**Mathematical Methods and Techniques in Signal Processing – I**
**Prof. Shayan Srinivasa Garani**
**Department of Electronic Systems Engineering**
**Indian Institute of Science, Bangalore**

**Lecture – 26**
**Introduction to random process**

So, let us get started with an overview of the basics of random processes because we will have to deal with these concepts in this course. So, it is not too difficult, it will require a background in basic probability. So, I urge you to study the basics of probability through other courses in NPTEL or otherwise and ramp up on this concept because if I spend my lectures on probability, we can spend a semester or probably three semesters on this course, but I will give you the basics of a random signals.

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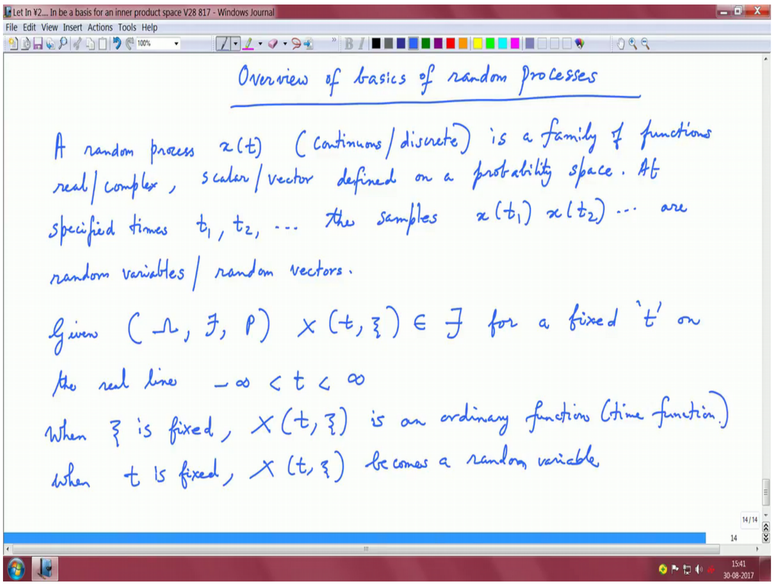


A random process x of t which could be possibly continuous or discrete is a family of functions that are possibly real or complex perhaps scalar or vector defined on the probability space. So, there is a very clear axiomatic definition of what a probability spaces. And at specified times t 1, t 2 so on, the samples x of t 1 x of t 2 dot dot are essentially random variables, random vectors. So, given the probability space omega the field and the measure x of t comma zeta belonging to the field. So, given the tuple which is omega, the field f and the probability measure p, x of t comma zeta is a function that belongs to this field for a fixed t on the real line minus infinity less than t less than infinity.

So, if you fixed t then basically this becomes a random variable and that belongs to this field else it is just an ordinary function. So, when zeta is fixed, X of t, zeta is an ordinary time function. When t is fixed, X t, zeta becomes a random variable that is given the three tuple omega, the Borel field f and the probability measure p, X of t comma zeta belongs to the field for a fixed t on the real line which is from minus infinity to infinity. And when zeta is fixed, X of t comma zeta is an ordinary time function; and when t is fixed X of t comma zeta becomes a random variable.

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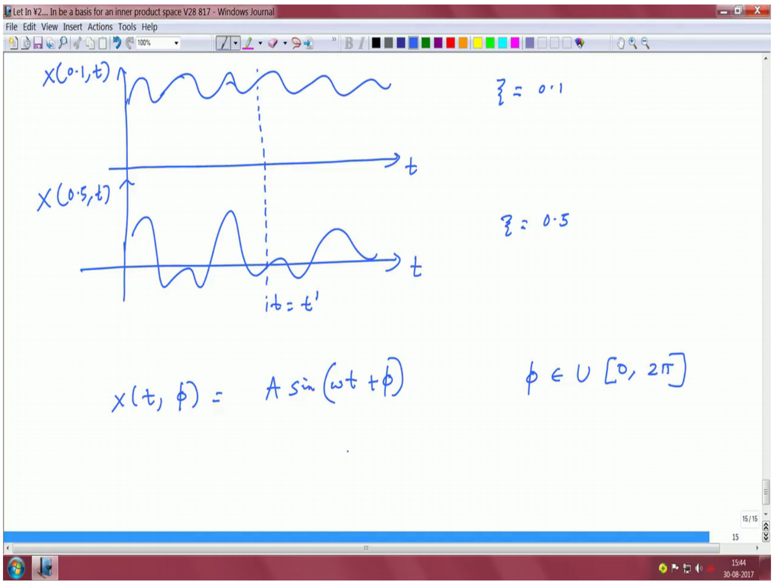


So, given if you think about waveforms, I choose perhaps zeta equals say 0.1, I get one waveform; 0.1 comma t. If I choose say zeta is say 0.5, I may get another random waveform. So, depending upon this parameter zeta, I may get different time functions. So, this is the idea. But if I fix a particular time say t equals t dash - a particular time is fixed right, then these points they are random variables right and that distinction is very important. I mean when we just learn probability, we just look into random variables and then we look at some properties associated with these random variables. But with signals we have to think about random processes which means given a value that it takes the random variable takes, I mean you basically depending on the parameter you have basically a time function.

So, imagine a process, which is A sine omega t plus phi. Suppose, I say that phi is uniformly distributed between 0 and 2 pi. Now, if you plug in phi equals 0 here, you have A sine of omega t. You plug phi equals say pi by 2, you get basically a cosine, you get different waveforms, but you fix a particular value of time and observe, this basically are random variables.

So, I think it is a very important notion. And often in signals can be thought about as random signals because they may be parameterization in terms of if of a random phase or a random amplitude possibly a random frequency, we do not know. So, these things could be basically random variables. And then once you specify those parameters zeta, then basically you have a function of time. So, basically a random process has two parameters zeta and t. And once we specify the random process, we are interested in some of the statistics of these random processes. So, we will just basically define some of these properties and then most more rigorously get into the statistical specification of a random sequence.

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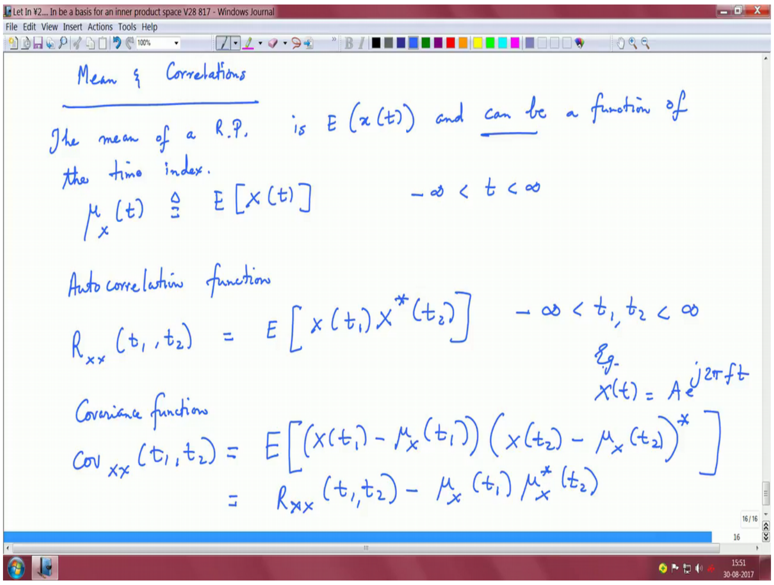


So, mean and correlations, the mean of a random process is denoted as expectation of x of t and can be a function of the time index. So, mu x of t is basically defined as expectation of X of t. So, I think one of the questions that may come to your mind naturally is how are you computing this expectation. I have this waveform in time, am I taking all the samples across time and taking an empirical average over this, or I take a collection of all these waveforms and then I average over the distribution of the underlying random variable or vector. So, these are two questions that naturally come into your mind right, how are you calculating this average right, is it just a normal time average that you think about or is it the statistical expectation.

So, depending upon that there is a time averaged mean and a statistical mean, so get this in your mind very carefully. I mean we will be not perhaps have to deal with this when you think about just the normal statistical mean, I give you some distribution Gaussian distribution. I ask you to compute its mean or you know uniform distribution compute its mean you can compute it you do not have a big problem in doing it. Here now you have to imagine you have a collection of all these waveforms. Now, do you pick one waveform that you get from this collection of waveforms just do a time average on that; or you collect all the waveforms, and take the statistical average over the underlying distribution of the random variables and then get what this average waveform is. These are two different notions which have to be crystal clear in your mind.

So, how you take the averages are sort of the details here. And mean can be a function of the time index as one can expect, the function the mean itself can be a function of time. Now, similarly to what we have considered for the mean there is something called the autocorrelation function. And the autocorrelation function is given by R x x of t 1 comma t 2 that is parameterize at two different times t 1 and t 2. And this is basically the expectation of X of t 1 with X conjugate of t 2 minus infinity less than t 1 t 2 less than infinity this is for complex waveforms. For example, you may have X of t say A e power j 2 pi f t in communication sometimes you will find signals of this type carriers for such signals for phasors you may have to consider it is a complex signal. So, therefore, you may have to consider the conjugate.

Similar to the autocorrelation function, we have the covariance function. And the covariance function is given by covariance x x of t 1 comma t 2 and this is basically you subtract the mean from the waveform x of t right at these times t 1 and t 2 and then basically compute the expectation. So, this is x of t 1 minus mu x of t 1 times x of t 2 minus mu x of t 2 conjugate. And it is very straightforward to verify this is R x x of t 1 comma t 2 minus mu x of t 1 times mu x conjugate. And these properties in terms of auto correlation covariance etcetera will be very useful in many of the tools that we will develop later on in the course.

For example the covariance will be very useful when we have to derive the Karhunen-Loeve transform - KL transform that means, I give you a pool of vectors from we know from a random process. And what can we say if you were to expand this signal in terms of an optimal basis in one of which is interdependent which is dependent on the data. And of course, optimality conditions have to be described more carefully, but we will see this application.

Then autocorrelation function this finds a role when you have to really think about the power spectral density of a process. And why are you interested in power spectral density of a process in communications, because let us say data is random data, I give you a random data which can be zeros and ones in some way. And this random data goes through some filter right, we assume its linear time invariant system LTI system. So, therefore, you have a transfer function, the data gets filtered through an LTI system. And then we are interested in the power spectrum at the output that means now this is basically is the data is being filtered through this filter. And then we were interested in the spectrum. And the channel for example, why I say about a filter because the communication channel may be fixed, you cannot control how air is, you cannot control how the magnetic medium is. So, the channel is fixed.

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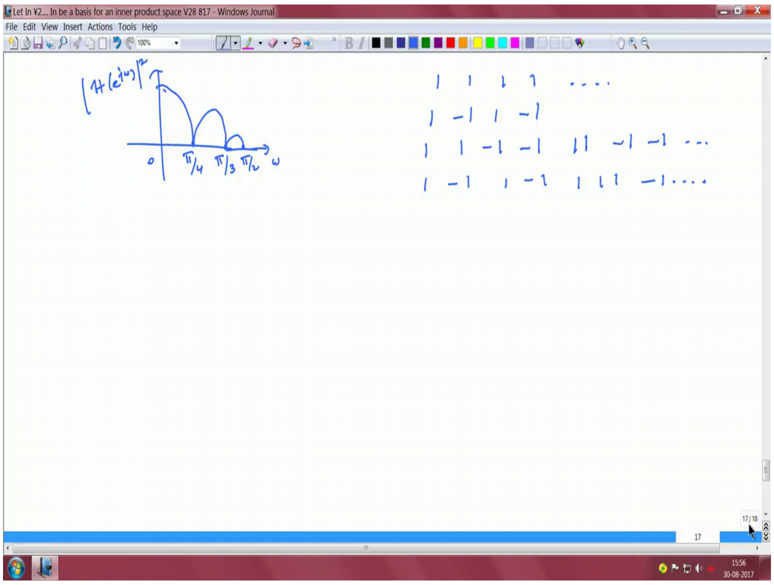


And if the channel has certain spectral nulls that means, if you if you imagine the channel has certain holes at some frequencies, I just say magnitude square, say this is suppose what you have. Then I want to ensure my data the power and the data is I do not allocate power in my data which lands up here. The channel has a spectral null, which means at this at frequencies pi by 4, pi by 3, pi by 2, there is 0 energy. So, even if you pump in power on your data, the data also has a spectrum right.

For example, if you think about 1 1 1 1 dot dot dot it is basically like a dc. If you think about data which is 1 minus 1 1 minus 1 basically it is it is having a high pass component right. If I think about data which is 1 1 minus 1 minus 1 1 1 minus 1 minus 1 dot dot am modulating in some ways. So, if I give you random data 1 minus 1 1 minus 1 1 1 1 minus 1 blah, blah, blah random data, this has a certain power spectrum if it has certain properties. And often I have to calculate, if I have to calculate the power spectrum I should calculate the autocorrelation of the sequence.

So, this is I give you a discrete sequence here, and for which have to compute dot of correlation of this discrete sequence, because you will know the power spectrum from this result from your from Wiener-Khinchin theorem. And if you know that this has some power at pi by 4 and pi by 3, then when I just multiply this spectrum with this spectrum basically its I am just losing my energy. Though I am having energy in the sequence at frequencies pi by 4 and pi by 3, because it channel has nulls are these frequencies and wasting my power by encoding it in certain way. So, this is basically the role of what modulation codes do for us.

And how you want to design your modulation codes to choose the spectrum right that is the motivation keeping that as a motivation this is important for us to compute autocorrelation functions. So, I said it would be discrete would be continuous. So, I gave you an example here, where you have these ones and minus ones which are discrete and you might want to compute autocorrelation of such sequences.

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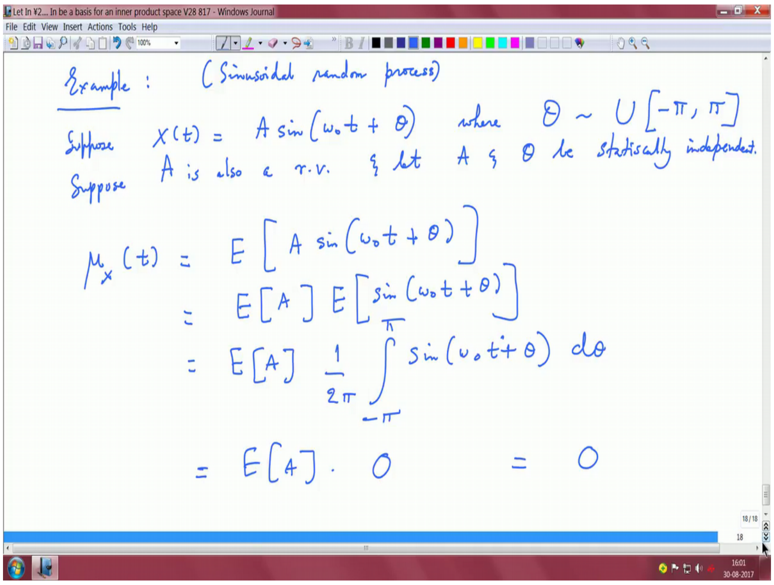


So, let us try to visit an example and then see how to compute these things it is not too difficult. So, let us consider a sinusoidal random process. So, suppose x of t is A sine omega naught t plus theta, where theta is uniformly distributed over minus pi to plus pi. Suppose, A is also a random variable possibly distributed according to some distribution; and let A and theta be statistically independent. Then if you want to compute the mean of this waveform, so I think I would imagine this picture there are two random variables a and theta right. And basically if you think about the sinusoid you can basically express it in terms of a phasor right e power of j omega naught t plus theta plus e power minus j by 2 j, you can express it in this form. And basically what you have is you have two phasors that are summed using this random variable. And you can think about several such practical scenarios.

Imagine I send you a beam of light right and then basically this is now there is a phase, there is a phasor, and then there is basically optical speckle that is happening, therefore this A could be a random variable whose length may be distributed according to some distribution. And then that is a practical scenario one can think about. So, if A and theta are both random variables and they are statistically independent and then you have a waveform here for a fixed omega naught omega naught is deterministic right.

So, the mean is basically the expectation of A sine omega naught t plus theta, and since there since A and theta statistically independent I could write this as E A times expectation of sine omega naught t plus theta. And this is expectation of A times I have to average it right over the distribution of theta. So, therefore, is 1 over 2 pi integral minus pi to plus pi sine omega naught t plus theta d theta because it is the uniform PDF, I pulled the 1 over 2 pi outside and we know this is E of A times 0, because your averaging the sinusoid over its cycle it is 0. So, irrespective of what the mean is for A this is basically zero. So, it is a sort of an interesting observation for this example.

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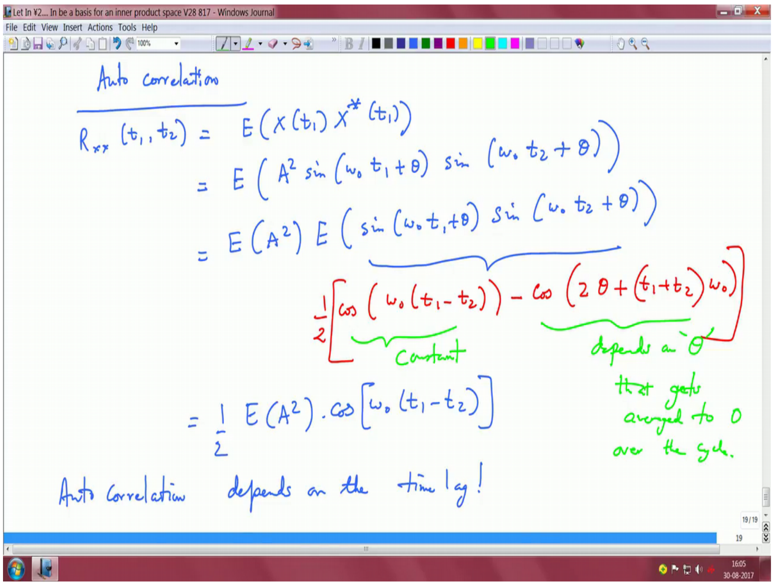


And we can also compute the autocorrelation which is R x x t 1 comma t 2 is expectation x conjugate t 1 write a expectation of t 1 times x conjugate t 1 which is expected value of A square sine omega naught t 1 plus theta times sine omega naught t 2 plus theta, which is basically E of A square expectation of A square times expectation of sine omega naught t 1 plus theta times sine omega naught t 2 plus theta. Now, this quantity here is essentially one half cosine omega naught cosine of omega naught t 1 minus t 2 minus cosine of 2 theta plus t 1 plus t 2 into omega naught. You can just verify this from your compound angle formula.

Now, this has the random parameter theta, this does not have the random parameter theta. This is a constant expectation of a constant is a constant this is basically a cosine function, you average it out in that in interval between minus pi to plus pi that just cancels out, because it is 0, the two cycles cancel out. So, basically what you end up is basically E of A square times one half is pulled out cosine omega naught t 1 minus t 2; that means the autocorrelation depends on the time lag.

So, we have got basically a sort of a feel for what is a random process essentially we have gotten a feel sort of an intuitive feel that it is essentially a time function it is parameterized by two quantities which is basically zeta and t. So, if I fix zeta, it is basically an ordinary time function. And if I fix a t, it becomes a random variable. And then you can do whatever you want to do basically look at investigate the statistical properties of this process. You can take the time average, you can take a statistical average, you can take a time expect time autocorrelation, you can take a statistical autocorrelation right, you can take all the moments time moments, statistical moments, you can do all these kinds of operations. And accordingly there are various notions whether this process is a ergodic and so on and so forth. If the time average is equal to the statistical average, it is ergodic in the mean and it is a very powerful concept, I think a lot of theorems in the foundations of communication theory rest on this notion of ergodicity.