Mathematical Methods and Techniques in Signal Processing – I

Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 30
Problem on MAP Detection

Let us get started with our interactive problem session. So, this problem was posted in homework 2 as part of the mathematical methods and techniques in signal in signal processing offered at ISCC in 2017.

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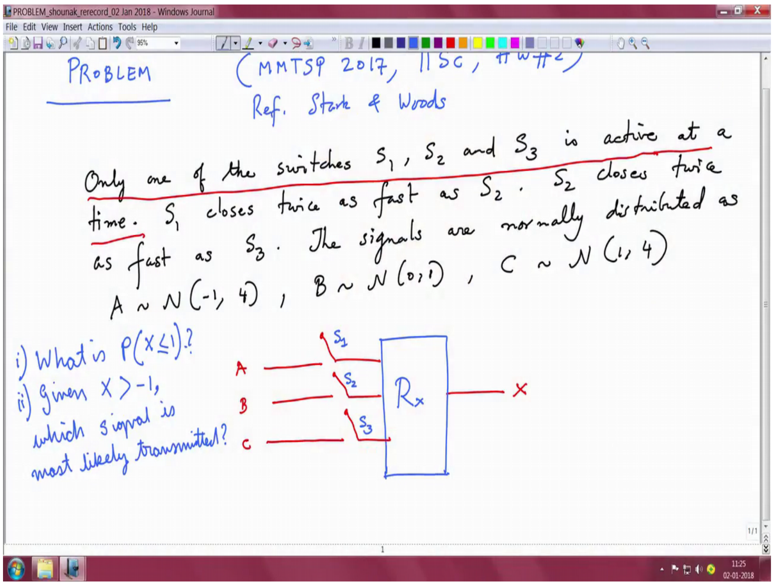

So, the reference to this material is stark and woods. So, here is the problem only one of the switches S 1, S 2 and S 3 is active at a time switch S 1 closes twice as fast as S 2 switch S 2 closes twice as fast as S 3; the signals are normally distributed as follows a is normal with mean minus 1 and variance 4 B is having mean 0 and variance 1 and C has mean 1 and variance 4.

So, basically you have these switches which will operate and then you get some received signal x what we want are the following.

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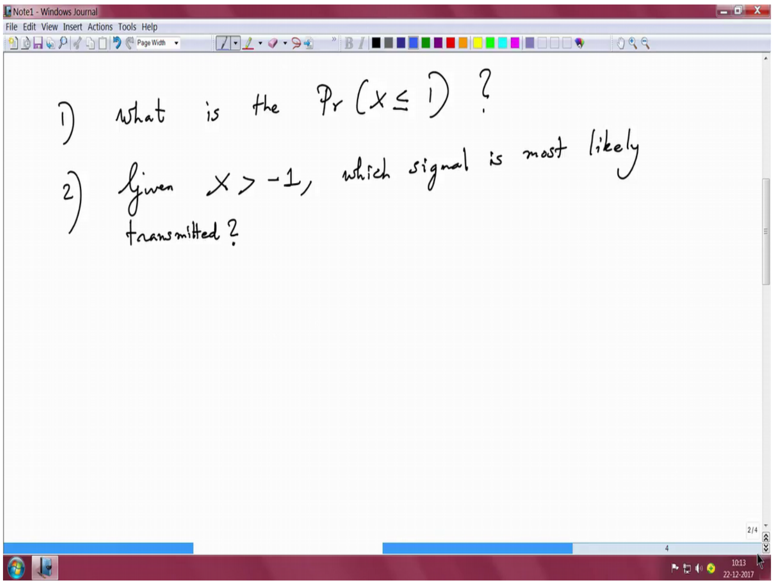

First what is the probability that x is less than or equal to 1. So, I receive some x what is the probability that x is less than or equal to one. The second question is given that x is greater than minus 1 which signal is most likely transmitted. So, using your basics in probability, how you would go about solving this problem. So, this problem will be solved by (Refer Time: 01:42)

So, this problem is a let me go through the problem once and then we will try to solve it. So, we have a receiver which is connected to 3 switches A, B and C where the switch is S 1, S 2 and S 3 which is which is sent to and x is the receiving side now a very important condition that I have underlined over here is only one of the switches S 1, S 2, S 3 is active at one time S 1 closes twice as fast as S 2 and S 2 closes twice as fast as S 3 and the random variables A, B and C are normally distributed with the with the following mean and variance as given here.

We have to first calculate what is probability that x is less than equal to 1 and next one is given x is greater than minus 1 which signal is most likely transmitted. So, we will start working with the first problem.

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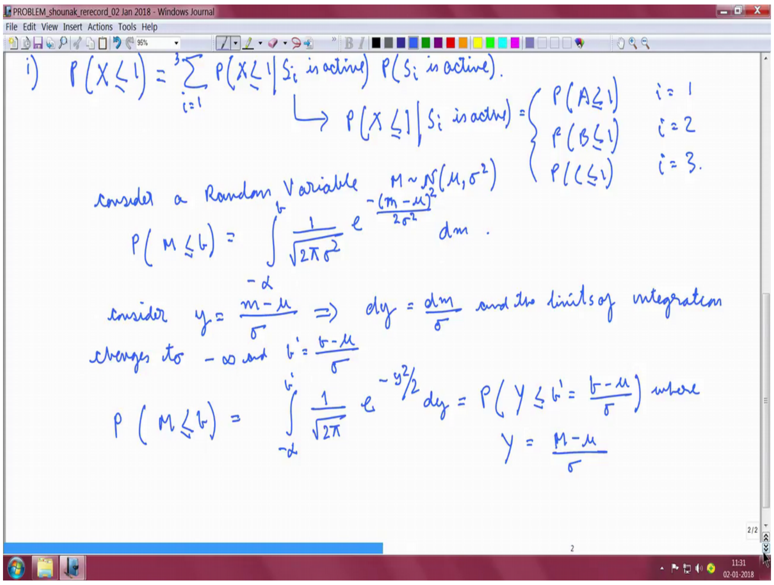

Probability that x is less than one is equal to. Now this is exactly the reverse my generation what we are doing over here. Now this particular quantity that we have over here this is nothing, but probability that x less than equal to 1 given S i is active means probability that a is less than or equal to 1 for i equal to 1 probability that B is less than equal to 1 for i equal to 2 and probability that C is less than equal to one for i equal to 3.

Now, the let me just write the normal distribution function. So, if we have considered a random variable m with normal distribution having mu mean and sigma variance we have now this is the standard normal distribution equation, we will do a small modification over here to make our life a little easier.

Let us consider y equal to m minus mu by sigma we will do a substitution this implies that d y is equal to dm by sigma and the limits of integration changes to minus infinity remains same and some B dash which is nothing, but B minus mu by sigma; that means, now if after doing this substitution we have probability m less than equal to B is equal to minus infinity to B dash which is this value over here one by root 2 pi into e to the power minus y square by 2 d y.

This is basically if you now look at r edges you will have y less than equal to B dash equal to B minus mu by sigma where y is equal to m minus mu by sigma. So, this; obviously, looks a little less messy as you can see this is A 0 mean and variance one probability distribution now we actually try to solve the problem. So, what we have to actually solve is these values these values over here that we have to solve just let me open a new page.

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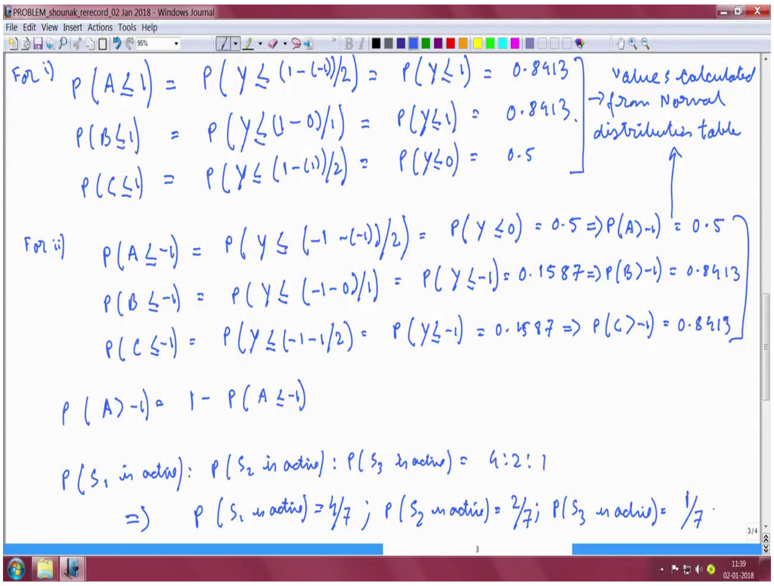

Probability that a is less than one now we will just substitute in the in the normal destruction function that we have done before is nothing, but probability that y is less than. So, we have over here m minus mu by sigma we are exactly going to do that substitution and let us just check the normal distribution mean and variance of a; this is given a minus 1 comma 4 over here so; that means, this falls down to 1 minus of minus 1 which is a mean and I put the variance 2, this is equal to probability that y is less than 1.

This comes out to be 0.8413, similarly probability that B is less than 1 is probability that y less than equal to 1 minus 0 by 1, it is equal to probability that y is less than one is a same value 0.8413 and next we have probability that C less than equal to 1 is equal to probability that y is less than equal to 1 minus 1 by 2 this is equal to probability that y is less than 0 equal to 0.5.

All these values are calculated from the normal distribution table. Now this is what we require for the first problem, I will also write what we require for the next second part of the problem and then we are going to assemble the problems together. So, for 2; so, this is 4 one that what we need sorry to just write this we have to calculate probability that a is less than minus 1 is equal to probability that y is less than minus 1 minus of minus 1 by 2 which is probability that y is less than 0 equal to point five probability that B is less than minus 1 is probability that y , but this is not the only thing that we need here we basically have to calculate probability of x greater than minus 1, right.

So, this implies that basically we are looking at the reverse order; that means, what we are going to use over here is that we just write e to write it over here we are basically want to calculate this thing. So, probability that A is greater than minus 1 is nothing, but one minus probability that A is less than equal to minus 1. So, that is what we are going to substitute over here for all A, B, C and just we have this thing.

Probability of A greater than minus 1 is 0.5, this implies probability that B greater than minus 1 is 0.8413 implies probability that C greater than minus 1 is equal to 0.8413 these values are also calculated from here. Now we actually solve the problem. So, if you look at the first equation that is that if we do that we have here.

So, what we have to do? We have already calculated P A greater than 1 less than equal to 1, B less than or equal to 1 and C less than is equal to 1, all the probabilities over here, we are going to put that over here, but we also need this probability that S i is active that we do not know it is not explicitly mentioned in the question, but what is mentioned is that S 1 closes twice as fast as S 2 and S 2 closes twice as fast as S 3.

So, what we are going to do we will do a simple ratio that is probability that S 1 is active is to probability that S 2 is active is to probability that S 3 is active is in the ratio 4 is to 2 is to 1 which implies probability that S 1 is active equal to 4 by 7 probability that S 2 is active is equal to 2 by 7 and probability that S 3 is active is one by 7 now we have all the tools that we need to solve question number 1.

So, now let us solve 1.

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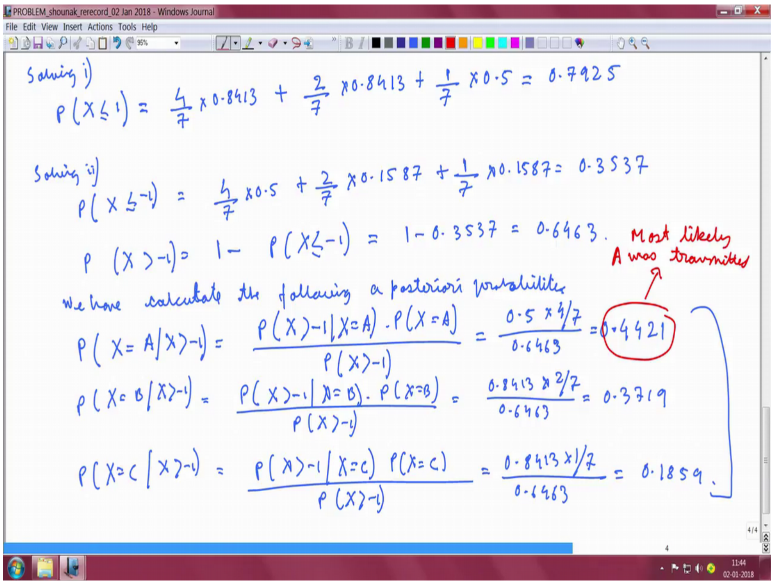

Probability that x less than one is; so, we have this 0 y less than or equal to 1, this a less than equal to one into 4 by 7 plus again 0.8413 into 2 by 7 plus 0.5 into 1 by 7. So, if I just write it over here; this comes out to 0.7925 we are done with the first part.

Now, in order to solve for the next part that is solving 2 in the same way, we will first calculate this thing probability x less than minus 1 is 4 by 7 into 0.5 plus 2 by 7 into 0.1587 plus 1 by 7 into 0.1587 which is equal to 0.3537, but we really need this right you read port of x greater than minus 1 this is 1 minus probability of x less than equal to minus 1 which is given by which is 1 minus 0.3537 equal to 0.6463.

But what we really need is once we have calculated is we have calculated probability of x greater than minus one, but what we need is what signal has actually passed through the receiver because we know that at a time only one signal has passed is allowed to pass. So, we have to calculate the posterior probability.

We have to calculate the following probabilities that is probability that that a was transmitted that is given by this given x greater than minus 1 standard conditional probability rule x greater than minus 1 given x equal to a into probability that x equal to a by probability that x greater than minus 1, this is equal to 0.5 into 4 by 7 by 0.6463 that comes out to 0.4421.

Similarly, we have probability that x equal to B given x greater than minus 1 is equal to probability that x greater than minus 1 given x equal to B into probability that x equal to a by x greater than minus 1, I will just write down the value over here 0.8413 into 2 by 7 by 0.6463 that is equal to 0.3719 and finally, x equal to C given x greater than minus 1 equal to probability that x greater than minus 1 given x equal to C or there is a small mistake here, I will just this will B .

Now, we have calculated these values, but; however, to make the decision we have rescue to select out of these which is a largest. So, as you can clearly see that A; that x equal to a given x greater than minus 1 is the highest so; that means, we can make a decision that this is the highest. So, we can make a decision that most likely A was transmitted this particular process of making decision making is called maximum a posteriori probability decision making.

So, this is very important in coding theory or wherever there involves a transmission. So, imagine that you have a channel where there is a transmitter site and there is a receiver site. So, what happens is in the transmitter site you send upon signaling receiver site you use the particular signal, but there may be some error happening inside the channel.

So, imagine that the first the transmitter side you can receive either you can transmit either A 0 or A 1 and in receiver side you going to receive either A 0 or A 1, but how do you know what has been transmitting with what you have received. So, for example, if what we have to calculate is if at the receiver side we get A 0, then we have to calculate what is the probability that A 1 has been actually transmitted or a probability that was 0 has been transmitted what we basically have to do is the following. So, if we have a channel like this. So, suppose this is my channel this is C.

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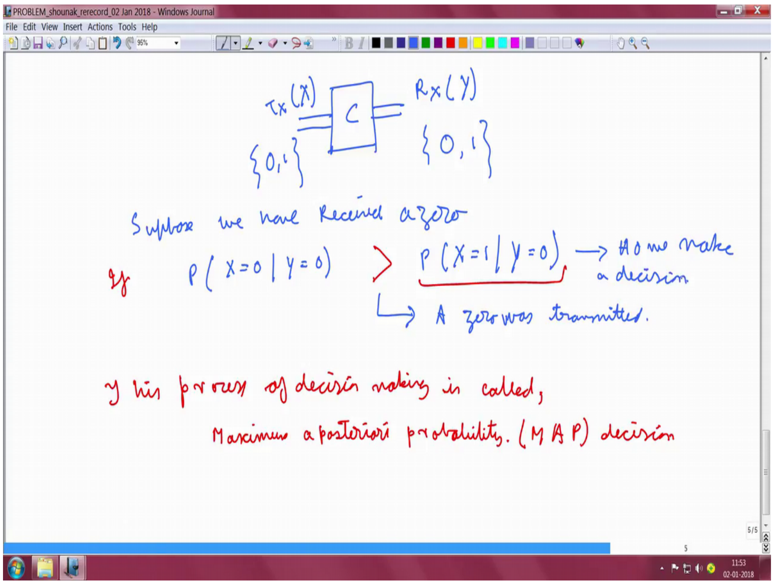

So, this is my transmitter side that is t x and these were receiver side R x.

So, here it can send a value 0 or 1 here; obviously, it will receive a value either 0 or 1, but it might have flipped because of the error, it might have flipped, I will just write this a little bigger. So, the digital mean that you have to do suppose we have received A 0 suppose we have received A 0.

So, let me write this as a random variable y and this as a random value will x suppose. So, probability we have to calculate x equal to 0 given y equal to 0 that is y was actually we say 0 was actually received at the y, we have to calculate priority at x equal to 0 and we also have to calculate probably that A 1 was probably transmitted given we have received A 0.

Now, if this is bigger; that means, what we have over here is this implies this implies A 0 was transmitted; that means, there is actually we have not undergone an error process, but if this was bigger if this was bigger; that means, if this was bigger then what we will have is A 1 was probably transmitted; that means, the B that we have received is actually an erroneous bit and we flip it back.

So, this process of decision making is called or in our language you can write it as a MAP. So, just let me, I will go through a recap the problem again. So, we have to first calculate that probability that x is less than equal to minus 1. So, what we did we use a reverse modulation generation over here for did x less than equal to one is given by this probability that x less than equal to one given S i is active in to whatever S i is active.

We calculate the probability of S i is active later. So, the first term on the r h S is given by this that for x greater than one given S i is active means; obviously, a has been transmitted if it is i equal to 1 and similarly for B and now we write the random variable function over here we do A, unless substitution given by this particular point over here and we have A 0 mean and variance one expression this will be helpful for substituting or calculating the values of A, B and C and that is what we calculate next probability of a less than equal to one B less than equal to 1 and C, A is equal to 1.

And we also calculate probability of a greater than minus 1 B; B greater than minus 1 and C greater than minus 1 and we also find probability of S i is active as this because explicitly, it was not mentioned, but we have was given a ratio that S 1 was we have this 4 is to 2 is to 1 ratio and similarly we have calculated probability S 1 is active is 4 by 7 S 2 is active 2 by 7 and S 3 active is 1 by 7 and then we solve the first part of the problem just substitute the values that we have calculated we get this.

In a second part, we also did a similar process as we calculated probability of x greater than minus 1 as this which is 0.6463 and, but our main question that was asked in the second part was what signal it might have transmitted. So, that we have calculated from probability x equal to A given x greater than minus 1 and we calculated these values for B and C also and we chose the one which was highest evidently A is the highest over here.

So, we make it decision that most likely A was transmitted. So, this type of decision making is called maximum a posteriori procedural probability which is very much used in communication these days. So, what we have is basically a channel which has the transmitter side and a receiver side the transmitter side can signal A 0 or A 1 and the receiver side can receive a signal 0 or A 1.

Now, let us consider the transmitter side as the random variable x and the receiver side random variable y suppose we have received A 0, then what was transmitted we make a decision in this following way we calculate probability of x equal to 0 given y equal to 0 receiving as 0 over here and if it is greater than probability that x was A 1 was transmitted given, we have received A 0 we make a decision either 0 was transmitted represent an error has not happened this type of decision making is called maximum a posteriori probability or m a p decision and the probability that we calculate is based on what the channel is. So, is a channel is a AWGN channel, we will have a normal distribution if it is something else like a standard basis we will have some different value of mean and variance or whatever we the distribution given.

So, I am not right explicitly mentioning the distribution over here this is the point where how we make a decision how we make a decision in the MAP process. So, in this particular problem you we have a slight difference we have 3 signals and we are receiving that now the following signal the same A B C signal on the receiver side. So, here which is the ternary alphabet in standard communication we have since we have bits it will basically a binary alphabet. So, with this we is end solving this problem.