Mathematical Methods and Techniques in Signal Processing
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Lecture – 32
Sampling theorem

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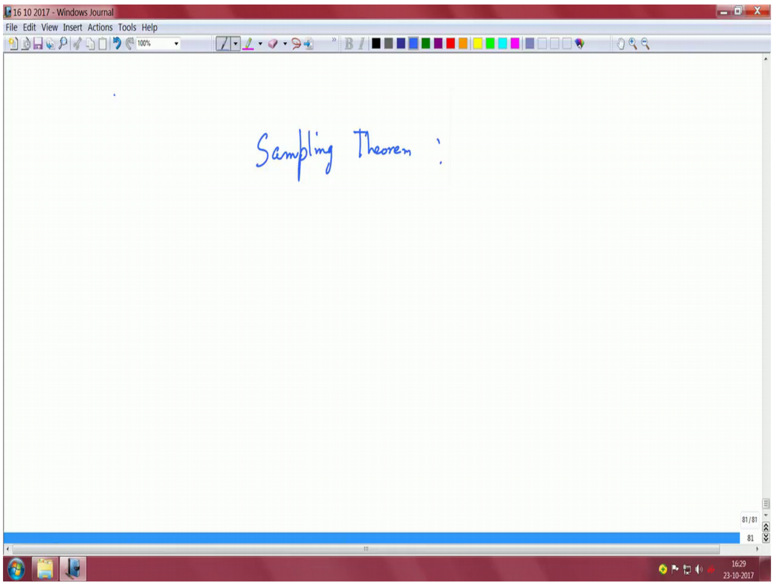

Now, I think with this we should now interpret the sampling process. We have just started with the conventional Dirac comb sequence and then linking that, we have the samples. Now let us actually interpret sampling.

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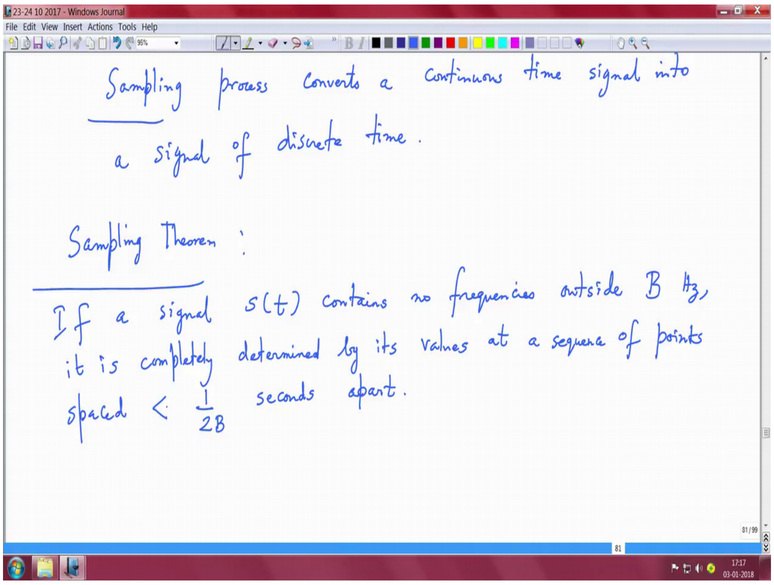

So the sampling process converts a continuous time signal into a signal of discrete time. Basically I have a continuous time signal and then I sample I just get some numbers; I just get a sequence of numbers.

Now, here is the basic idea, in the sampling theorem. And we saw that idea when we started off with Shannon's basic idea right. So, if a signal S of t, this is not delta; If a signal S of t contains no frequencies outside B hertz, it is completely determined by its values at a series of points to make its as a sequence of points here spaced 1 over 2 B seconds apart. I think this is a very important result.

That means, for a given sampling rate f s, perfect reconstruction is guaranteed for bandwidth capital B which is strictly less than f s upon 2. If f s is your sampling rate, then your perfect reconstruction is guaranteed, if your bandwidth B is capital B, is basically less than f s upon 2.

So, here well determined by its values at a sequence of points based less than 1 over 2 B seconds apart because, this is in time right, in frequency it is greater in time it is less.

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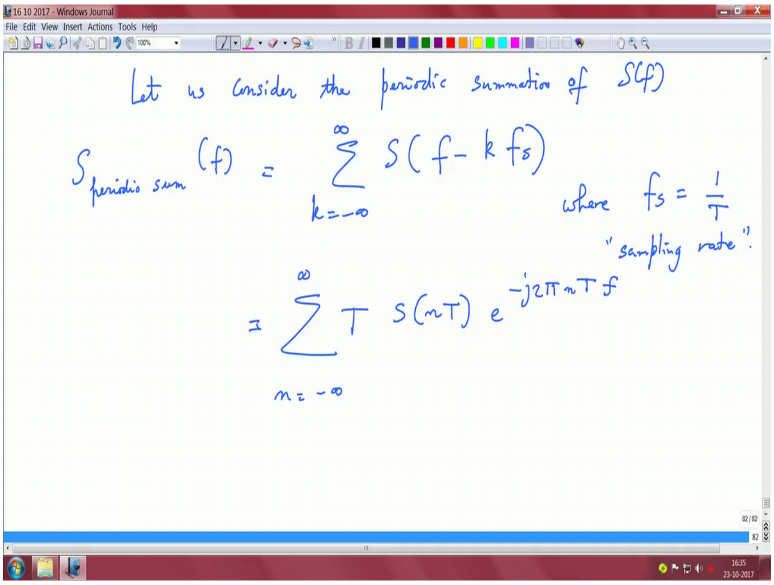

Now let us consider the periodic summation of the spectrum of S of T. This is basically k equals minus infinity to plus infinity, S of f minus k times f s, where f s is 1 upon T which is basically your sampling rate. And you can interpret when k equals 0, it is a base band.

Now this is written in the form n equals minus infinity to plus infinity T times S of n times T e power minus j 2 pi n times capital T times f.

We saw this in the last step when we derived it. We started off with a periodic sum, and then we interpreted how we can write it in this form. If you just go 2 slides here.

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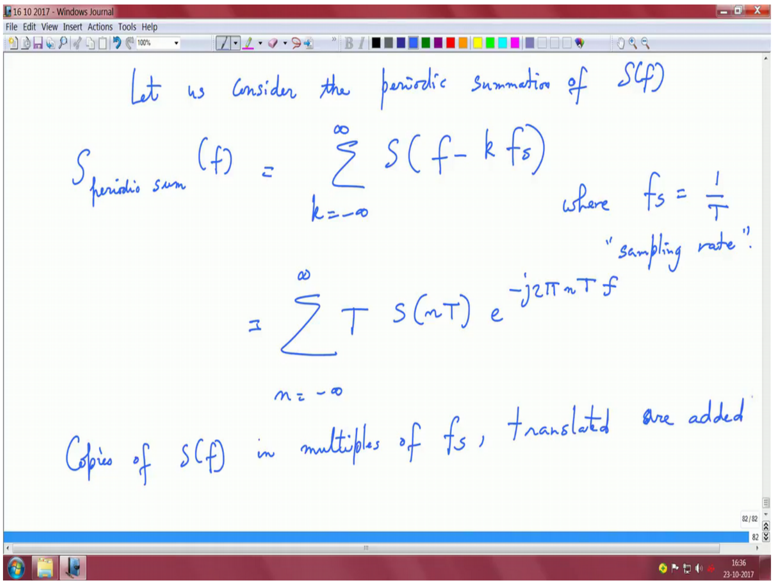

This is exactly the formula. So, we write it in this form.

So, what do we see here, I mean basically copies of S of f, multiples of f s translated are essentially added. This is what we mean by this periodic sum. I have a spectrum; I take a translation in multiples of f s. And I add the copies.

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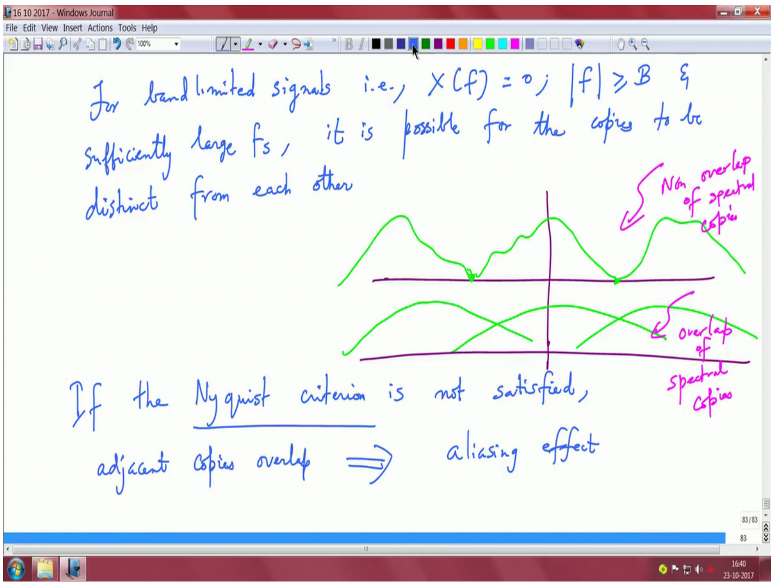

Now, for band limited signals, that is s of f is equal to 0 for frequencies outside B and sufficiently large f s, it is possible for the copies to be distinct from each other. Try to insert interpret this as follows.

You have some spectrum, and ideally you would want to have copies that are distinct. You can imagine this is a copy, and the copies have to be really distinct. Otherwise if you get them overlapping, then you have troubles.

You see the interpretation of this, and this is your aliasing; now if the Nyquist criterion is not satisfied, adjacent copies overlap, and this will imply aliasing effect.

So, you can think about this is overlap. Now this is one of the reasons why normally people like to limit. So, even if the band bandwidth is really not limited, you would want to limit the band by putting an anti aliasing filter or typically a low pass filter before you sample. This is very important.

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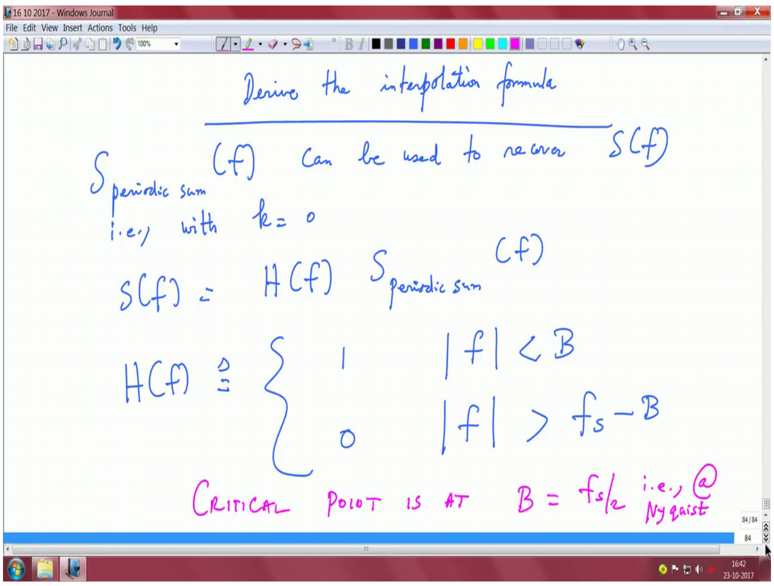

Now, let us derive the interpolation formula, Now, this periodic some copies of the spectra can be used to recover S of f, that is; with k equals 0.

So, we can say that S of f is some H of f with S periodic sum of f, but I say H of f is defined as a filter which is 1 for frequencies less than B and 0 for frequencies outside f s minus B. This is very critical. And the crossover point happens when B equals f s by 2.

These are the Nyquist rate. If not at Nyquist then it is fine, we do not have anything to worry. But if we are at Nyquist, then we have to be worried about.

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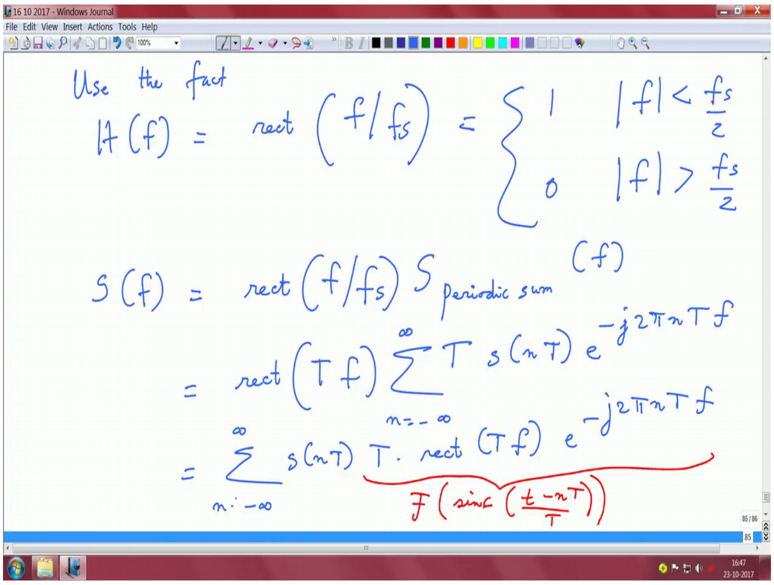

So now, H of f is basically rectangular filters, which is basically 1 inside the band; and 0 for all frequencies outside, and have deliberately left the point when it is equal because that is a discontinuity point. But if you want you can fold it into one of these inequalities. Now X of f is rectangle, f upon f s times S periodic sum of f, and we linked what this periodic sum is.

So, this is a rectangle T times f because capital T is 1 upon f s times summa n equals minus infinity to plus infinity, and we can write this as periodic sum as T times s of n times capital T with the modulator, we derived this earlier. So, now, we fold everything here. So, this is basically summation n equals minus infinity to plus infinity, S of n times T, times T, times rectangle T times f times e power minus j 2 pi n times capital T times f.

Looks really messy at this stage it looks, I am just folding this constant inside the summation. Now look into this. It is basically the Fourier transform of sync t minus n times capital T by t. And now, I think you can almost see that how the result is correct, I mean you just take the inverse Fourier transform on both sides of this equation.

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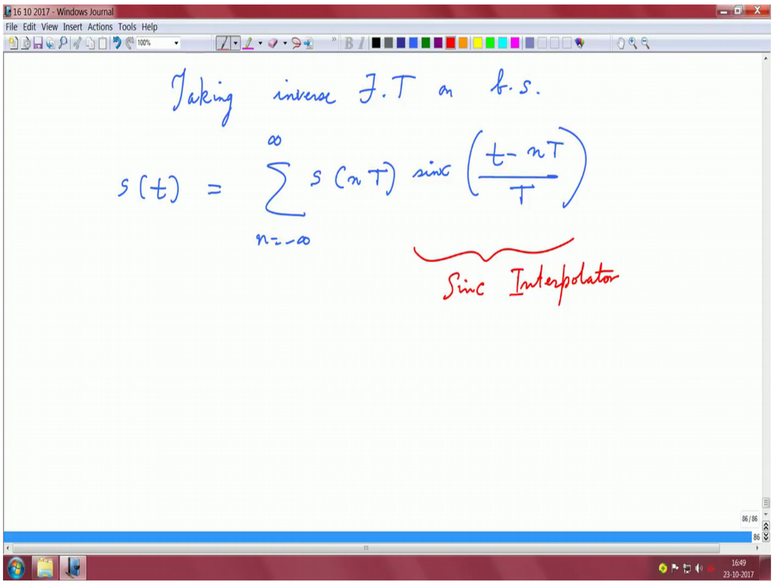

We have x of t equals summation n equals minus infinity to plus infinity, x of n times T sync of t minus n times capital T by capital T.

I mean again because Fourier transform is a linear. So, Fourier transform is a linear. So, therefore, I think I have to use a variable s which is consistent here, I think I can make this S of f so that it is consistent with the rest.

So now, this is fine. So, this is good S and this also happen. So, basically to get the baseband S of f, I use this relationship. And then I expand the periodic some in this form, I fold the rectangle function into the summation, and I can interpret this as basically the Fourier transform of the shifted copies of the sync function, and now you have summation over several Fourier transforms. Summation of something with the Fourier, I take the inverse Fourier transform because it is linear, it automatically stems and this is basically your sync interpolator.

So, this is a reason why people talk about ideal low pass filter towards reconstruction and ideal low pass filter is essentially doing this job of sync interpolation. Now I think one of the important discussions related to sampling would be the following considerations.

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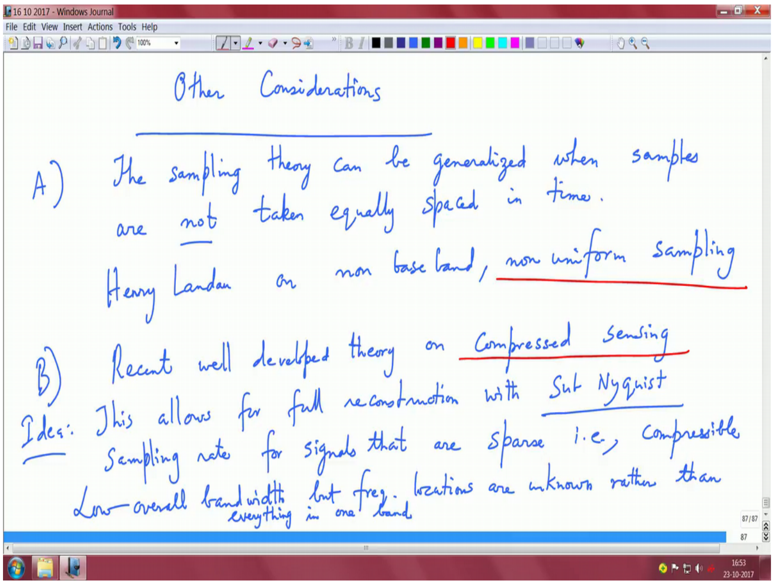

So, I would say these as other considerations. A. The sampling theory can be generalized when samples are not taken equally spaced in time. So, so far we considered samples which are 1 upon 2 B apart, 1 upon 2 B seconds apart if B is your bandwidth in hertz.

Now, samples need not be equally spaced. So, there is work by Henry Landau on non baseband, non uniform sampling. You can kind of generalize the sampling theorem assuming uniform sampling to a more general form with non uniform sampling or irregularly spaced samples. And of late, there is recent well developed theory on compressed sensing or it is also compressive sampling. The idea is, this allows for full reconstruction with sub Nyquist, is very important. Sampling rate for signals that are sparse that is they are compressible. So, what it means is basically you have low overall bandwidth, but frequency locations are unknown rather than everything in one band.

So, these are some variations of the sampling theorem. And one can generalize this from 1 d to 2 dimensions, when you can imagine signals in space etcetera in 2 dimensions picture signals etcetera. Color can be continuous. What we think is 256 or 65000 odd colors etcetera, it is our perception.

Color can be really continuous because all of these are frequencies which are basically continuous and if you think about an image that is essentially basically can be sampled from continuous picture in space.

So, I think these are some considerations and one can deal with sampling theory completely from scratch. It is probably a semester of course work. But in the interest of time and towards this course, I have just given you a brief idea of the sampling theorem, the Nyquist criterion and the ideal interpolator and derivation of the interpolation formula.

Starting from sampled signal, how can you reconstruct the continuous time signal? And now once we understand what sampling is, how we can get perfect reconstruction, we are interested in sampling rate conversion. As part of multi rate signal processing that is given digital samples which are sampled at one rate, how can we convert from one sampling rate to another sampling rate without going back to the continuous time signal? And that is the focus of the study for the rest of the modules. We will stop here.

Thank you.