Mathematical Methods and Techniques in Signal Processing -I
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Lecture – 35
Decimation and interpolation filters

In the last lecture, we covered the basics of decimation and expansion. We looked into the time domain and the frequency domain effects of the down sampling and the up sampling process. So, in today's lecture, we will go a little deeper into the aspects of down sampling and up sampling along with filtering, and then cover a few details on the interconnecting systems. Whether you want a down sample first up sample later or exchange these operations and we want to see under what conditions these are possible, but let us try to recap, what the down sampling process does, right.

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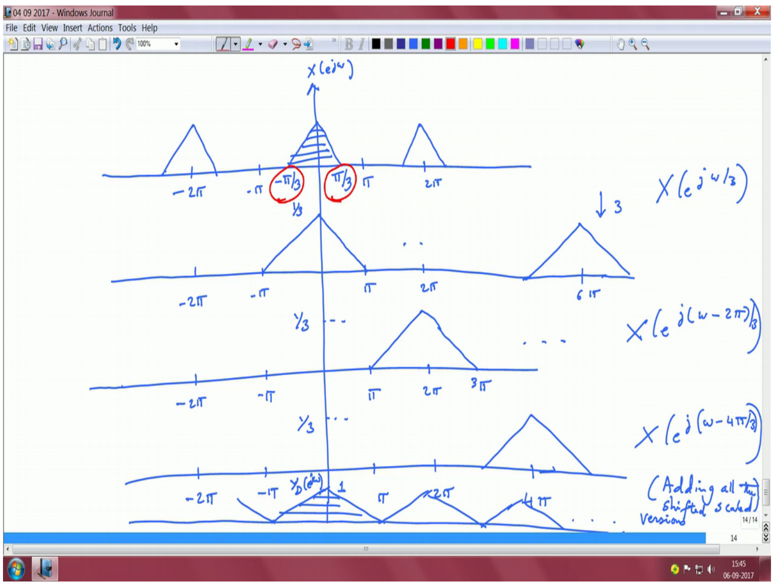

Here is an example where we consider the spectrum of a signal. I mean we have a baseband signal here, let us say this is between minus pi by 3 2 plus pi by 3, it exists between minus pi by 3, and plus pi by 3 and it is basically periodic with 2 pi right. Let us assume that we have something like this, or we just have just three copies here. Now once you down sample in the time domain, we learned from the last class that basically the spectrum actually stretches right, there is a stretching, and there is a scaling and you see the scaling happening here, its scales to one third and we saw, it is 1 over M for a down sampling by a factor of M. So, this is one third it stretches.

So, from minus pi by 3 point is pushed to minus pi, pi by 3 point is pushed to plus pi and so on, I mean and you have these. So, you have the base, I mean this is X e j omega by 3, then you have translations of these at multiples of 2 pi and translated, at multiples of 2 pi integer multiples of 2 pi. So, it is 2 pi k and you have three copies here. So, basically you look at the spectrum of X of e j omega minus 2 pi by 3 and then you look at X of e j omega minus 4 pi by 3 right, and basically you add all the copies together, right. And once you add all the copies; this is what you have at the output of the down sampler this is your Y d of e j omega, right.

Now, as you can see through this example, I mean I have critically chosen minus pi by 3 here, I mean this is a very important point, it was carefully chosen so that when you stretch it, it is basically just getting towards the minus pi and plus pi points and the spectra do not overlap. If you stretch and spectrum goes beyond the minus pi and the plus pi points there would be aliasing, and if there is aliasing you cannot reconstruct the signal right, without doing any other operations. So, simply you know if you just you know invert, you will not be able to get rid of this aliasing error. So, this leads us into an important condition to avoid aliasing which can be succinctly stated as follows.

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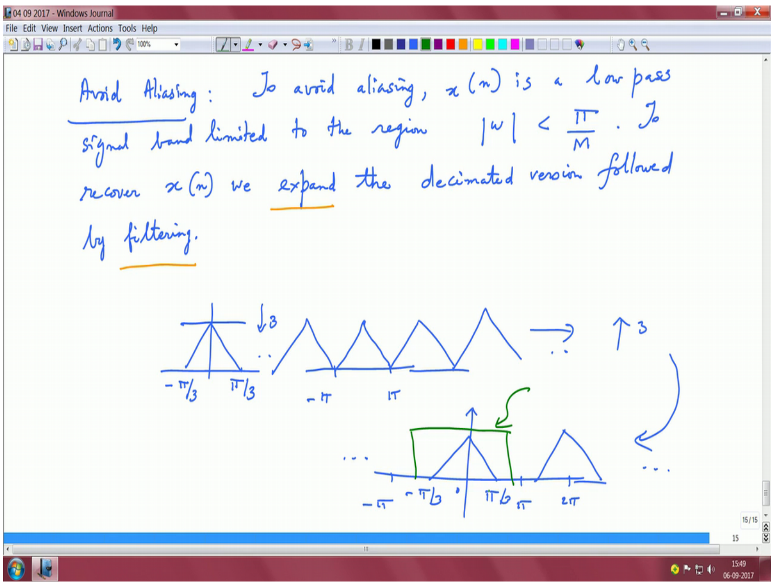

To avoid aliasing x of n is a low pass signal band limited to the region modulus of omega is less than pi upon M right when you down sample by M you stretch it, stretch the frequency axis by M. So, therefore, it has to be band limited to the region for all frequencies, absolute frequency less than pi upon M; otherwise the spectrum really overlaps and causes aliasing error. To recover x of n, we expand the decimated version followed by filtering, and this is a very important step. Two steps to follow, because what happens is, because you down sampled you are stretching. So, basically if you have a down sample signal say from minus pi upon 3 to plus pi upon 3, right. So, there are copies.

So, you will stretch this to minus pi to plus pi, and of course, there will be copies, and therefore you have to account for those and if it is 2 pi periodic you know all these copies will overlap into the baseband, and you get the signal here. So, you may get things like this on either side, and when you expand basically you would have a spectrum from minus pi by 3 to plus pi by 3, the same thing happens for the rest of the cases right, and this is a 0 and this is a 2 pi so on, and you will basically have to have a filter to just filter this out, critically from minus pi by 3 to plus pi by 3. So, this filter is not exactly on to scale.

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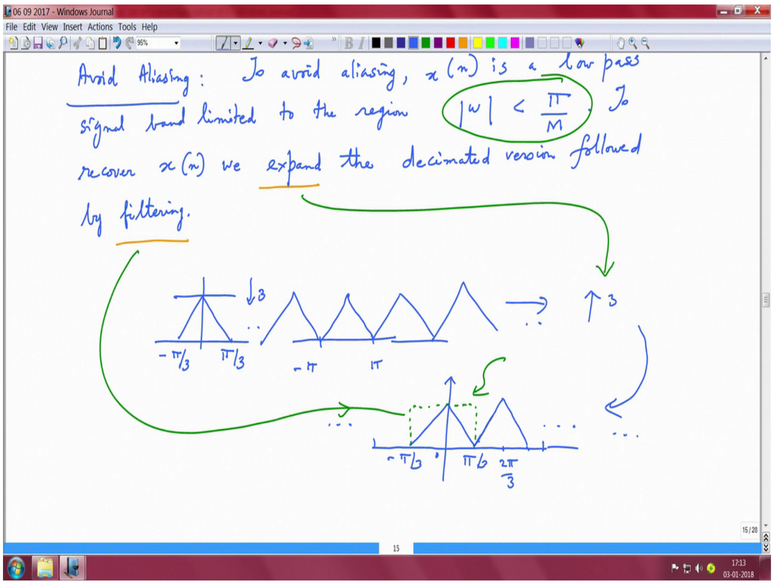

So, you just have to have a filter to filter this process out. Well I think this axis is really not on the right scale. So, we should have a translate here, centered at 2 pi by 3 and then you have the rest of the copies. So, this is what I meant by filtering, and this is what I meant by up sampling, because you want basically get your baseband signal right, and you can appreciate that if you want to avoid aliasing, then modulus of omega; that is the absolute frequency should be less than, strictly less than pi upon M and this is a very important condition.

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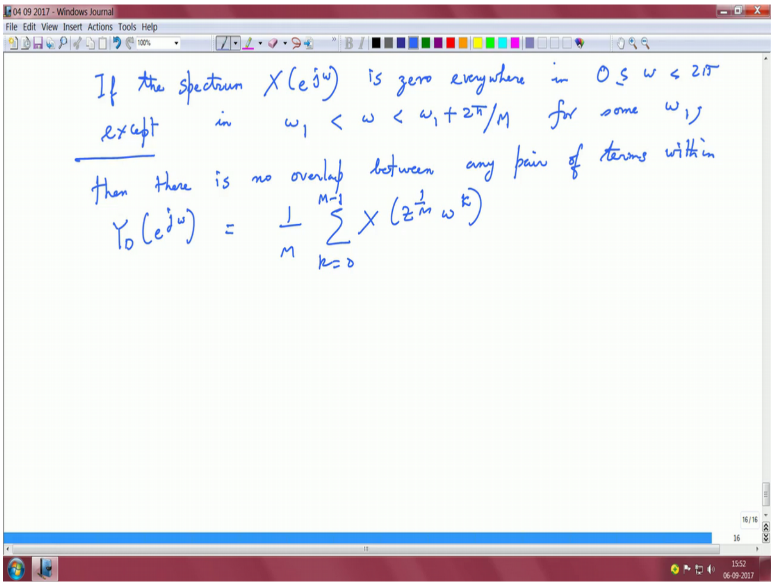

So, if the spectrum X of e j omega is 0 everywhere in 0 less than or equal to omega less than 2 pi. Except in some band omega 1 less than omega, less than omega 1 plus 2 pi upon M for some frequency omega 1, then there is really no overlap between any pair of terms within Y d of e j omega; that is the frequency response of the down sample version given by this equation. So, this clearly indicates that you can reconstruct this signal by expansion followed by filtering right. I mean we saw the case where omega 1 is 0 trivially in the previous example and you can just basically have it translated with the frequency of omega 1. Now, that leads us into the next question, how do we go about designing filters for this, right, and before we get into the detail, certain details, we want to first understand something regarding decimation and interpolation filters.

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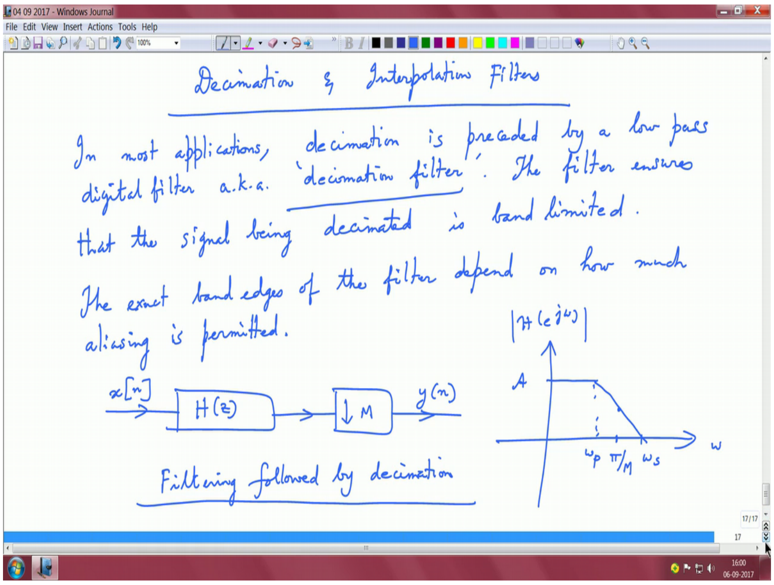

So, there is a little bit of nomenclature sort of thing here, I mean people say down sampling and then they may say interpolation. So, this interpolation is to be sort of made a distinction from up sampling, because interpolation is when you have a filter, which basically fits in the values after up sampling, because up sampling will introduce 0s and this interpolation filter will basically feed the values for those zeros right, that is an important distinction that we want to make. So, therefore, it is good to treat down sampling and up sampling, and any filters that we have, we should say their decimation filters or interpolation filters. In most applications decimation is preceded by a low pass digital filter also called as decimation filter.

So, what really happens is important, the filter ensures that the signal being decimated is band limited. Decimation is preceded by a digital low pass filter, because the filter ensures that the signal being decimated is band limited; otherwise if you first down sample and then you do filtering, you are basically losing a lot of signal out there and when you filter on top of it, I think your common sense would tell you that you are losing information out there right. So, before you reduce the sampling rate, first you have to ensure that you have what you have what you need right. And the exact band edges of the filter depend on how much aliasing is permitted, this is very important, because you may think, well, can I just filter first then down sample or down sample first then filter, right.

These things these questions will naturally arise in your mind, but your common sense at this point should tell you that nominally if you were to down sample or reduce the sampling rate, you should have a decimation filter before that you should have a filter before that. So, that you band limit the signal and then you down sample and then how much would you band limit and the exact band edges of the filter should depend upon how much of aliasing is permitted right, and this you have to account for, because you are doing first filtering and a down sampling will stretch the spectrum, and how much you are cutting off the signal via your filtering will depend on how much of aliasing you will tolerate, because it is followed by down sampling.

These are very important practical considerations, and when we will see through some examples when we talk about filter designs etcetera and you will appreciate what we are trying to do here. So, put in a nutshell, we have x of n discrete time signal, we have H of z which is basically some filter digital filter, you down sample by some rate M and then you get the signal y of n. And if you can think about what your modulus of edge of e g omega or the magnitude spectrum is probably like this. Let us say this is your pass band omega p, this is omega s your stop band and this point here is somewhere around pi over pi upon M right.

Some gain I just did not put all the details in, but again this is according to the specification, you can probably assume this point somewhere is pi upon M, because when you stretch this point is basically at pi and then there will be some amount of aliasing from this portion from this stop band. So, therefore, you can you can sort of think how much of aliasing you might want or how much you can tolerate, and that should basically be made accountable in the design of this digital filter. So, this is basically filtering followed by decimation. Now we can also think about the other way around, what is relationship between up sampling and filtering, right.

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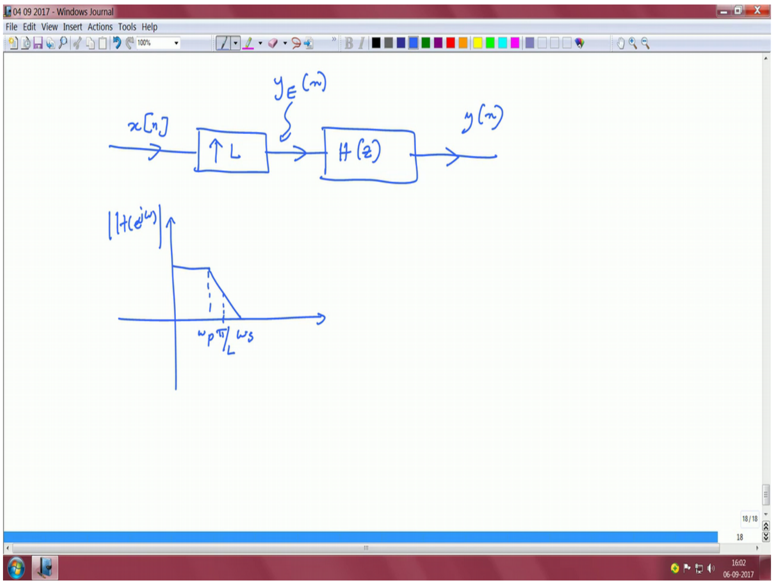

So if you have a discrete time signal here, you up sample this first with a rate l, and then you have this filter H of z that operates on the up sampled signal right, at this stage it is basically Y Y e of n, e standing for expansion. Again it is common sense, because if you have a filter first and then you up sample, you are just putting some zeros, your filtering your doing some operation and then you are up sampling you are inserting 0s in the time domain and what do you want that for right.

So, your common sense would tell you that you first increase the rate to restore, whatever you want restore the rate, if you want to say, if you have down sample then you want to restore the rate, you up sample it first, up sampling introduces zeros on top of it you put an interpolator right, and that will basically fill in for the 0s that you have introduced in the up sampling process and one can think about the spectrum of this magnitude spectrum of this filter in a similar way. So, make a p and probably this point here is your pi upon L right, you up sample and then you cut off this portion. Now I think once we get this picture, I will leave this as a small exercise.

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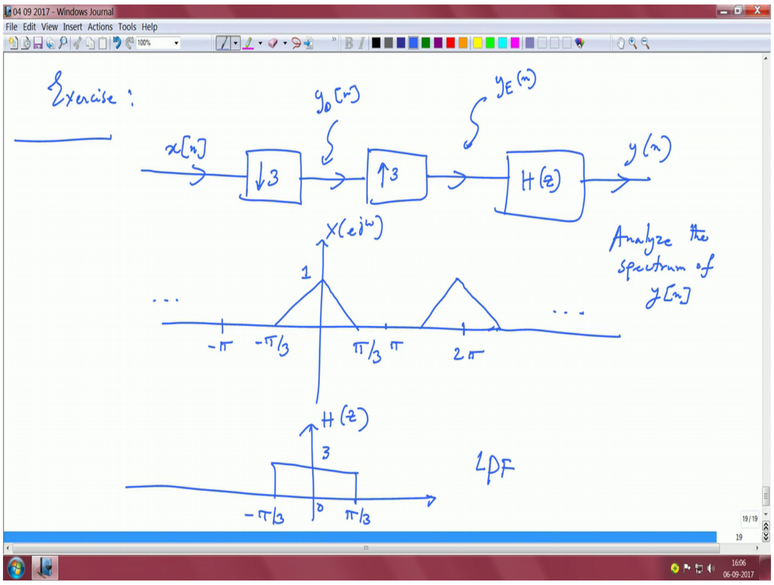

Suppose I give you a discrete time signal x of n, I down sample this by a factor of 3 then I expand this by a factor of 3 followed by some filter H of z. I get a signal y of n. Suppose the original signal has spectrum which looks like this. So, minus pi by 3 2 plus pi by 3, same example we considered last time 1. So, e g omega, so this is periodic 2 pi dot dot dot on either side, let us say this is pi. This is minus pi, this scale is not exactly perfect, but I think you can see the conditions here and my H of z is a filter which is basically a brick wall filter, but a gain of 3, it is a low pass filter, it is a brick wall filter here. And I want you to basically analyze the spectrum of y of n ok.

So, analyze the spectrum of y of n, I mean you have to look at the interim points also this is y decimation, this is y expansion followed by filtering. So, I think two important concepts, I think three important concepts are covered here; one what is happening after down sampling right, you have to basically understand that there are four operations happening. The spectrum here, the base spectrum that you have here would be stretched right and there will be shifted copies translation at the first (Refer Time: 23:48) 0 component, it is basically a stretch for the remaining M minus 1 components; M equals 3 in this case that is for the remaining two components you will be translating at 2 pi and 4 pi right.

So, you will have j omega minus 2 pi upon 3 and j omega minus 4 pi upon 3 basically plot that spectrum, there will be a scaling here of 1 by 3 right. So, that would be how the spectrum would look like and then you have to add all the copies, because it is a periodic signal with 2 pi, 2 pi periodicity in the spectrum. Then you expand this, expansion will basically compress the spectrum and then apply a brick wall filter right. So, I have sort of given you all the hints what you need to do this exercise and solution would be a supplied; so, just for your understanding.