transform, which is linking the time domain samples and the frequency domain in an inverse l given the frequency domain samples, we can get back to the time domain and that is given by this equation 1 upon N, k equals 0 to N minus 1 X of k omega N power minus k times m. So, this is basically our DFT this is our inverse discrete Fourier transform and if you think about this matrix the entry in the kth row and m-th column is basically e power minus j 2 pi k times m upon N.

So, we can think about this as a matrix multiplication right I mean if you stack all these points, if you take an end point DFT x 0, x 1, x 2. So, on till x of capital N minus 1 and then on N and then you have these time domain samples small x of 0 x is 1. So, on till x of n minus 1 you can think it as sort of a matrix multiplication, where you are taking the time domain samples multiplying with this matrix comprising of these entries and then you have these frequency domain samples which are x of k.

Now, X is W times x I will just ignore this subscript ok. So, you can link the time domain samples here, these are time domain samples these are frequency domain samples and this is our DFT matrix ok. So, this is our DFT matrix, we linked the time domain and the frequency domain samples.

Let us see with a simple example how this is and if our intuition makes sense right.

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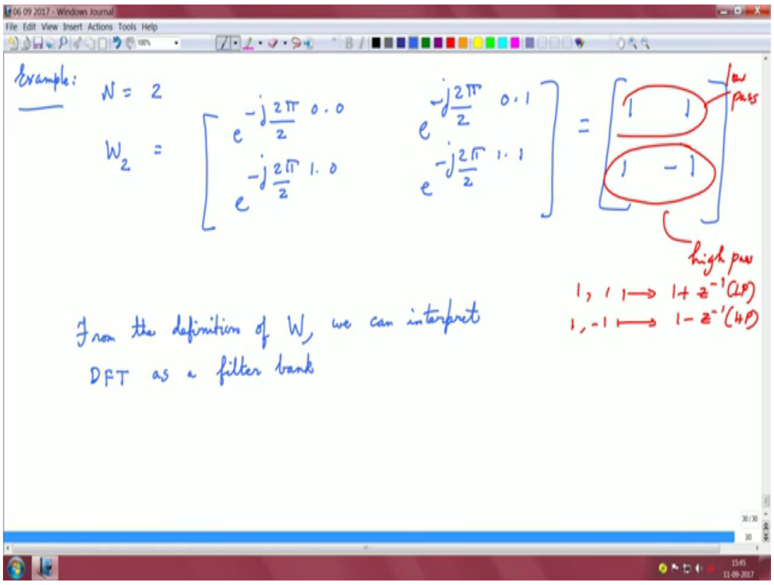

If we take N equals 2 then W 2 this matrix is e power minus j 2 pi upon N N is 2 here, this is k m is 0 0 here this is e power minus j 2 pi upon 2 this is a constant factor 0 times 1, this is e power minus j 2 pi upon 2 this is 1 times 0 this is e power minus j 2 pi upon 2, 1 times 1 and if you just simplify this you get 1 1 1 minus 1, this is e power minus j pi which is cos pi minus j sine phi which is minus 1 and the rest happened to be 1 because this evaluates to 0.

Now, then this matrix is for what it is, but if you take a slightly closer look at this matrix, observe this row and observe this row right. We are just taking a 2 point DFT here and if you observe the 2 rows of this matrix 1 1 is basically like the coefficients of your low pass filter right. If you take the impulse response 1 1 1 comma 1 basically translates to 1 plus Z power minus 1, which is a low pass filter and 1 comma minus 1 is basically like a high pass filter right it is 1 minus Z power minus 1 you can just plot the spectrum of this really quickly and then you can conclude that this is low pass and this is high pass.

Now, this sort of gives us an indication that the DFT is doing already something for us in terms of filtering right, if you look at the rows of the DFT matrix, you are going all away from low pass filters to high pass filters. You can do this example for W 3 W 4 so on and so forth and you can you can conclude.

Now, this simple example here gives us an indication that from the definition of this matrix W, we can interpret DFT as a filter bank. So, now, we will go a little further into the analysis basically what we do is we consider the sequence x of n and we basically think about passing this sequence through a delay line which forms us, which form sub sequences and then we just look into the aspect of filtering these sub sequences through some filter and then we will try to link all these variables together and see how we can interpret the discrete Fourier transform as a filter bank.

So, let us start with this basic idea.

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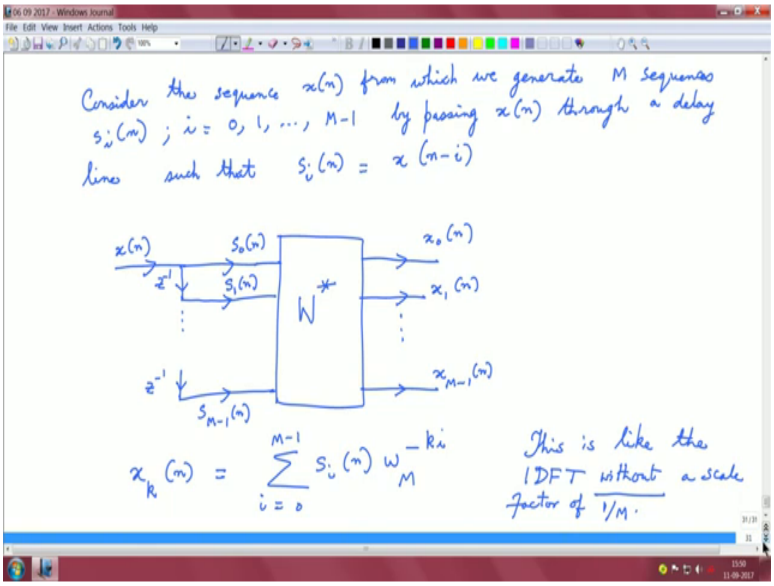

So, consider the sequence x of n from which we generate M sequences S i of n i 0, 1 dot dot dot till m minus 1 by passing x of n through a delay line such that S i of n is x of n minus i, for all values of I going from 0 to m minus 1 to this is our setup.

So what we do is as follows we have x of n no delay is S naught of n and we have 1 unit delay here this S 1 of n so on S m minus 1 of n and we do slightly something different here instead of passing through the normal discrete Fourier transform, we will filter it through the IDFT which is w conjugate and we get x naught of n x 1 of n so on till x m minus 1 of m this is sort of like your sub band signals.

Now, it is pretty straightforward for us to link x k with these sis x k of n is basically sum i equals 0 to m minus 1 S i of n w m power minus k i; and note this factor minus because it is the conjugate w conjugated therefore, this is what we have. So, basically we are combining we are linking a signal here with the input signals that we have before this filter bank ok.

So, the difference is w conjugate is what we have this is like the IDFT; IDFT except with a factor of 1 over m right. So, this is like the IDFT inverse discrete Fourier transforms without a scale factor of 1 upon m, 1 upon m ok. So, this is our setup.

Now, we will dig slightly carefully now what we do is basically take the Z transform on both sides of this equation and then we will try to link the variables together.

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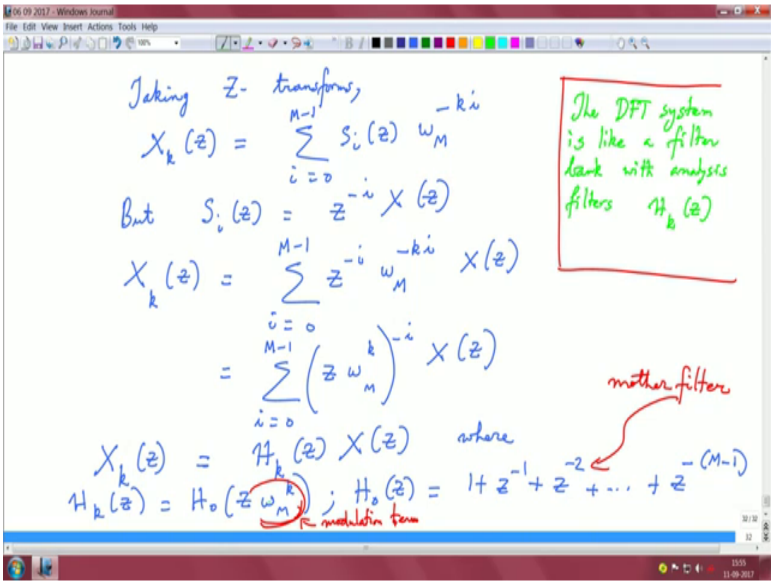

So, taking Z transforms X k of Z equals summation i equals 0 to M minus 1 S i of Z w m power minus k times i. But the sequences sis are basically delay sequences right and we can link them to the Z transform of X via delay element as follows.

So, therefore, we can say X k of Z equals summation i equals 0 to m minus 1. So, just plug this 1 here. So, it is Z power minus i, w m power minus k times i times X of Z right and we simplify this to Z times omega M power k to the power minus i times X of Z

Now, we can say X of k X of x k of Z can be written in the form H k of Z times X of Z where H k of Z is H naught of Z omega m power k and H naught of Z is defined to be this filter 1 plus Z power minus 1 plus Z power minus 2 dot dot dot plus Z power minus M minus 1.

I think this is sort of pretty straightforward for us. So, visualize I mean what we did is this X X Z is basically a constant we just pull this out we just have summation because there is no index i here right we can pull this out and this summation of M copies of Z omega m power k or minus i we interpret that summation as basically Z being replaced Z in this sum 1 plus Z power minus 1 plus Z power minus 2. So, on replaced by Z omega m power k that is what is happening here and why we need to bring this h naught this is like our base filter or our mother filter or our low pass filter here right.

We got our intuition with a 2 by 2 matrix, we started off with a 1 plus Z power minus 1 and you know if we extend this for an endpoint we get Z power minus 1, Z power minus 2 so on and so forth. So, we have this as our mother filter this is basically the mother filter and rest of the filters can be obtained by changing the parameters of the mother filter because it is being modulated by omega m power k by having a modulation term, this is basically modulation term. So, by having a modulation term, we can construct a modulation factor we can construct rest of the filters from the mother filter.

So, what this shows is the following the DFT system is like a filter bank with analysis filters, H k of Z right this is my analysis filter and I stack all these analysis filters and I get analysis bank.

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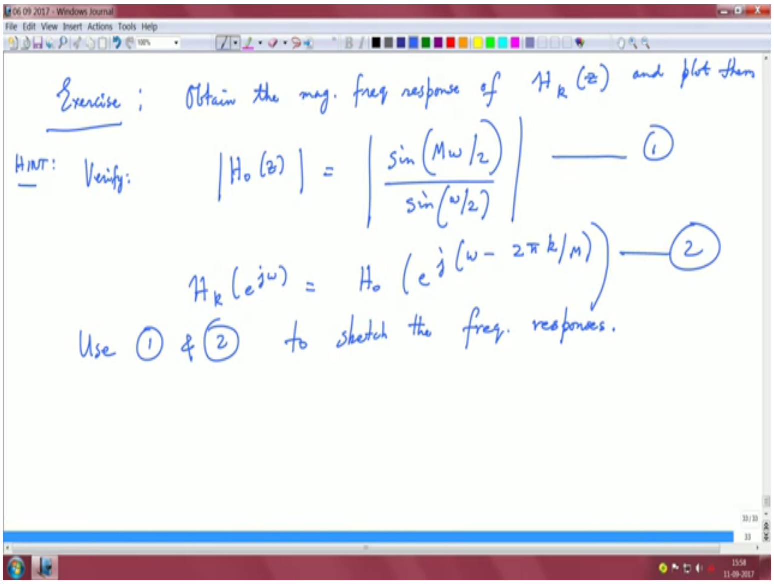

So, with this I give you a simple exercise here, which is for your homework obtain the magnitude frequency response of H k of Z plot them.

So, basically you will have to verify that, magnitude of H naught of Z is mod sine M omega upon 2 divided by sine omega upon 2 and H k of ej omega is basically a frequency translates, which are uniform shifts of 2 pi k upon m for each of the values of k.

So, basically verify that magnitude of H naught of Z is given by this and then use 1 and 2 to sketch the frequency responses. So, this what gives you an idea how the frequency responses look and if these filters are overlapping or non-overlapping and in some of these details right and this gives you an idea where how these filters are evolving v d modulation of omega m power k factor ok.

So, this is this is the simplest filter bank.

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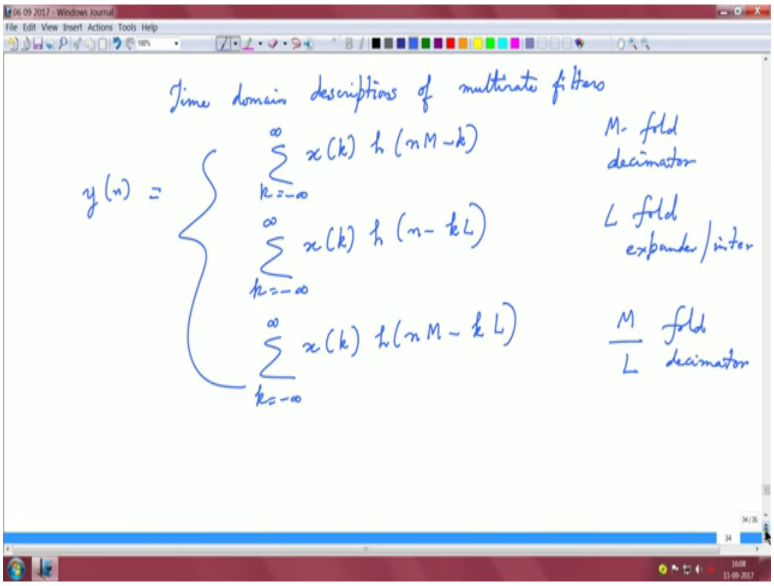

So, before we get on to next topic we have to basically understand a sort of summary of the time domain descriptions of multi rate filters. Now y of n equals summation k equals minus infinity to plus infinity, x of k h of n M minus k, and this is response if it is an M fold decimator n minus k L.

If it is an L fold expander, I would say interpolator answers of this form x of k times h of n times M minus k times L if it is an M upon L fold decimator. And whichever is convenient to you if you were to go in the time domain or in the frequency domain, you can you can feel free to adopt whichever path you would like to choose that makes it easier for you to understand and interpret the flow of the signal through multi rate operations along with filters ok.

I think what we have to interpret here in these multi rate operations is I mean we have filters along with it right. We filter followed by down sampling or we up sample and then we follow this with a filtering operation or we do an M by L decimation post filtering. So, these are sort of filtering operations that we do, and one has to interpret this h of n as impulse response of that filter that governs multi rate operations that either precede or follow the sampling rate conversion operations as the case might be.