Mathematical Methods and Techniques in Signal Processing - 1
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Lecture – 31
Fourier transform of Dirac comb sequence

Let us get back to 1930s, 40s perhaps and people who are thinking about the following questions from communications perspective and many other applications. I am given a continuous time signal, analog signal; is it possible for me to sample this analog signal and use the samples for further processing in the process of sampling can I reconstruct the analog signal perfectly right; these are the questions which were hounding these people in the 1930s and 40s for most of the communication engineers. And one such brilliant idea came from Shannon and I will I will I will just state the ideas here.

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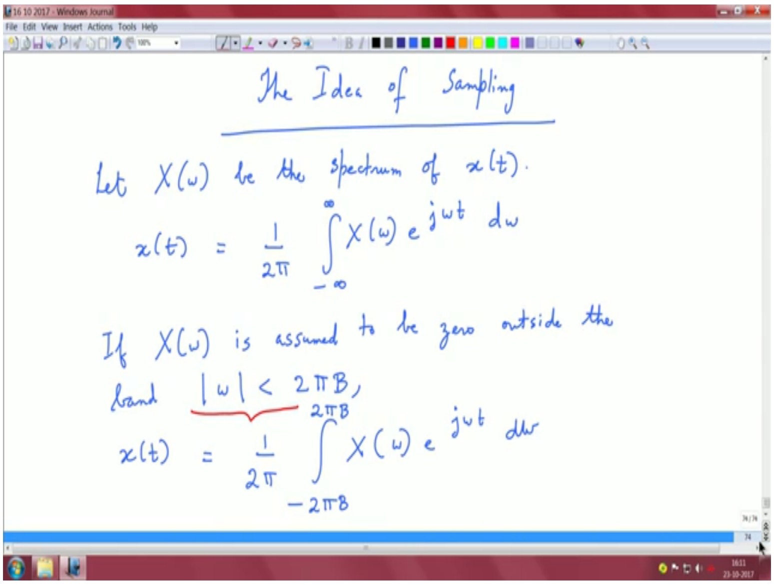

So, let us start with the following let X of omega be the spectrum of x of t right x of t is a continuous time signal. x of t is one upon two pi integral minus infinity to plus infinity X omega e g omega t omega and we know this; this is X omega is the Fourier transform of x of t and then you can get the x of t from the inverse Fourier transform.

Now if X of omega is assumed to be 0 outside the band modulus of omega is less than 2 pi times B right then we can write x of t as one upon 2 pi integral minus 2 pi B 2 plus 2 pi B X of omega e j omega t d omega.

Just straight forward. I mean you can think about why we need to make this assumption here about the band limitedness. When most signals are band limited I mean you think about speech signal you think about most natural signals are band limited and even if they are not band limited by filtering you can make them band limited ok.

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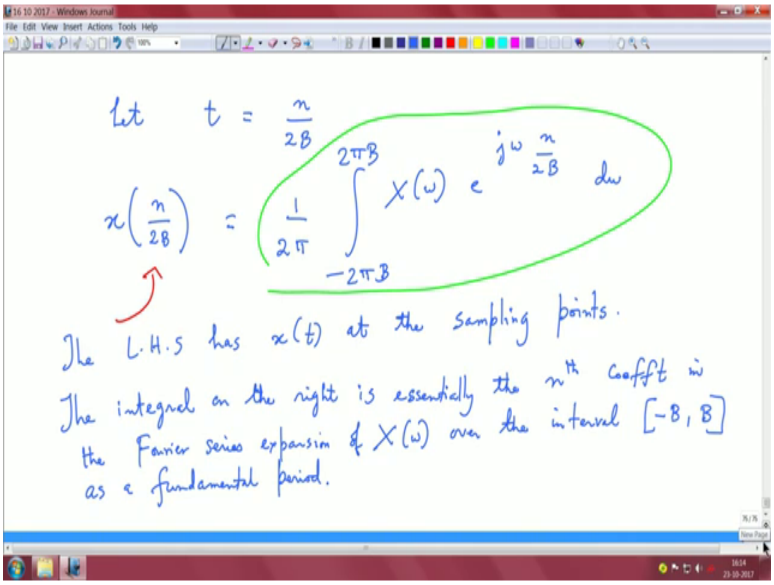

Now, let small t be n upon 2 times B therefore, I can get the samples x of n upon to B as follows. So, this is basically 1 upon 2 pi minus 2 pi B to 2 pi B. So, basically the continuous time t is replaced by n upon 2 B. This is X of omega e j omega n upon 2 B d omega right. Now you have to really interpret this the left hand side has x of t.

Add the sampling points the integral on the right which is this is essentially the nth coefficient. In the Fourier series expansion of X of omega over the interval minus B to plus B as a fundamental period right.

I mean this is the way we interpret this equation.

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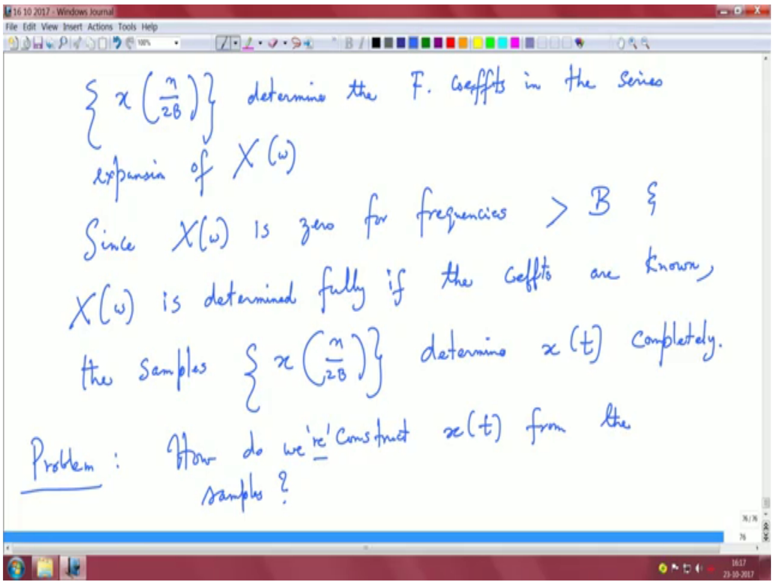

Now, x of n upon 2 B; I think this sequence that we have for n equals 0, 1. So, on these determine the Fourier coefficients in the series expansion of X of omega. Now, since we assumed that X of omega is 0 for frequencies greater than B; and X of omega is determined fully if the coefficients are known all right.

We can say that the samples x of n upon 2 B determine x of t completely right. We started off with the spectrum and then linking with the continuous time signal then basically sampling and then interpreting this as basically the Fourier coefficients the series expansion of the spectrum right. So, one of the questions; that naturally comes to us is, how do we construct or if you want you may add the word re con reconstruct x of t from the samples? Right our intuition tells us we should have some kind of interpolation filters right.

We should we have the samples and there should be some interpolation filter which has to go through these samples and it has to be with right interpolation filter if you are to reconstruct this right. So, we will try to get towards this idea more formally within a mathematical framework ok.

Though you have a basic idea; now I take a spectrum I sample the I i take the spectral and I look at the Fourier coefficients in the series expansion of the spectrum and I get these values and how do I interpret these values as right and the other question is I have the samples can I get my continuous time signal back.

So, let us start with some very basic stuff that you will that you know from signal processing right.

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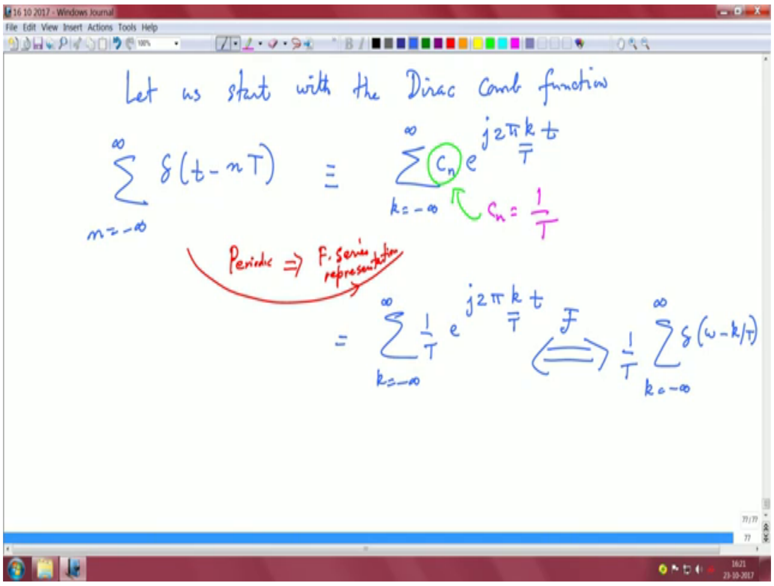

We will start with the Dirac comb function comb, because its really a comb I mean Dirac function. Because the delta function and if you have a train of the delta functions its basically like a comb right comb that you use for your combing your hair. Now summation n equals minus infinity to plus infinity delta of t minus n times capital T is this Dirac comb sequence and the sequence is periodic and since the sequence periodic it has a Fourier series representation.

I mean you can just get the exponential form right you can you can write it as some c n times e power j 2 pi k upon capital T times small t and you can you can figure out this coefficient c n right and if you if you solve for this Fourier series representation ok. Now, this c n you can compute it to be 1 upon t ok. So, very straight forward just plug in the formula for the exponential series c n equals integral from t naught minus t by 2 to plus t you will just get the integral form and.

Then you know you will get to solve this c ns straightforward.

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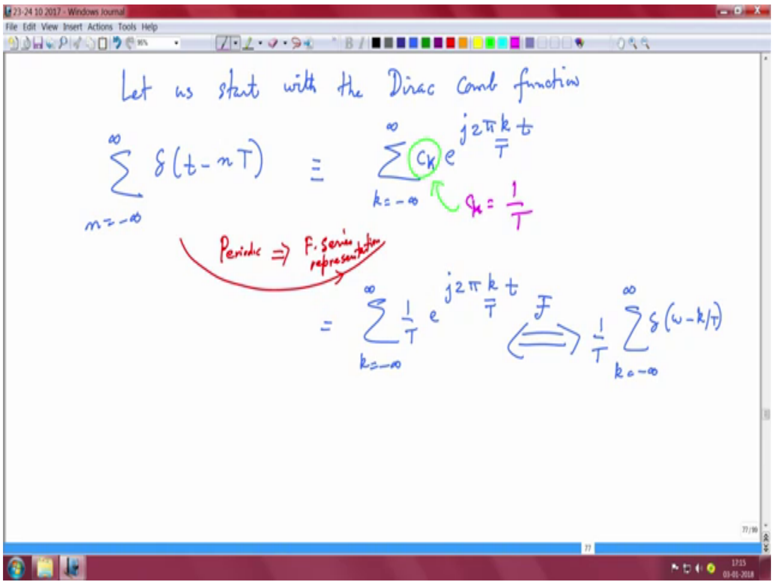

And small typo here and this has to be c k here lot c l. So, let us change the subscript this is a c k and this is the c k here ok. Now basically we can say that this is 1 upon t e power j 2 pi k upon capital T times this t and this is basically Fourier pair and you can write this as 1 upon t sum of k equals minus infinity to plus infinity delta omega minus k upon t right Fourier transform pairs.

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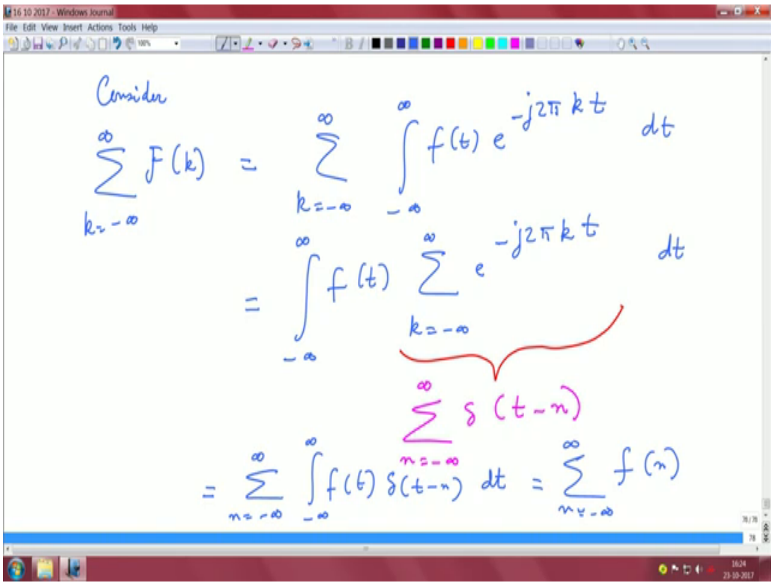

Now, let us consider sum of k equals minus infinity plus infinity f of k just the Fourier coefficients. So, I have the summation k equals minus infinity to plus infinity. Now f of k

I replace it by this integral, because this is how I would compute the Fourier coefficient right Fourier transform f k. Now, this can be written in this form assume I can do the exchange. So, I have minus infinity to plus infinity f of t summation k equals minus infinity to plus infinity power minus j 2 pi k t.

Now, you can say that this summation can be replaced as I will put this in pink summation n equals minus infinity to plus infinity delta t minus n note that capital T is 1 if you go in this form here, but a capital T here; if the t equals 1 you would have this form right. So, therefore, I can simplify this equation as now again I do this exchange back sigma n equals minus infinity to plus infinity then integral minus infinity to plus infinity f of t I think when I have an integral I have to put a d t here right it is something to be careful about ok

Now; so, I exchange this back again assume this exchange is possible. Now this is delta t minus n d t and this would exist when t equals n therefore, this is basically sum of f of n right. We are now sort of we have linked the time samples to the frequency sample the summation right.

Now, let us sort of go one step forward.

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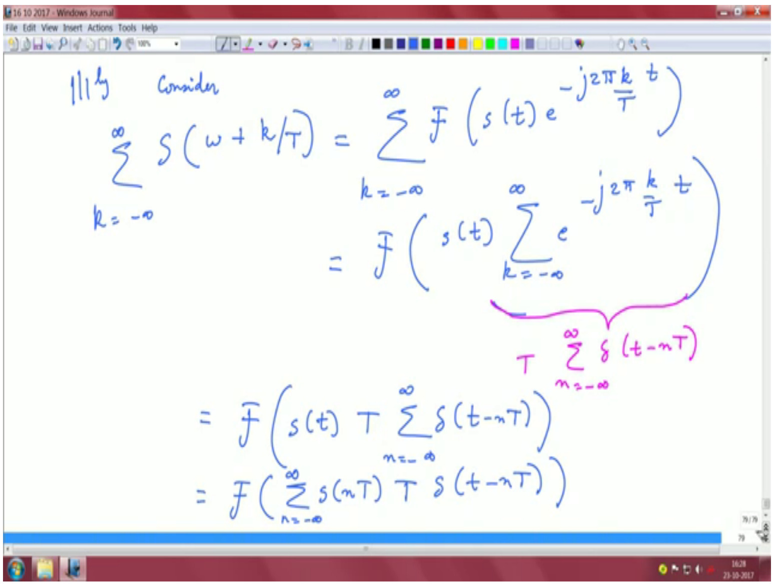

Similarly, consider a periodic sum S of omega plus k upon t this is basically sigma k equals minus infinity to plus infinity I have a spectrum here; basically I take some periodic copy essentially of the spectrum right is what is happening here this is how I would interpret this sum.

Now, this is basically Fourier because Fourier transform is a linear transform right this is I can I can take the Fourier here Fourier of S of t e power minus j 2 pi k upon capital T time small t this is just basically the modulation property right. So, this can be interpreted as the Fourier of S of t is taken outside because there is no index k here summation k equals minus infinity to plus infinity right e power minus j 2 pi k upon t capital T right, because the Fourier transform is linear I can just this is this is a making use of linearity here at this step.

Now, you can interpret this as T times sum of n equals minus infinity to plus infinity delta t minus n times capital T right because its because from our relationship from starting with the Dirac comb sequence. Now, this is basically the Fourier transform of S of t times capital T times sigma n equals minus infinity to plus infinity delta t minus n times capital T. Now these things are existing for t equals n times capital T right otherwise it is vanishing. So, therefore, this t I can replace as n T and roll everything in the summation.

So, I can write this as a Fourier of summation n equals minus infinity to plus infinity S of small n times capital T times capital T times delta t minus n times capital T right. Because this exists a t equals n T not otherwise. So, therefore, I will I will pull this here.

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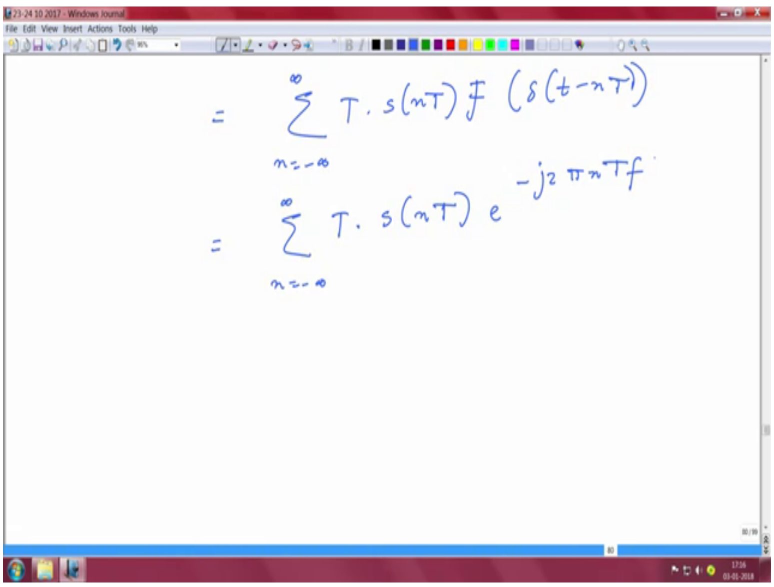

Now I can say that the last part is basically sum of n equals minus infinity to plus infinity t times S of n times capital T times of Fourier of delta of t minus n times capital T and I can interpret this as t times S of n T; that is I am having the samples of the signal and then linking them with this modulation term well if you follow the convention that omega is 2 pi f we just have this to be f here.