**Mathematical Methods and Techniques in Signal Processing - I**
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**Lecture - 20**
**Linear approximation of signal space**

So, with this idea of gram Schmidt orthogonalization or ortho normalization; when you consider a norm to be one we land up with an interesting concept of how to approximate a signal linearly in a signal space ok.

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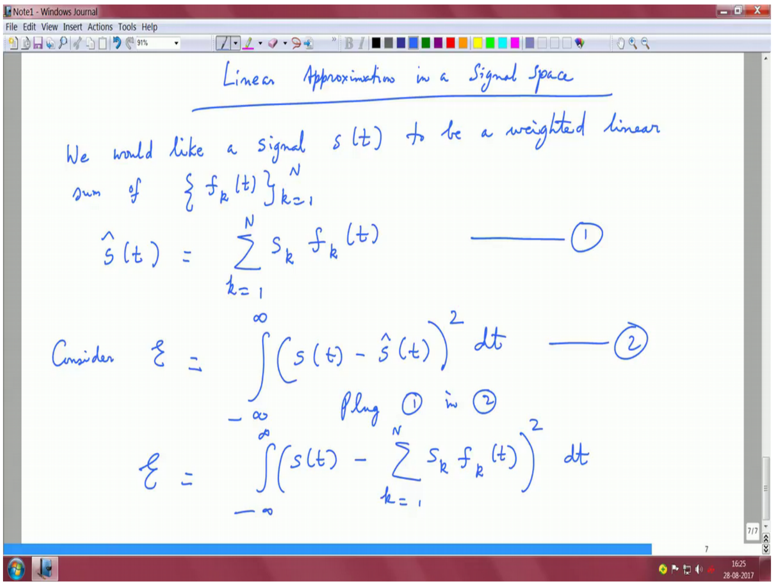


So, the question is we would like a signal S of t to be a weighted linear sum of some bases signals in bases signal k equals one to n that is we want an approximation to S of t which is in this form.

S hat of t is summation k equals 1 to n S k f k of t. So, as I said the question is how do how can we do this what is the bases f k of t how do we choose the bases f k of t right and what should be this components S k right. So, this is one of the questions that we get. So, we will see this as a as a minimization of a square error procedure to compute these coefficients S k which appear before the bases f k of t and this automatically and the second question is basically; how do we generate f k of t given a set of signals S 1, S 2, S 3, S 4 and so on am giving a collection of signals and I want to get an orthonormal bases to these set of signals that is the first question and given those components.

How can I get S k that we will think about the projection, but we will see if this really how this translates to this projection will think about in 2 different ways. So, for this, let us formulate the square error right at this moment I am not assumed any stochasticity in the signal and if you bring in stochasticity, then you have to kind of average it over the distribution in we have to think about some measure over the distribution of the random variables in the signal.

But I am just ignoring this at the moment. So, the square error is basically from looking at the error of the signal which is basically S; S of t is my original signal S hat of t is my linear approximation to the signal in terms of these bases functions right and I want to take the square deviation integrate this signal from minus infinity to plus infinity, it could be just the limits could be just from minus pi to pi or 0 to 2 pi or whatever finite limits you want to consider, but you know without laws of generality I just wrote minus infinity to plus infinity; now we want to minimize this square error.

So, now once you set this up setting up these objective is important I mean like in any optimization problem right you have to have some objective why this square error is a is a question right, why not the cube error why not the power 4 error square error has a very has a number of interesting properties and one of them is it is a quadratic function. So, therefore, if you want to optimize something you land up with the unique solution it is a convex, you have to think about it in terms of geometry right the square error is basically a quadratic surface is a parabola.

So, and if you think about the parabola, you will have only one minimum point. So, therefore, if you apply your calculus, if you want to solve for some optimal parameters over the surface, you can uniquely get to that point unless if the surface has is more complicated where you will have to figure out a norm, what should be the global minima of that surface etcetera. So, this is a these ideas; you will sense it in optimization, but in signal processing heavily people have followed square distractions square error right and we will see this how this trick is going to be useful for us now.

So, once we set this up as a square error and this square error is just time averaged over this you know just integrate the square I mean am not even talking, I am saying it is the energy not even just a scaling here by the time and taking a limit as the time goes to infinity right I am not even defined it in terms of power just the energy here. So, e is basically integral minus infinity to plus infinity S of t minus I just drop in log one into its my plug 1 in 2, I have the sum k equals 1 to capital N S k f k t, take the square d t. Now this is very straight forward though it looks like integral and some summation inside.

So, we just have to apply calculus.

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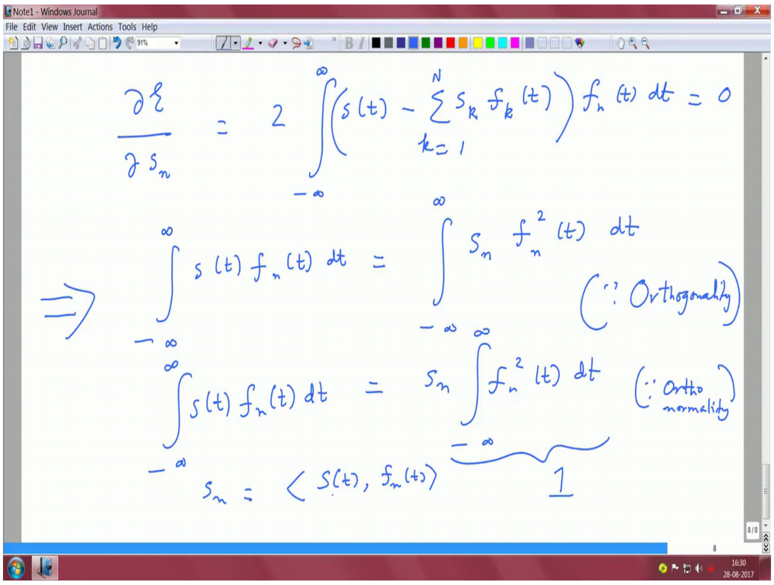


We take the derivative with respect to sum S k right and sum S k, I have to I have to take the derivative. So, what do I get? So, I have 2 times. So, maybe I could make this S n here just. So, that since I have some case there in the last slide some summation goes from k equals one to n may be I take the derivative with respect to n this is without loss of any generality.

So, I take derivative of the error with respect to S n right. So, basically I have minus infinity to plus infinity S of t minus sum k equals 1 to N capital N S k f k t like; this is what I have and then I have to take a partial derivative of this argument here with respect to S n right see if I have to take with respect to S n I deliberately avoided the S k here. So, this is this is something that you have to be clear-ful about when you do if you if you started off with some indices going from k equals one to n may be used some other index here that you would want to take the derivative with respect to right. So, now, we have fu only that with f suffix n will be there and rest will just vanish because it is a partial derivative right.

So, basically we just land up with f n of t d t this has to be set to 0 now I get a very straight forward equation here right. So, now, integral minus infinity to plus infinity the 2 will just cancel out you do not have to worry about. So, S of t with f n t f suffix n t dt is; now if you take the integral here, see you are going to have sigma. So, now, you take f k with f n if you have them to be orthogonal to each other then they just you know the f n f f for k not equal to n they vanish and for k equal to n they exist right.

So, basically I will write it in this in form this is going to be you going to have an S k S n essentially and this is basically f n square d t right, if these signals are orthogonal then, but now if you want them to be orthonormal right, I would say this is S n; this list just going to be f n square d t this could be one because of orthonormality.

Things are now very simple it is just the inner product of this signal S t with f n t this is how we would compute S n which also sort of makes sense if you thought about represent it is basically giving you the coordinate of that of the signal S of t in terms of these bases f n t f suffix n t clear is one way of look at it the other way is just sort of recasting it in terms of a square error taking this partial derivatives and then and then computing what you need to compute now our job is not yet done because we need to compute.

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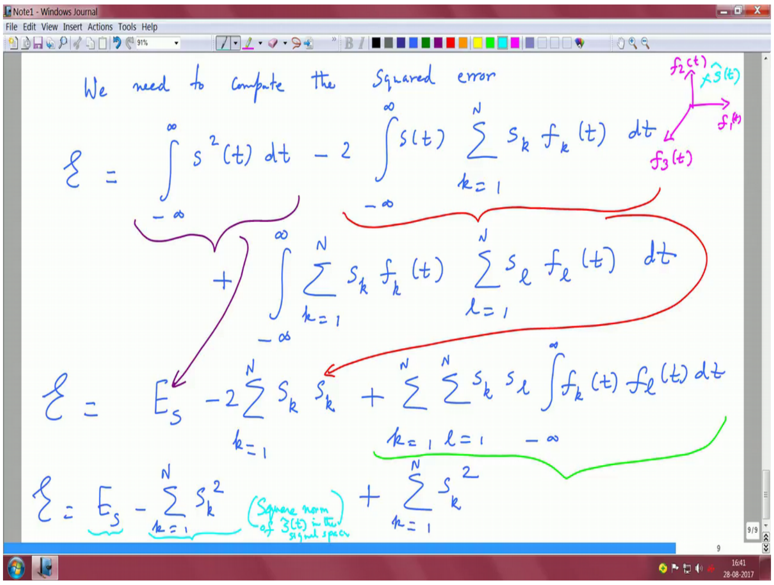

The squared error right because we said we are approximating it as a weighted linear sum of some orthogonal signals right orthonormal signals and now we want to compute what our squared error is.

So, E if you just expand this out is a square of t d t, right, just take S; S of t minus this you have this quantity here right you have this equation 2 and you just have to expand equation I mean equation 2 in this form here basically just what expand this out. So, you will have a square t S of t with this term here and then this whole term square right a minus b whole square you just what expanded this out.

So, this is minus 2 times minus infinity to plus infinity S of t summa k equals 1 to n capital N S k f k f k t d t plus integral minus infinity to plus infinity summa k equals one to n sum S k f k t times sum l equals 1 to n; S l f l t d t. So, what I did instead of taking this whole square I just expressed it as 2 different summations as just the same just the running variable here is k and l. So, you do not have k equals one to n and another k equals one to n you just replace it with one k equals one to n j equals one to n and we will see how this basically simplifies. So, this square error now just focus on this term here right this term is just the energy in the signal S right energy in the signal is basically square of the signal you know integral minus in plus infinity S square t d t is the energy in the same.

Now, this is interesting because now assume that you can exchange the summation and the integral and you have to be careful when you could ex exchange the summation and the integral I mean this is the finite supportive sigma of k equals one to n. So, this is finite therefore, you could you could exchange this; the astonish theorem which basically gives you a condition how you could exchange these double sums etcetera from analysis. So, assuming such an exchange is possible right.

So, now what we have to think about is you have k equals one to n S k and think about st with f k t. So, it is basically the projection of the signal S t with f k t sum for all the components k equals one to n right. So, you just land up with sigma k equals one to n this S k will be there and take this integral inner product S k S of t with f k t you get S k there right because that is our definition right if you just go back here this is our definition S n is basically the inner product of the signal S of t with fn t. So, therefore, this step is straight forward. So, bearing mind I have considered all real coordinates here.

So, therefore, if it is complex you will have to be slightly careful what you have to do here.

So, basically this term can be simplified as this and we have the last term. So, the last term is also not too hard. So, first we will just pull the summation is out k equals one to n summa sum l equals one to n I pull out this. So, I have S S k I have S suffix l right I just get this integral. So, it will be extremely careful when you exchange the summations and a integrals I mean single process we just take for granted to do this, but again routes to this is analysis. So, if I did a math course then an analysis 1 or 2 you will appreciate all these certain aspects of exchanging limits and exchanging integrals exchanging summations or summation and integrals etcetera.

So, I want it with those things because it is part of analysis a separate course. So, we land up with this integral minus infinity to plus infinity f k of t fl of t d t. Now we are home for good right I think I would just want to keep this arrow here I did not write this term. So, now, you can you can basically simplify; simplify this because this will vanish if they are ortho or if they are orthonormal, right. So, therefore, integral when k equals l this integral would exist else it would just vanish for k not equal to l. So, therefore, this is basically k equals one to n S k; S k; S k square.

So, I just had to take a 2 here that I missed. So, now, our error is basically the energy in the signal minus summa k equals one to capital N S k square and I think you have to ponder about it at this moment as to why this result is interesting now if you had these signals f one f 2 f 3 right you can consider them. So, if you had the signals f one f one t f 2 t f 3 t. So, on you can and we said that mutually orthogonal and they are orthonormal, it is an orthonormal bases therefore.

So, you can you can you can write it like this. So, this S hat of t where I am assuming 3 components here I mean if you think about n such bases you have to think about in n dimensional signal space. So, my S hat could be some signal that is a point in the space; that means, it will have some components of f 1 some component of f 2 some component of f 3 right and that is how I would represent that signal as a weighted.

Sum of these bases right and the error that I do in such an approximation is now what is basically the energy in the signal which is Es minus summation k equals one to n one to capital N S k square this is basically norm square of this vector representation this is square norm of S hat of t in the signal space I think it is a pretty interesting result I mean it seems intuitive that if you have an over complete bases or if it is a complete bases this error should be 0 if it has a complete bases the error should be 0 if it is not a complete bases you have residual error right.

So, now the questions that one have one has to ask is can we construct a complete bases for a particular signal set is that possible what sort of bases what sort of properties and this is one of the important questions in signal representations be it wredgelets be it wavelets be it dct be it any transfer we should you can come up with your own set of set of functions set of basis functions. So, provided this; this is first property that you will have to satisfy if you call this as a signal bases towards representation.