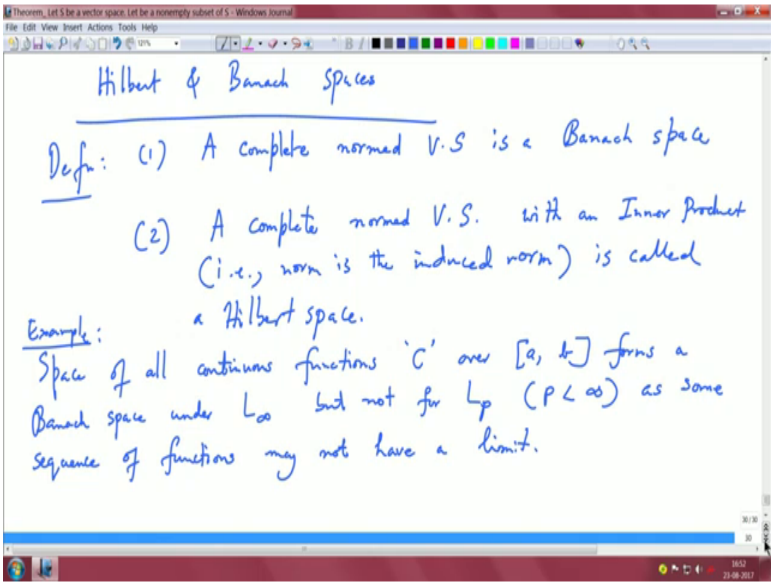
Mathematical Methods and Techniques in Signal Processing – I
Prof. Shayan Srinivasa Garani
Department of Electronic Systems Engineering
Indian Institute of Science, Bangalore

Lecture – 18
Hilbert space and linear transformation

So, I would also like to briefly mention about Hilbert and Banach Spaces.

(Refer Slide Time: 00:18)



And we will deal with these Hilbert Spaces extensively in signal processing. So, we will see what the definitions are. A complete normed Vector Space is a Banach Space and a complete normed Vector Space with an Inner Product. So, a complete normed Vector Space with an Inner Product, this is called a Hilbert Space.

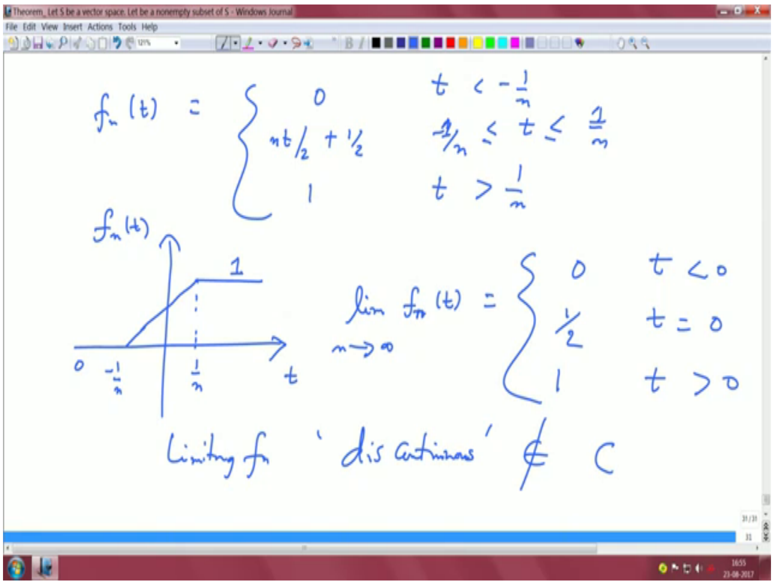
So, let us see some examples as to as to what we mean by this ah. So, VS, a Vector Space by itself is not endowed with a norm. So, we have to bring in some notion of norm to this vector space.

Then, it becomes a normed vector space and the idea of completeness is if we think about the limit of the sequence right; limit of this function or limit of that of the that vector limiting vector that has to lie within the space. If it lies within this space then, it is complete; otherwise, it is not right.

A good example of this is the Space of all continuous functions call it C over the interval a comma b forms a Banach Space under L infinity , but not for Lp P less than infinity as some sequence of functions may not have a limit.

So, let us see this carefully right.

(Refer Slide Time: 04:07)



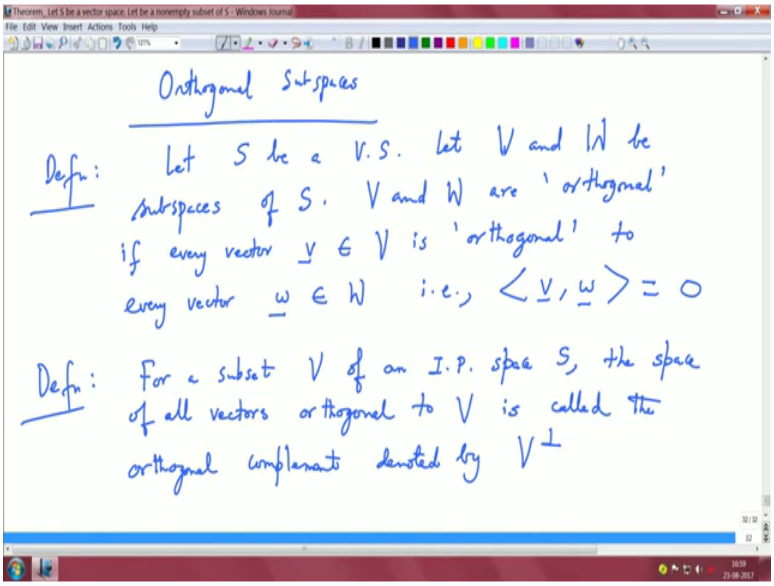
To some sketch let us think of f n of t is basically 0. t less than minus 1 over n and it is nt upon 2 plus one half in the interval minus 1 over less than or equal to t less than or equal to 1 over n. And this is 1, when t is greater than 1 over n.

So, you look at between minus 1 over n and plus 1 over n. This is basically what you have and this is a sketch of this function fn t and if you look at the limit of this function. This is basically a discontinuous function and since, it is a 'discontinuous' function, this does not belong to the space of all continuous functions. So, this is a good example for the case where, the completeness property is not satisfied right.

So, now if it has to be a Hilbert Space, it has to be a Banach Space endowed with an inner product. That means, if you take product of these two, I mean you should be able to define this inner product function. If is you cannot define; then, there is a trouble right. So, if it is endowed with an inner product; then, it becomes a Hilbert Space and we will see extensively how these Hilbert Spaces are useful for us in various signal processing applications ok.

So, this basically sets us pretty much into various notions of these spaces in you know basically starting with vector space, endow endowing the vector space with a norm. Then, getting into the inner product all right and then, sort of extending these to the notion of what a Banach Space is and what a Hilbert Space is? I mean this is sequential flow.

(Refer Slide Time: 07:40)



There are 2 other interesting spaces that we are interested in. One is basically Orthogonal Subspaces. We will see what the definitions are. Let S be a Vector Space. Let V and W be subspaces of S.

Now V and W are Orthogonal, if every vector v belonging to the space V is 'orthogonal' to every vector w belonging to the space W that is the inner product of v with w is 0.

And we might want to construct such orthogonal subspaces because we want them to be distinct right and this property we will see where and how we can construct such orthogonal subspaces etcetera when we deal with wavelet transforms and so on. So, all this we are studying here will be basically applied, when we when we when we when we study transforms and see how we can create such sub subspaces.

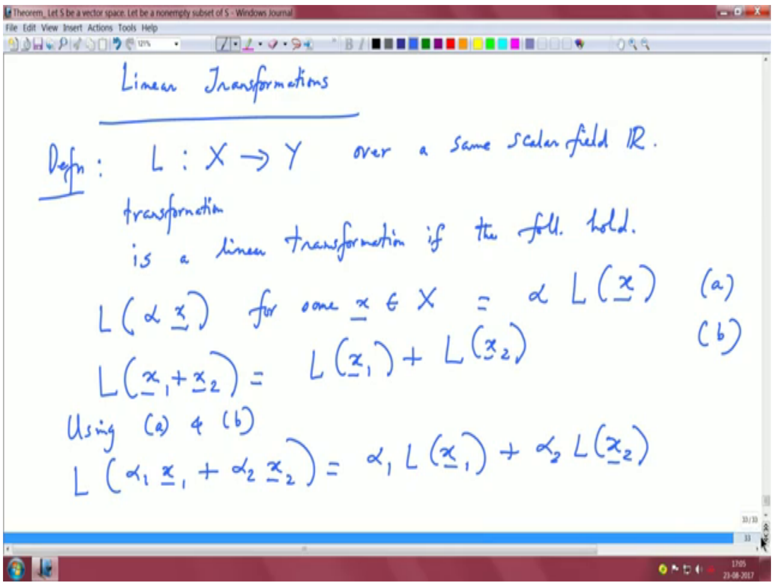
And there is also another definition for orthogonal complement. For a sub set V of an inner product space S, the space of all vectors orthogonal to V is called the orthogonal complement.

And it is denoted by V perp. Take a sub set and let that be the space of vectors and if it is and the space of all vectors which are orthogonal to this V. They are basically called orthogonal complement given by V; V perp.

So, these are basically the definitions and with this we should be ready to construct such spaces. I think the goal in signal processing is now, we have an intuition. What is this vector space? What it could be endowed with? What properties we want to study in this vector space?

And from which we want to bring in this notion of norms and an inner products and so on and so forth. And our goal really would be to construct such spaces that is very important and we will see this, when we when we go into in to more advanced lectures.

(Refer Slide Time: 11:46)



So, basically let us look into Linear Transformations and some of the some of the definitions associated with these maps. Let, L be defined as a mapping from X to Y and basically L is a transformation going from 1 vector space to the other vector space over a same scalar field R.

This transformation is a linear transformation, if the following properties hold. L of alpha x for some x belonging to this space X is basically alpha times L of x; the mapping of this vector x under this transformation.

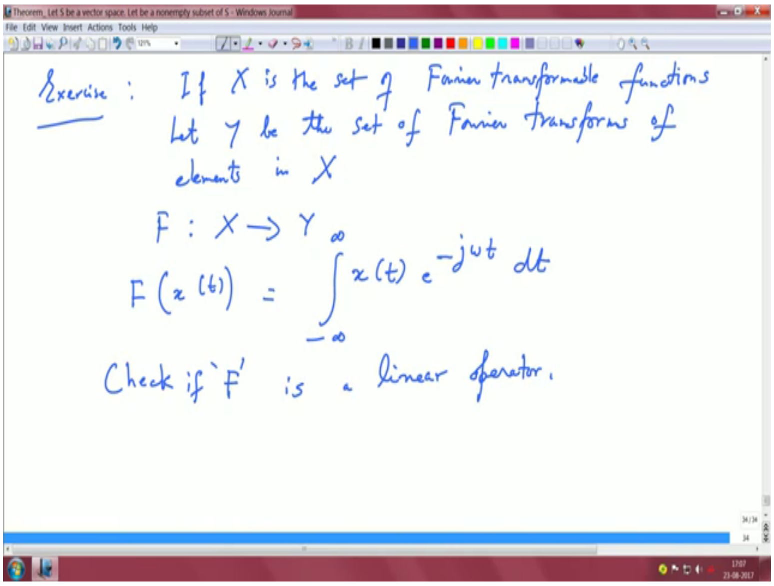
And then, L of x 1 plus x 2 is basically L of x 1 plus L of x 2 right. Let us call this is a, this is b. So, using a and b L of alpha 1 x 1 plus alpha 2 x 2 is alpha 1 l of x 1 plus alpha 2 L of x 2.

Now, can you just recall, the superposition principle that we just started off when we described linear systems in one of the early review lectures. We just assumed something; we just said this is basically a linear system if it satisfies these properties.

Now if we define this, formally like this to get into this notion of what this linear transformation is over 1 vector space and you know it is basically transformation which is acting on 1 vector space which transforms these vectors into another set of vectors in another vector space right.

And that we saw, this is as a mapping. Now using this notion of transformation we can really appreciate what our definition of linear you know linearity was in systems theory.

(Refer Slide Time: 15:35)



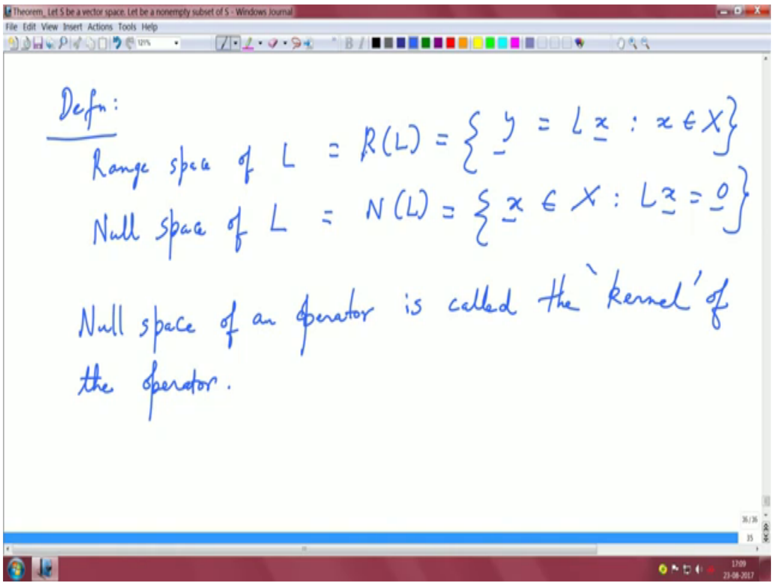
Alpha is a scalar. Now, keeping this in mind, if X is the set of Fourier transformable functions. What all functions can have a Fourier transform. Let Y be the set of Fourier transforms of elements in X.

So, let X be the set of all Fourier transformable functions and Y be the set of all Fourier transforms of elements in X. So, basically F is basically this map from X to Y and

Fourier of x of t is given by this integral minus infinity to plus infinity x of t e power minus j omega t dt right.

Check if this mapping ‘F’ is linear? And you can do this for Laplace or V transforms and so on and so forth. So, you will get an idea, what this is about? And there are a few other Definitions.

(Refer Slide Time: 17:43)



Is something called the Range space of L which is given by R of L which is a set of all y which is L times x for every x belonging to this space X.

This is the range space. Similarly, we have something called the Null space of this transformation which is noted by N of L which is a set of all x belonging to X such that Lx is basically this 0 vector and Null space of an operator is called the kernel of the operator. These are some definitions.

So, now, given some linear transformation you can figure out, what the range space is? What the null space is and you can examine some of these properties? So, I think with this we are we are sort of set in terms of all the essentialities, we need from linear algebra towards this course.

So, this is basically sort of a quick review that we sort of had from linear algebra because this itself this topic of linear algebra itself is a semester long lecture series right. And here we just we are not doing linear algebra course here, but just enough of that which is required for us to set the phase for this course. And then, we will extensively use what all we have studied so far in the rest of the lectures.