Mathematical Methods and Techniques in Signal Processing
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Lecture - 23
Problem on Signal Geometry

So, let us have some interactive problem solving sessions by students who have taken this course. So, you will see some illustrations and examples into problem solving which is useful to understand and digest the concepts learnt during the lectures.

I am taking PhD student at IISC. So, today we are going to see look into a problem that discusses the signal classification. So, we have studied signal geometry in the class. So, we are going to apply it for some kind of a practical example. So, here we have a source that transmits one of the 4 signals; S 1, S 2, S 3 or S 4. So, it transmits one at a time.

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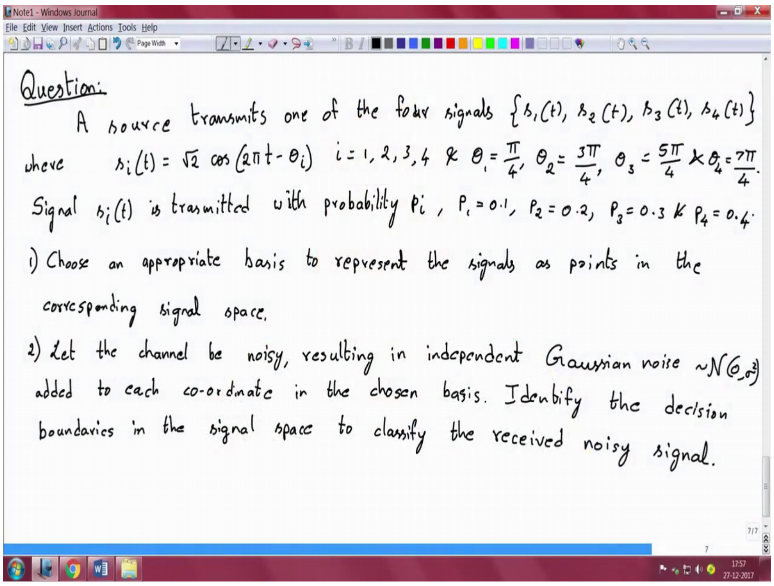

Where each signal S i is given by a cos signal cos 2 pi t, but shifted by different phases. So, the first signals is off setted by pi by 4 offset, the second one by 3 pi by 4, third one 5 pi by 4 and fourth one by 7 pi by 4. So, and each of these probability signals are transmitted with certain probability either 0.1, 0.2, 0.3 or 0.4 respectively. So, the question here we have is; so, we have this signals and the question asks us to represent these signals using appropriate basis in a signal space and the second question second part of the question; it asks us to identify the regions within the signal space in order to make a proper classification of a signal. So, we will get into the details of this question.

So, first part one.

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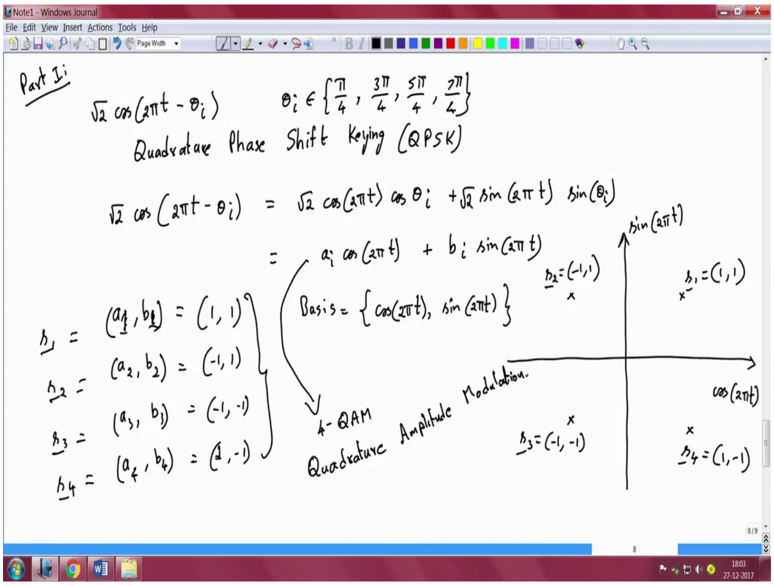

So, notice that the signal root 2 cos 2 pi t minus theta i and theta I, they belong to pi by 4 3 pi by 4 5 pi by 4 and 7 pi by 4. So, these signals these are called quadrature phase shift key in signals of QPSK. So, we can actually expand this cos term as theta as this root 2 cos 2 pi t cos theta pi plus sin 2 pi t times sin theta i.

So, this root 2 is missing. So, this can this cos theta and sin theta they are dependent on the signal i. So, this can be written as a i cos 2 pi t plus B i sin 2 pi t where the coordinates a i comma B i depending on the phase shifts we can write them as one comma one or I would say a 1 comma B 1 as 1 comma 1, a 2 comma B 2 is minus 1 comma 1, a 3 B 3 is minus 1 comma minus 1, a 4 comma B 4 is 1 comma minus 1 and if you notice this signal is written as a summation of 2 orthogonal signals cos 2 pi t linear combination of signals other than signals cos 2 pi t and sin 2 pi t and in fact, you can easily verify that cos 2 pi t and sin 2 pi t also normals signals also that is energy each of the signals is one.

So, essentially here we have the basis is nothing, but this 2 cos 2 pi t and sin 2 pi t and these form the signal points I can call them as S 1 S one S 2 S 3 S 4. So, these 4 forms a points in the signal space. So, in order to visualize what it means here we have sin 2 pi t cos 2 pi t we have S 1 equal to 1 comma one S 2 is minus 1 comma 1 S 3 is minus 1 comma minus 1 and S 4 is 1 comma minus 1.

So, this represents the signal space that we have signal space and the 4 signals S 1, S 2, S 3, S 4; they represented as points in this particular signal space. So, now, moving to a quick comment, here this particular form is called a quadrature amplitude modulation here we have 4 points. So, it is called 4 QAM. So, now, moving to the part 2.

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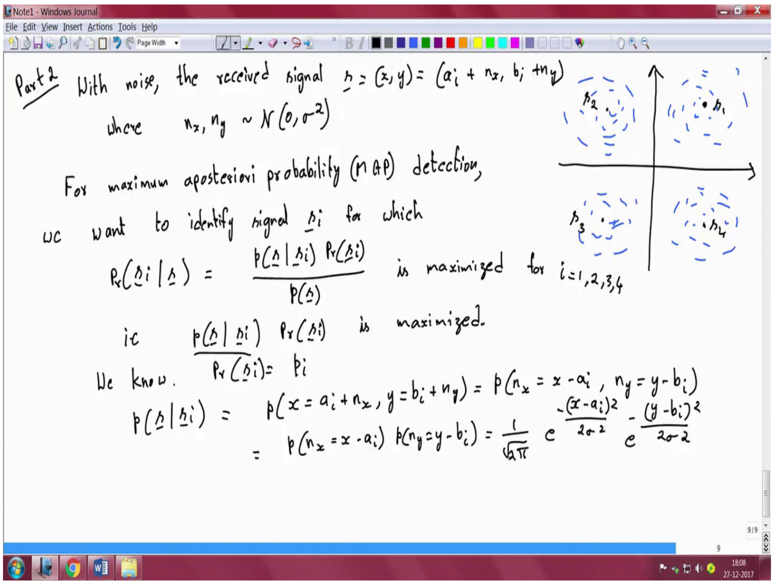

So, with noise, the received signal you can write. So, if x comma y are the values, they are going to be of the form a i plus n x or comma B i plus n y where n x and n y are Gaussian noise samples with 0 comma sigma square 0 mean and Varian sigma square.

So, essentially what it means is we have these 4 signals S 1 S 4; however, because of this noise each coordinate, there is going to be some noise that gets added. So, the actual received samples may will not fall exactly at this basis, but it could be anywhere outside it could be issue it at somewhat like this. So, if we plot all these received samples, it forms a conciliations like this centered around these individual points S 1, S 2, S 3 and S 4. So, for optimal detection, what we need to do is we need to maximize a posteriori probability. So, this is a criteria that we choose. So, we need to; so.

So, essentially we want to identify a signal S i for which its posterior probability given by probability of S i given the signal which is given by probability that using Bayes theorem we write it as probability of S given as i times probability times S i by probability of S is maximized.

So, what i it essentially means is we have received a signal S and we want to maximize this aposterieri probability for different choices of i and see which signals maximizes this aposterier probability and that that signal is that we will make. So, that is optimaltation. So, essentially because of the denominator is same for all i's we want to maximize probability of S given S i hence probability of S i is maximized. So, we know here in order to compute these we have a probability of S i equal to p i the values are given and in order to compute this probability of S with this other term.

So, we just expand it. So, probability of S given S i is nothing, but probability that x is a i plus n x comma y is B i plus n y. So, which we can write it as probability that the noise sample n x is nothing, but x minus a i and the noise sample in the y coordinate is nothing, but y minus B i and we know this it was given that these noise samples n x n y are independent. So, we can write them as probability of n x probability n x equal to x minus a i p times probability n y equal to y minus B i and using the Gaussian distribution we can just write them as 2 pi e power minus x minus a i square by 2 sigma square times e power minus y minus B i squared by 2 sigma square.

So, this gives us the terms that we that are involved in computing this aposterieri properties and towards making the decisions. So, now, consider the decision boundary between 2 signals.

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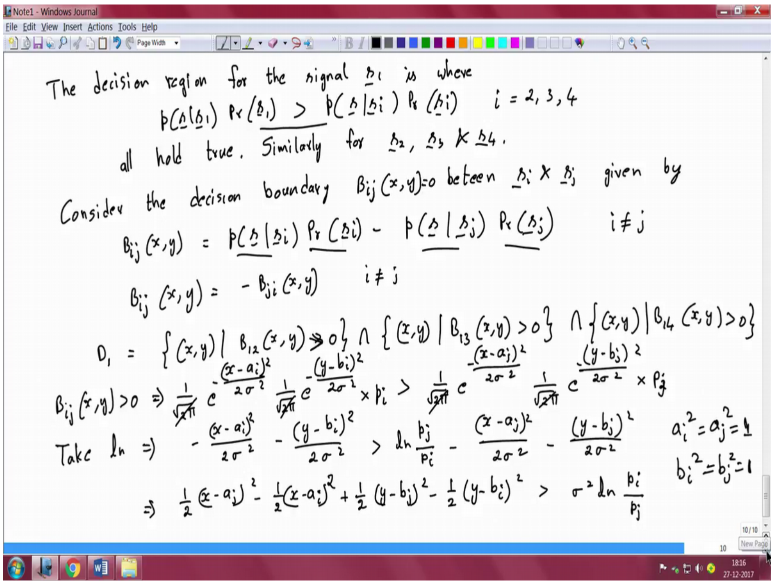

So, the decision boundary or let us say we call the decision region. So, the decision region for the signals let us see the pick S 1 a particular signal S 1 is where this probability that we have written down earlier times probability that S 1 is greater than the same probability for any other signal S i for all.

So, whenever these conditions hold true for all i equal to 2, 3 and 4 we say that the received signal S is very we make a decision to receive a signal S is based on the signal S one. So, similarly we can extend it for other signals this definition S 2 S 3 and S 4. So, now, based on this inequity that we have we can define a decision boundary between the 2 signals. So, essentially this B i j of x y it forms a decision boundary we can is given by. So, I say B i j equal to 0 is a boundary between with between signals S i j and S i S j regions which is given by B i j x comma y which is nothing, but I just take this up difference between these 2 terms probability times S i minus probability S probability S j and looking at this we can easily say that B i j of x comma y is minus of b j i of x comma y. So, this is for all i not equal to j.

So, what this essentially means is the decision region for the signal S 1 is the intersection of all these sets or all these basis B 1 2 x comma y greater than or equal to 0 or I will just stick to inequality intersection x comma y B 1 3 is greater than 0 intersection x comma y B 1 4 x comma y is greater than 0. So, this is essentially decision region for the signal S 1. Similarly we can define the decision region for S 2, S 3 and S 4, it is going to be in the signal form where it is going to be the intersection of 3 sets now to conclude this part we have to compute this decision boundaries. So, let us look at this condition p i j x comma y is greater than 0. So, I compute these terms or I substitute this from our earlier derivation.

So, we have one by root 2 pi e power minus x minus a i whole square by 2 sigma square times one by root 2 pi e power minus y minus B i whole square by 2 sigma square times probability of x i is p i minus the similar term for S j x minus a j whole square by 2 sigma square tends one by root 2 pi e power minus y minus b j whole square by 2 sigma square is a times p 2 is greater than 0 or I will just say this. So, we take log. So, what we get on both sides.

So, what we get is. So, these terms cancel out here and p i instead of p 2 it is sorry it is p j and taking log of them we have minus x minus a i whole square by 2 sigma square minus y minus B i whole square by 2 sigma square is greater than logarithm of p j by p i minus x minus a whole square by 2 sigma square minus y minus b j whole square by 2 sigma square. So, this further can be written as half of x minus a j whole square minus half of x minus a i whole square. So, I am just rearranging these terms and multiply it both the sides by sigma square plus half of y minus b j whole square minus half of y minus B i whole square is greater than sigma square log p i by p j.

So, this is what we have and further simplifying. So, also note that a square is same as a j square; say equal to 1. Similarly B i square equal to b j square equal to 1 because a i and a j are either plus or minus one. So, certain all this terms a j square a i square b j square B i square and x square from both these terms. So, all of them get cancelled out and you will be left with a linear equation which will be of the form x times a i minus a j by 2 plus y times B i minus b j by 2 is greater than equal to sigma square by 2 logarithm of p j by p i.

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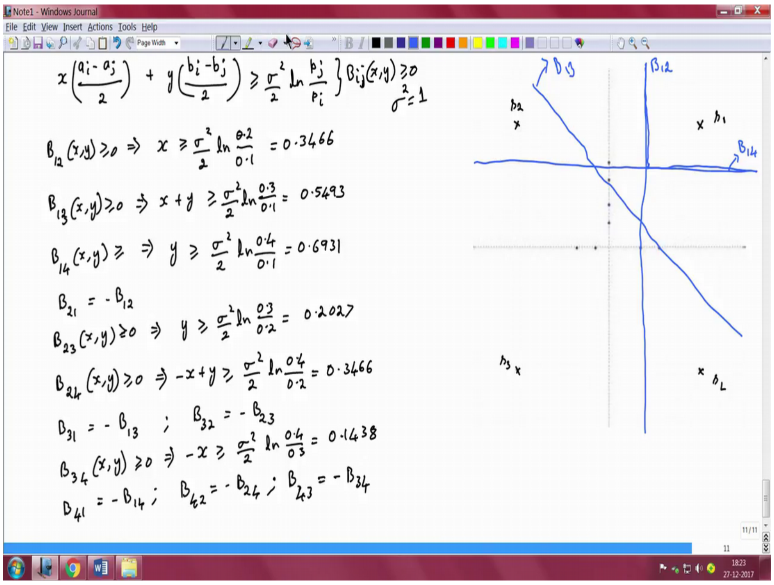

So, this is nothing, but B i j of x comma y greater than or equal to 0. So, this is the decision boundary between the signal i and j.

So, we will just come substitute these points. So, for that; I mean to demonstrate that I brought this graph paper. So, we have the signals S 1, S 2, S 3 and S 4. So, let us look at each individual boundaries. So, if we look at B if we look at the boundary B 1 2 of x comma y greater than or equal to 0 this is given by.

So, between the signals 1 and 2 the only x coordinate changes and y coordinates are the same. So, essentially B i and b j are B 1 B 2 are the same, but a 1 and a 2 are different and hence will be left with just a term x is greater than sigma square by 2 l n p 2 is point 2 by point one and let us assume sigma square equal to 1.

So, we will use this for convenience of plotting. So, this comes out 2.3466. Similarly, we compute the boundary between one 3 implies. So, here a 1 and a 2; a 3 are different B 1 and B 3 are also different. So, we will have both x and y terms is greater than or equal to sigma square by 2 l n point 3 by 0.1. So, this comes out to 0.5493 B 1 4 x comma y.

So, between 1 and 4, only the B you one and B 4 are different a 1 and a 4 are going to be the same. So, we will have only the y term is greater than sigma square by 2 l n point 4 by point one this comes out 2.6931 instead of taking these values.

So, that I can plot here; I will show you and we know that B 2 one is same as B 1 2 with a negative sign and B 2 3 x comma y. Similarly following the same procedure we have between 2 and 3 only the y coordinate is different. So, if you have we have y is greater than sigma square by 2 logarithm of ratio of the probabilities which is 0.3 by 0.2 and this comes out 2.20 to 27.

Similarly, B 2 4 between 2 and 4 both x and y coordinates are different and if you substitute you will get this and for 3 one we know it is B a negative of one 3 and B 3 and 2 it is negative of 2 3 and B 4. So, between 3 and 4 it is only the x coordinate that is different. So, you will have minus x greater than or equal to sigma square by 2 logarithm of point 4 by point 3 which is 1438 and for 4, it can be computed from the remaining boundaries. So, we have 4 one is minus B 1 4 B 4 2 is minus B 2 4 B 4 3 is minus B 3 4. So, all these terms are already computed here all these boundaries. So, let us plot these boundaries one by one individually.

So, I will use a different color. So, let us look at B 1 2. So, B 1 2 is x greater than equal to 0.34 or x equal to 0.3466. So, the boundary looks like this B 1 2. Similarly let us look at B 1 3 x plus y is 0.5493. So, boundaries going to look like this and B 1 4 is y greater than or equal to 0.6931. So, these are the boundaries that we have and we classify the signal as S 1 when it lies within the intersection of all these boundaries. So, essentially it is this region. So, so for that what I will do is; yeah essentially it is this region let me remove the unnecessary boundaries that are not in use.

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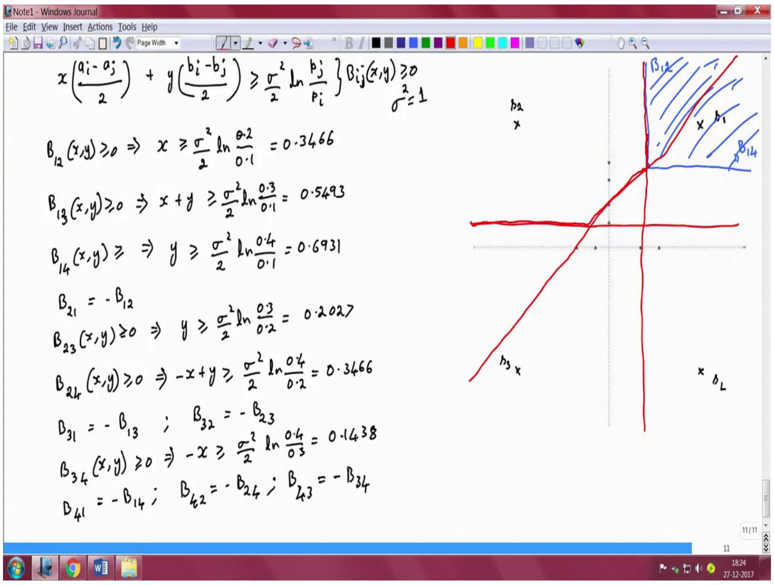

So, this is the decision region for S one. So, let me share it a bit. So, this is the decision region for S 1. Now similarly for we can continue this for S 2. So, between 1 and 2, we have this boundary that is already present and between 2 and 3. So, we have it is greater than 0.2027. So, it is going to look like this and between 2 and 4,it is minus x plus y is 0.3466 it is going to look something like this and the intersection of all these regions is going to be this.

So, let me re and some of the other boundaries they are not necessary. So, I will just erase these extra portions of the boundaries. So, this red portion that gives the decision boundary for S 2.

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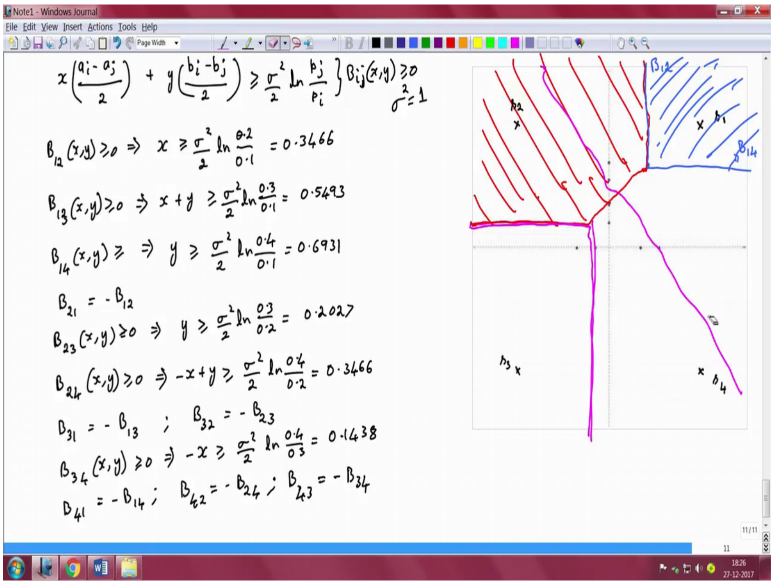

So, this is for S 2 similarly for S 3, I will use a different color. So, between S 2 and S 3 we already have this boundary between S 3 and S 1, we have a boundary which goes like this and between S 3 and S 4 the decision boundary is 0.1438. So, it is minus x is greater than or is equal to 0.1438. So, it is going to look like this.

So, now the intersection of all these regions is essentially this. So, let me remove again the remove the unnecessary boundaries that we saw yeah. So, this is the decision region for S 3.

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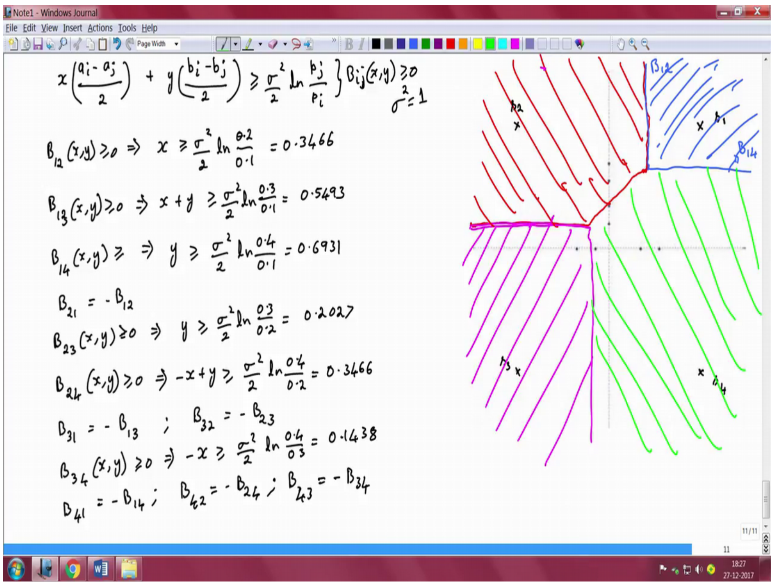

And remaining space is decision region for S 4. So, this gives the decision boundaries for various signals S 1, S 2, S 3 and S 4 within the signal space that we have seen. So, yeah.

Thank you.