Mathematical Methods and Techniques in Signal Processing – I

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Lecture – 28
Stationarity of random processes

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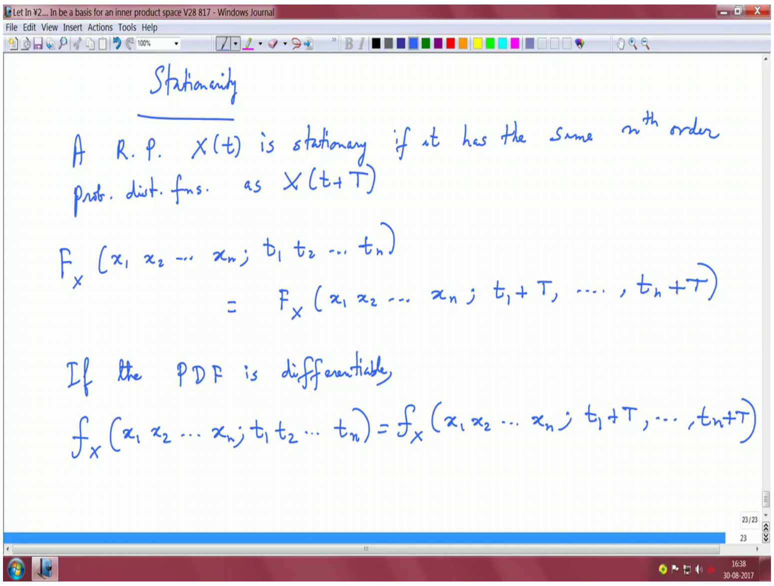

A random process X of t is stationary, if it has the same nth order probability distribution functions as X of t plus capital T, I mean I shift this waveform by some time period say capital T, then the probability distribution functions should be the same that is F x of x 1 x 2 so on till x n specified at times t 1, t 2 so on till t n should be the same as F x of x 1, x 2 so on till x n specified at times t 1 plus capital T dot, dot, dot till t n plus capital T.

If I shift the time and I do some observations on the probability distribution functions they have to be the same if this process is stationary. So, if the probability distribution function is differentiable, then you know the densities have to be the same. So, this is a very very important idea, I mean important concept, I would say stationarity and if you think, this process is stationary; then one can think is a very strict condition, then if you observe all the moments they should also be the same; compute the mean, compute the second order moment, third order moment etcetera, etcetera, etcetera, etcetera, you know that it only they have to be there same. So, it is very difficult to satisfy this property of stationarity.

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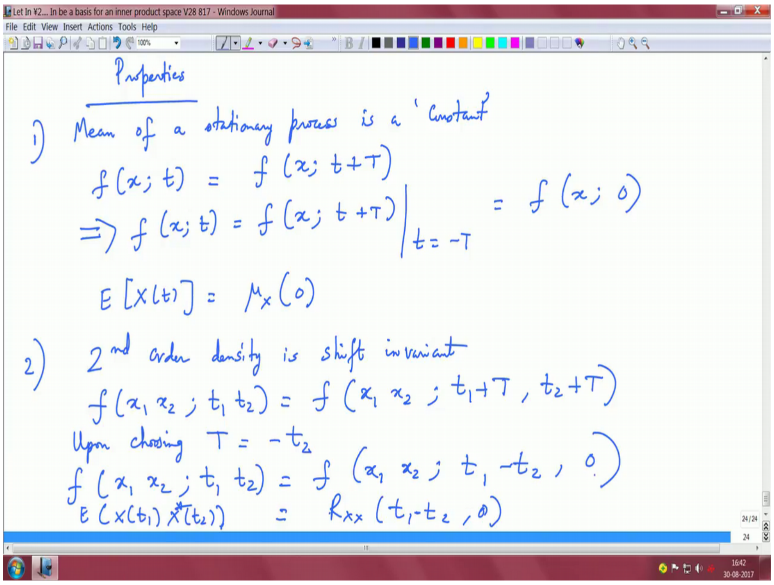

So, there are a few other basic properties which I would like to sort of mention mean of a stationary process is a constant. So, if you take of f x t which is the density that has to be the same with a time translation. So, which implies that you could compute this shifted density at time t equals minus T; that is I just do a time translation backwards minus T therefore, this is what I get.

Now, expected value of this process X of t which is stationary is mu x of 0 and this is basically constant. The second order density is shift invariant; so, we have f x 1, x 2 specified at times t 1, t 2 is basically if it is stationary we say it is x 1, x 2 at times t 1 plus capital T some t 2 plus capital T. Upon choosing capital T is say minus t 2; f of x 1, x 2 at times t 1; t 2 is f of x 1, x 2; this is going to be t 1 minus t 2 with 0; which is basically R x x of t 1 minus t 2 comma 0; which is what the expectation of thought of covariance basically this is going to be this.

So, if you want to prove the second order density is shift invariant basically we start with the density function specified at times t 1 and t 2. Then we look at the density function which is translated by a time capital T. So, we choose some capital T is minus t 2; this becomes t 1 minus t 2 and this is basically 0 is what the density that we get. And if you look at the autocorrelation function specified at times t 1 and t 2; this basically just depends upon the time lag between these 2 points.

In the second order density is basically shift invariant. Now there is a weak form of stationarity.

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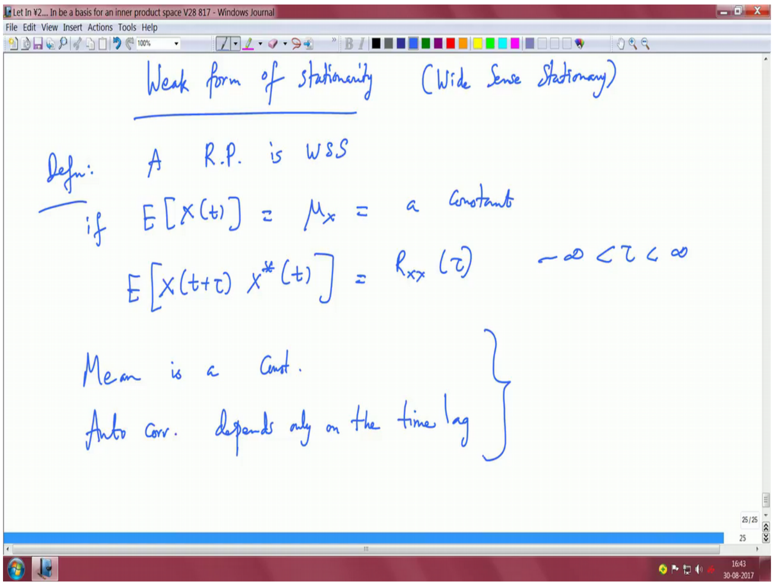

It is also called wide sense stationary process. So, is the definition for this a random process is wide sense stationary, if the mean is a constant and the autocorrelation function depends only on the time lag; between the two waveforms for each of the time lags from minus infinity to plus infinity. That is a mean is a constant and the autocorrelation depends only on the time lag. And if these two properties are satisfied, then it is called a wide sense stationary process.

And you can see that it is a weak sense stationarity because we are only specifying it in terms of the mean and the autocorrelation of the process. But in the case of the strict sense stationarity, we had to say that the probability distribution functions at different time specifications had to be the same irrespective of a time translate; it is very difficult to achieve in practice. So, therefore, this is a little more relaxed version of the stationarity.

So, there are other notions of stationarity such as quasi stationarity and so on and so forth and we will not bother to discuss about this for this course. There are like cyclo stationarity so on and so forth, there are many other forms and we will not bother about these details for this class, but if you take a digital communication course you may have to deal with this or a speech course; for example, you may deal with quasi stationary, if you think about in the context of communication signals, you may deal with cyclo stationarity and so on.

But here we will just restrict ourselves to just strict sense stationary process which is basically stationarity implies that is the meaning of stationarity is it is a strict sense stationary process and if it is weak form of stationarity, we mean these to be wide sense stationary.

Now, with this I have a few examples.

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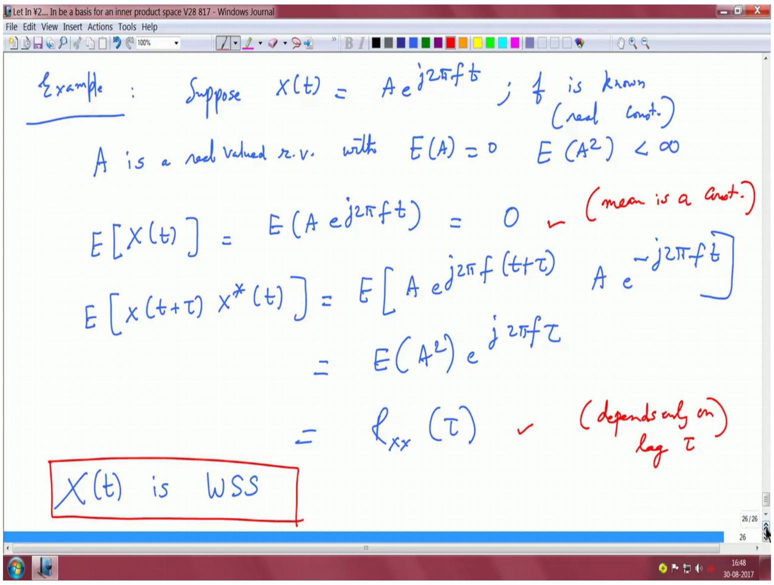

So, let us consider the following example suppose X of t is A e power j 2 pi f t; f is known and it is some real constant, A is a real valued random variable with mean 0 and energy is finite which means look at the variance. Because it is a second moment and since the mean is 0, it was second moment happens to be the variance and that is basically finite.

So, if you look at the mean of this waveform which is expected value of A e power j 2 pi f t; which is basically 0 because expected value of a is 0. And if you look at the auto correlation; that means, you consider A lag of tau. So, this is e power j 2 pi f t plus tau times A e power minus j 2 pi f t because you know to consider the conjugate here this happened to be e of A square e power j 2 pi f tau.

So, this process that we have X of t is A e power j 2 pi f t where A is a real valued random variable this is a wide sense stationary process because the mean is constant which is 0 and this is autocorrelation function which depends only on lag tau clear. So, with this we will look into some exercises.

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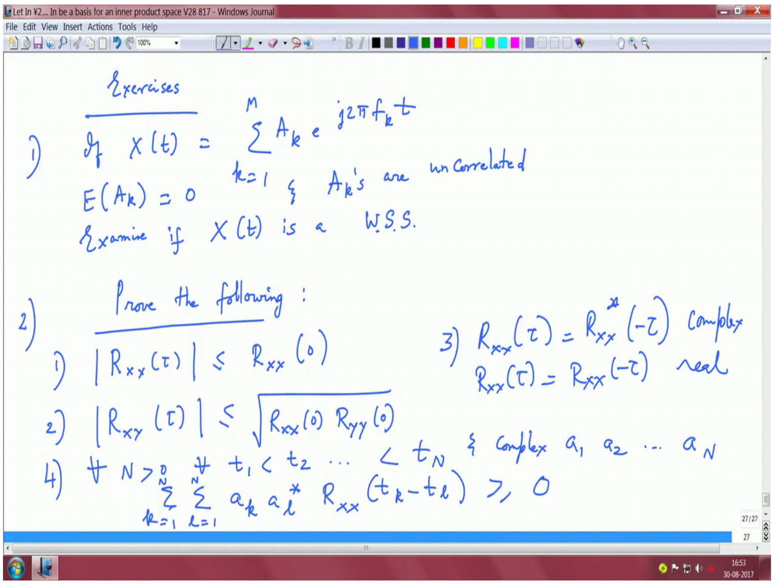

If X of t is given by a mixture of such waveforms right A k is a random variable whose expectation is 0, the mean of A k is 0 and A k is are uncorrelated.

Examine if X of t is a wide sense stationary process we pretty straightforward for you to compute the mean of X of t and then look at the autocorrelation function and then basically conclude if this is a wide sense stationary process we also want you to prove the following properties modulus of r x x tau is less than or equal to r x x of 0 mod r x y of tau is less than or equal to square root of r x x of 0 times r y y of 0.

Then r x x of tau is r x x conjugate minus tau this is for complex signals and for real of course, you could guess that r x x of tau here is r x x of minus tau this is for real and for every capital N greater than 0 for all times t 1 less than t two. So, on till t capital N t s of x capital N and complex a 1 a 2. So, on till a suffix n this quantity some are k equals 1 to capital N some are l equals one to capital N A k a l conjugate times r x X of t k minus t l this is positive semi definite; that means, this quantity is always greater than or equal to 0.

So, you have all the definitions. So, this is basically some homework exercises and the solutions will be supplied after you submit the homework.

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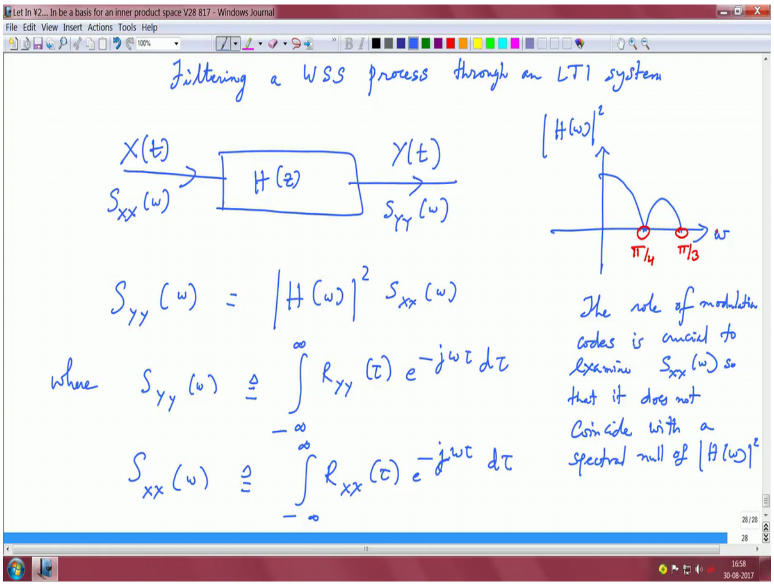

So, now with the tools that we have on random processes is your basically definitions about various statistical aspects of random processes is to let examine filtering of wide sense stationary process through an LTI system and this is a very important result in an in signal processing.

So, if I take a random process whose you know I say X of t is the process and whose power spectral density is S x x of omega and I filter this random process through an LTI system. And we observe the power spectral density at the output, there is an important relationship that links the power spectral density at the input and output with the system transfer function and that property is the power spectral density at the output is the magnitude of the transfer function square times the power spectral density at the input, where you have the power spectral density which is defined as the Fourier transform of the autocorrelation function.

This is for both the input and the output power spectral densities and this is a very very important result because if the channel h of h of omega has a spectral null at certain frequencies then if the power spectral density of the input if this has some finite power at these frequencies then we are wasting power therefore, it is very important and vital to shape the data such that the power spectral density of the data does not coincide with the spectral null of the channel.

So, this is where the role of modulation chords becomes very crucial role of modulation codes is crucial to examine the power spectral density of the input. So, that it does not coincide with a spectral null of the channel they will do very straightforward to prove I think this is basically from your undergraduate material basically you can take become you can basically figure out what the output is in terms of the convolution because it is an LTI system you can convolve the input with the filter and then basically compute the quantities S y y of omega.

First compute R y y of tau then compute S y y of omega and then link all the quantities together; a very straightforward derivation I would not do this it is left as a homework exercise. And I think what is very important in this study is that we do not want to modulate our data in such a way that it coincides with a spectral null, this is very very crucial.

When you design codes coded sequences in communication systems you do very careful that you do not land up with a where; that means, you are wasting power, right, if you if S x x of omega has some finite values at frequency components pi by 4 and pi by 3, then I have power at those frequencies in the data and if it coincides with spectral null; that means, I am just wasting my power by coding the data in such a way that it lands in spectral null. So, therefore, understanding the channel response is very crucial to designing these courses and that is basically the philosophical implication of this theorem.