Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 19
Gram Schmidt orthonormalization

Let us get started with this theorem. So, in the in the last module, we learnt that if you have a set of vectors that are mutually orthogonal, then they are linearly independent, but it is not the other way around. And using this property, let us see how we can express a vector in terms of it is coordinates using an orthonormal set.

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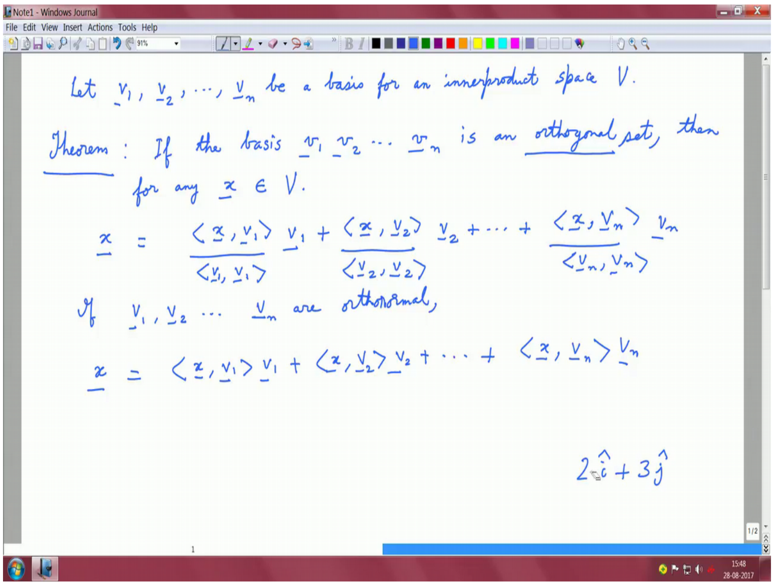

Let v 1, v 2. So, on till v n be a basis for an inner product space V ok. This inner product space V, and let v 1, v 2 so on till v n be a basis for this. So, statement of this theorem is as follows.

All of these theorems are really not very hard that just easy theorems, the basis set v 1, v 2 so on till v n is orthogonal, then for any x belonging to this space V, you can write this vector x as x with v 1 take the inner product divided by the normalization value for v 1, in the direction of v 1, plus the projection of x upon v 2, divided by the inner product of v 2 with v 2 in the direction of v 2 plus 1 till. The projection of x with the v n divided by the norm of v n in the direction of v n.

Now, if the product if the product of v n with v n in the direction of v n right. And if v 1, v 2 so on, till v n or orthonormal that is their norm is a 1. So, this inner product essentially is basically right it is the square of the length, and the length is basically 1 square of the length is 1 so therefore, all of this is 1.

So, basically you can write x, in a simple form. So, it is basically the take the projections or take the inner product of this vector x, with all of these other orthonormal vectors, and that gives you the scalar component, and then you then take the linear combination of these inner products, along the direction of these individual vectors right. So, this is the theorem, the proof is pretty straightforward , but I think before we proceed with the proof, I think what you have to understand is a sort of an intuitive feel for this representation.

Now, what is happening here is exactly like what we thought about, in the representation of vectors right that means, we take the projection of the vector, in the to the each of the bases and, then that gives you basically the amount of the length that you have to consider, in the direction of the basis.

So, for example, if you think about 2 i plus 3 j right, if you think about 2 i, plus 3 j, which means you know your 2 two units in the i direction, and 3 units in the j direction and these inner products are essentially those units in that direction right. So, this is this is a basic idea, and now once you think about this in terms of vectors, it is probably not too difficult to imagine these in terms of functions right.

So, the function an arbitrary function can be written as a linear combination of some functions, and this is basically linear combination of these basis functions, and you ever figure out what is the amount of you know what weight is do you give to that particular basis all right, and that is basically the amount of the length in some measure right an in terms of inner product of that that function with the basis or that vector with the basis.

So, throughout this as we think about vectors we have to also think about in terms of signals as well, and we will see that they are linked with each other, and that is the that is what we will we will do.

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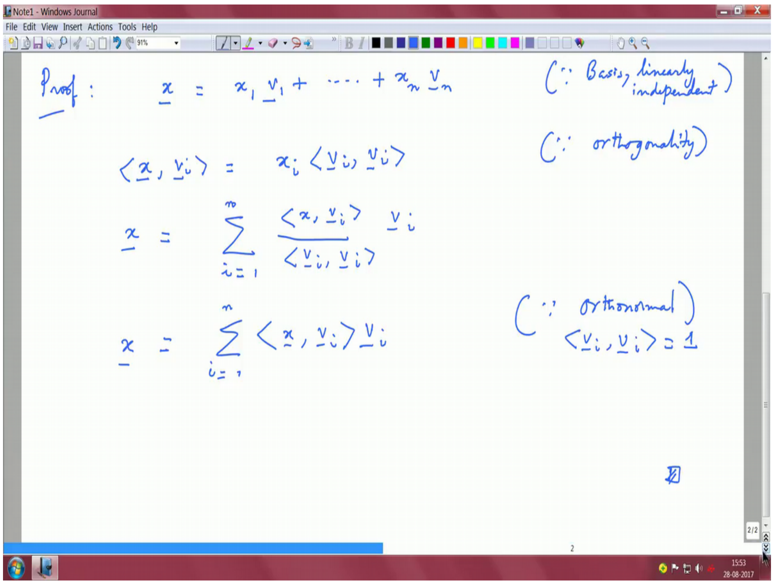

So, the proof is very straightforward very easy proof. So, what we do is we write x in terms of some components that connect the basis right. So, a vector x can be written as x ,1 in the direction of v 1, plus x 2 with the direction of v 2 plus so on. Till x n in the direction of v n right, and this is because basis it is a basis therefore, they are linearly independent.

Now the trick is just take this vector x, and look at it is inner product with each of these v i right. So, x with v i is basically x i with inner product of v i with v i right, because v i with v j will just vanish off, because of orthogonality all right. So, therefore, we can write x in this form which is sum, and say this is because for orthogonality i equals 1 to n.

So, now I want to express x i using this equation that I have, this is basically the inner product of x with v i, inner product of v i with v i, and then in the direction of v i. Now this can be more compactly written, in this form i equals 1 to n x with v i, in the direction of v i, if that orthogonal that is v i dot vi is 1, and v i dot v j is basically 0, I mean of course, we assume that here, and then we just normalized it. And this is pretty straightforward. So, what we have done now we have taken a vector, and in terms of a orthonormal basis we have expressed this vector, and we can exactly compute what the coordinates are ok.

To pretty straightforward result I think what is very important, i sometimes even wonder how decart a imagine this, ortho orthogonality, orthonormality idea through this coordinate geometry, I mean you see a lot of advancement as you think very in very formal terms, when you think about linear algebraic framework, but when you think about really some of these you know very early mathematicians, and this sense of intuition is something that you should have to be able to kind of realize how to construct things.

So I mean if you were told, in your 5th grade or 6th grade that this is the reason why perhaps you have to consider the cartesian plane, you know maybe x y being mutually perpendicular right, mutually perpendicular axis. So, if you were taught this theorem definitely we would find graphs very hard to even plot a function right, but I think if you now you can really appreciate the connections between orthogonality linear independence, and then basically writing this in terms of coordinates; so with this theorem.

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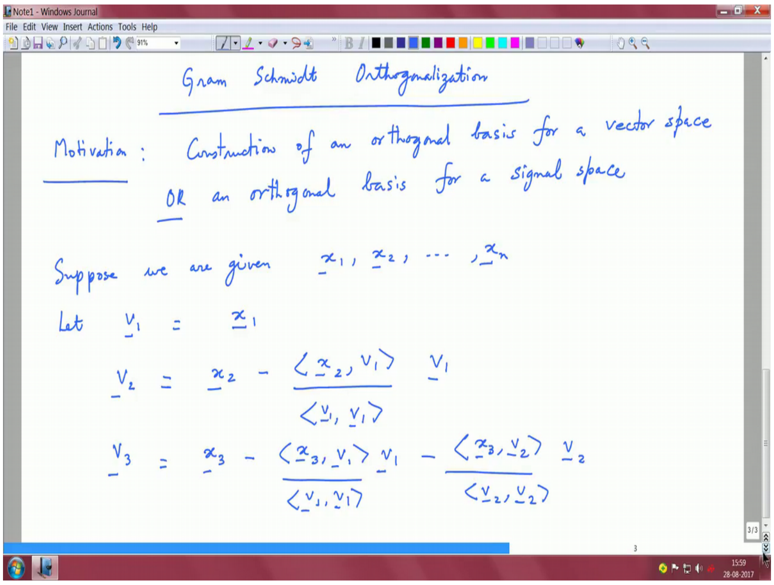

We will now proceed to an important step called the Gram Schmidt Orthogonalization. So, what is the motivation for this for this procedure right, some given you I am given a bunch of signals, and I want an orthogonal basis to represent the signal in terms of the basis signals, or basis functions or going back I give you a bunch of vectors. And I want to get a basis for to represent any vector any vector in this in this set, right some of them could be linearly dependent, some of them may not be linearly dependent.

But I those that are linearly dependent, somehow I want to figure out to express them in terms of a basis. So, with this motivation we develop this notion of Gram Schmidt Orthogonalization procedure, which is very useful um to basically construct an orthogonal basis for a vector space, or an orthogonal basis for a signal space ok. So, the motivation is construction of an orthogonal basis for a vector space, or possibly an orthogonal basis for a signal space.

So, let us suppose where given vectors x 1, x 2 dot x n. So, here is a procedure what we could do, and where we could start actually towards getting an orthogonal basis for v. So, let us start with this vector, let v 1 with this vector basically v x 1, I will talk about normalization later on, but let us just think about orthogonality, and get this intuition, and then we think about this more rigorously.

So, v 1 this vector is basically x 1. So, I give you 1 vector right, what is the basis for that vector itself right there is nothing more. So, therefore, I can say v 1 is x 1. Now v 2 is I am given another vector from this set from this pool I pick up this vector x 2, then I project x 2 in the direction i x 2 with v 1, in the direction of v 1.

So, what exactly is happening here is basically I am removing that component of I am removing that component of the vector in x 2 in the direction of v 1. So, think about this intuition, I have i j k usually orthogonal, if I do not if I want to get the direction of I want to remove that in the direction of i, I want to remove that in the direction of the j. So, basically I have to remove all these pies to get what I really want right. So, that is that is the very simple idea behind this, this construction and let us see how this works right.

So, then I go with v 3, this is x 3 minus take x 3, take the inner product x, we take the inner product of x 3 with v 1, and remove that component, take x 3 with v 2 normalize it remove this in this comb in this direction of v 2 so on, right.

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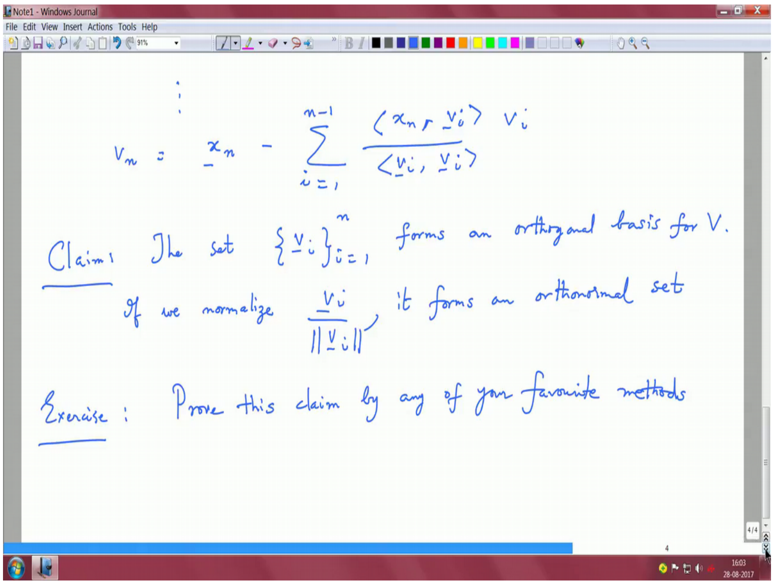

And if you do this for v n you have x n minus sum i equals 1 to n minus 1 x n minus x n with v i, take the inner product of x n with vi, vi with vi, in the direction of v i right. So, to remove all the n minus 1 components in their respective directions, and that is what you have right.

So, the claim is the set v i, i equals 1 to n forms an orthogonal basis for this vector space V, and if we normalize, it forms an orthonormal set right, if you normalize this it forms an orthonormal set. So, I will leave this proof as an exercise. So, the idea is straightforward I already gave you the idea you can try this with straightforward algebraic proof to work this out, else try alternative methods such as induction logic to show this procedure is correct.

But before we sort of wrap this just get a picture of why this should work right, I mean if say if you just. Even just routinely work through the algebra, take the inner product of v 2 with v 1 all right. So, you have x 2 with v 1 minus this thing right, and then v 1 with v 2 with v 2 with v 1, v 1 with v 2 is basically you know you want it to be 0 it is going to vanish there right.

And then so you will have v 1 with so v 2, with v 1. So, this is x 2 with v 1, minus this component v 1 with v 1 so, v 1 with v 1 this would cancel, you will have x 2 with v 1, x 2 with v 1 and which is basically 0 right, which is straightforward to think through.

So, I will leave this proof to you. So, the idea is as follows I give you i j k so on that are all mutually orthogonal if I if to start with so, the idea is if I want k i have a remove i, and j, and if hired i and j together and I do not want, I have removed that component in the direction of i that exactly is the idea here right. So, the proof is very straightforward, I will leave this as an exercise homework exercise the solutions would be supplied.

To prove this claim, by any of your favorite methods, so we will just revisit an example.

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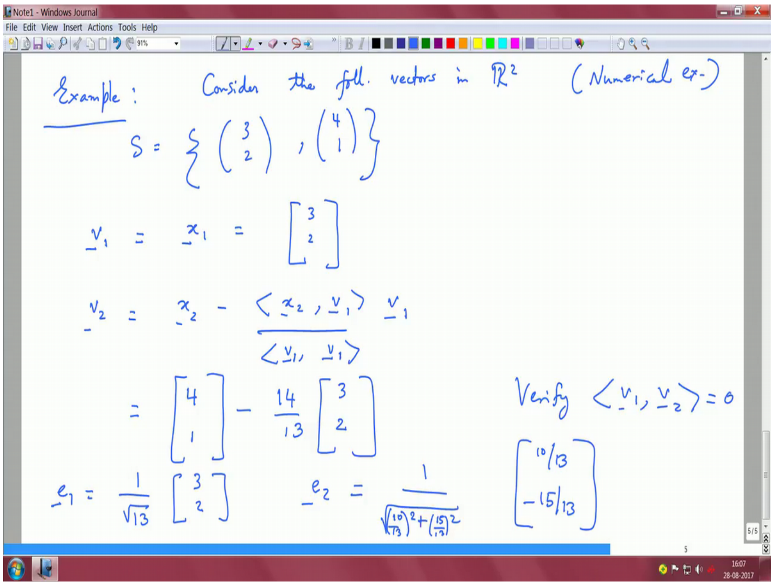

Just for the sake of a numerical exercise. So, consider the following vectors in R 2 just 2 dimensional vectors, so, I have 3 2 is 1 of them 4 1 is another. So, if you want to get an orthonormal set, we just follow the Gram Schmidt Orthogonalization ideas. So, we basically start with v 1 v 1 is my x 1, let us do this the orthogonality, and then we have a normalized later right.

We start with this vector which is 3 2, and v 2 is basically x 2 minus the inner product of x 2 with v 1, upon the inner product of v 1 with v 1 in the direction of v 1. So, if you just work out the numerics, this is vector 4 1 minus, if you work out this math.

It just happens to be 14 upon 13 times the vector 3 2, and you can just verify that v 1 with v 2 is 0 just for the sake of it, and you land up with the basis e 1 is 1 upon root 13 times this vector 3 2, and you have a root 13 here because you have to normalize this. So, that this vector is of length n this e 1 is normalized right.

So, therefore, it is 3 square is 9 2 square is 4, 9 plus 4 is 13, and that is where you have this, root 13 factor, and e 2 is similarly 1 upon this vector is 10 upon 13, and minus 15 upon 13, you just have to take the sum of the squares of these 2 components ok.

So, this is basically completion of the gram Schmidt orthogonalization procedure. So, this is a very important step because at this stage given a set of vectors that are possibly linearly dependent, we can through construction come up with an orthogonal or an orthonormal basis for this collection, such that any vector in this collection can be expressed in terms of this orthogonal set or this orthonormal set right, and orthonormal set is very important because, we can get the coordinate representation of the vector clear. So, with this in mind we can start off with a signal processing exercise and we are we have all the background n now from our linear algebra, and vector spaces.