Mathematical Methods and Techniques in Signal Processing – I.
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Lecture – 17
Linear independence of orthogonal vectors

In this lecture we will learn more about orthonormal vectors and inner products. Now, with this we are set to another important result and we state this as a theorem, if vectors P1 P2 so on till Pm are mutually orthogonal. They must be linearly independent. So, I think before we delve into the details of the proof of the theorem, I think we have to philosophically ask this question what really why we need this result right. If you were to go back to your middle school as you are plotting these graph sheets and coordinates have you ever wondered why your x y axis had to be at 90 degrees to represent these points?

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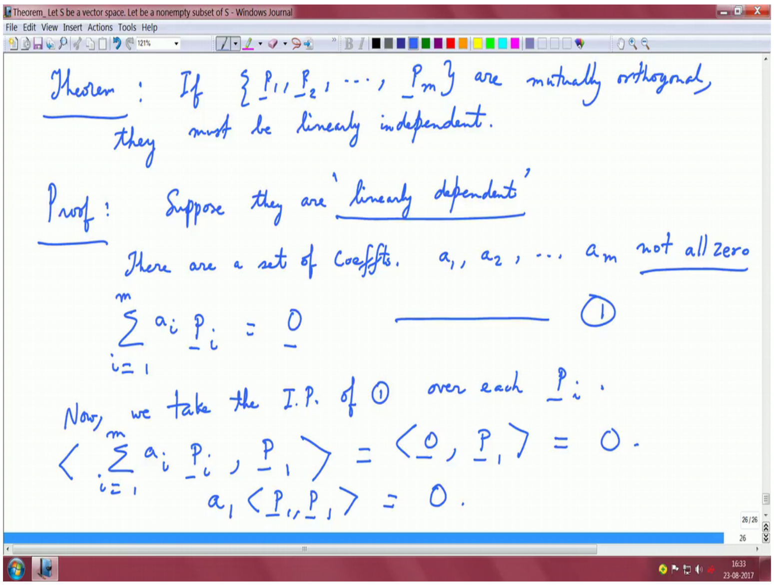

I could also start with my basis which is I and I plus J and then still get the span of all the points, but what was it that, what was it that made us look into these mutually perpendicular axis to represent coordinates. So, this is a philosophical question that we have to ask now and what we routinely did in our middle school, you know plotting these coordinate axes and then basically looking at these graphs and etcetera. We have to think about it in a more deeper context when we have to get into signal spaces and is it possible to think about signals as points in a signal space such that the coordinate axis are essentially mutually orthogonal and this is the idea where we are sort of drive into and therefore, we need this result ok.

And this will help and if it is mutually orthogonal then they are linearly independent therefore, you can basically and if it if it forms if it spans the space and its linearly independent and spanning the space, then basically we can represent all the points in this in this space. So, and that is basically the philosophy behind how we want to approach this ok. So, let us prove this theorem. It is probably not too difficult standard proof . So, we start with the supposition that they are linearly dependent and then we will try to prove this by a contradiction right.

So, let us suppose these vectors are linearly dependent. So, if they are linearly dependent , there are a set of coefficients a1 a2 so on till a m , not all zero, this is very important if all of them are zero then of course, there is a there is an issue here because it becomes linearly independent because since they are linearly dependent, let us assume that there are set of coefficients a1 a2 so on till a m, not all of them being zero such that sigma i equals 1 to m a i Pi , there is a zero . So, let me call this equation 1.

Now, the trick here is basically take the inner product of all these Pi s with equation 1 and come up with some simplified set of equations assuming that they are mutually orthogonal. And then, see if our assumption that they are linearly dependent is violating or not right. So, now, we take the inner product of 1, of this equation 1 over each of a over each Pi. So, when you take 0 with Pi, this is basically Pi right. So, basically what I do is sigma i equals 1, I will just take 1 case and it is very easy for you to think about the rest , take it with P1 , this is basically 0 with P1 . Inner product of 0 with P1 is basically 0 and this quantity if you simplify, now they are mutually orthogonal and if you take P1 with P2 it is 0, P1 with P3 it is 0, so on. So, therefore, you will be only left with a1 here. So, this means a1 and the inner product of P1 with P1 has to be 0 right. Now, once you do this trick you will you will you will you will you will do you will get an idea for the rest of it.

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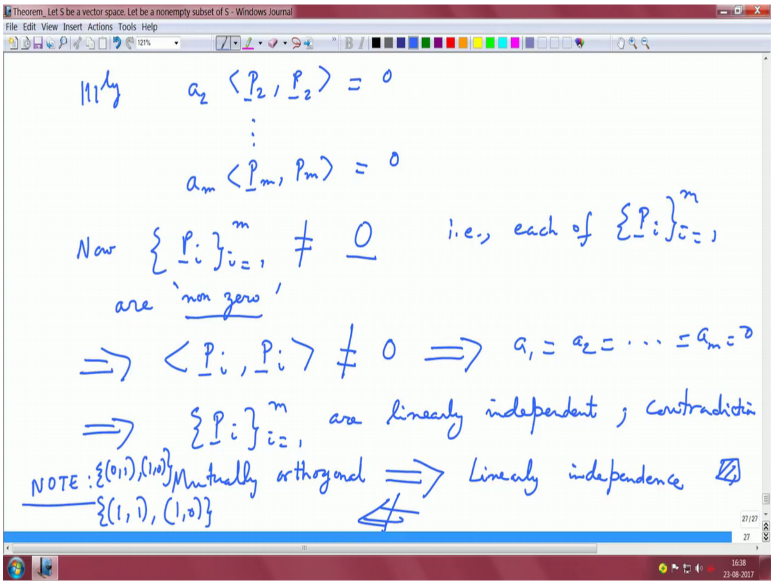

Similarly, a2 with a2 times, inner product of P2 with P2 equals 0, dot dot dot a m with inner product of Pm with Pm is 0. Now, these Pis, i equals 1 to m they are not equal to this null which is an important condition, but all of them are nulls right, that is each of these Pis, i equals 1 to m are non 0 which means now we have derived the set of equations a1 with inner product of P1 with itself is 0, then we have these equations a2. So, on till am all of these have to be 0, but we said that these Pis are essentially non 0; that means, their inner product is not 0, which implies inner product of Pi with Pi is not equal to 0. This means, the only way it is possible is all of these have to be 0. Now, this means we are coming into the condition where they have to be linearly independent, which means Pis which is basically a contradiction to where we started off and therefore, this is done ok.

So, this is an important result, I think which you should not end it the proof is very trivial, but I spent quite a bit of time in this proof because you have to get this picture very clear, that if it is mute, if the vectors are mutually orthogonal this implies that they are linearly independent and the other way round it is not possible is not necessarily true; that means, if you find vectors that are linearly independent then they are not necessarily mutually orthogonal. An example of this is basically think about the vectors 1, 1 and 1, 0 right.

These vectors are not mutually orthogonal, but they are linearly independent, but if you look at the natural basis which is 0, 1 and 1, 0, they are mutually orthogonal and that implies linear independence. So, now, I think with this you should be pondering in your mind if I want to construct a basis, what is useful for me? It is easier if I go with the route of figuring out a set of vectors that are mutually orthogonal and that can span the space. So, if I can do that, then I get the basis for the space and from which I can I can represent vectors in that space and it is possible to ensure that these vectors that you can derive the set of that you can create, a set of vectors that are mutually orthogonal, there is a procedure for doing that and that we will get into the next step clear so far till here.

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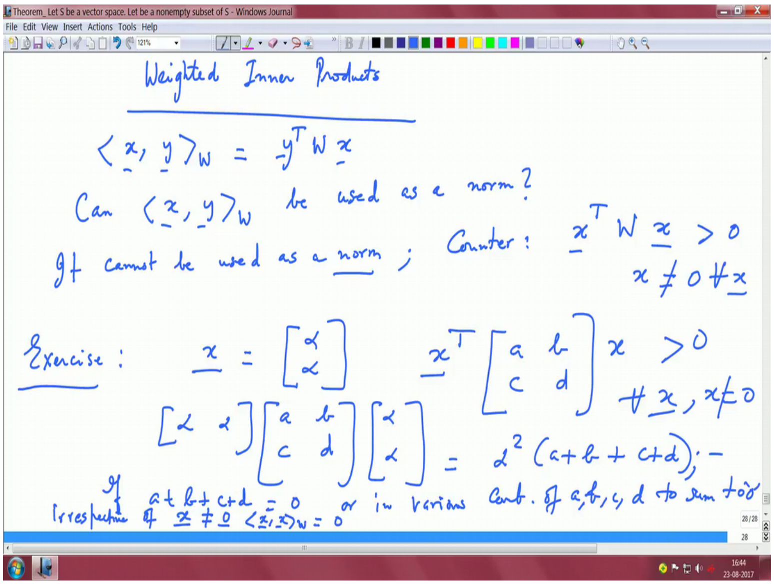

Now there is this notion of weighted inner products. So, you might philosophically ask a question, suppose I look at 2 vectors x and y and I weigh them by a weighing matrix W right. So, I am assuming some matrix w times, x is what I have ok. So, of course, this has to be this inner product can be defined over this can be the inner product can be a function from the Cartesian space of 2 vectors, 2 complexes as well so. In fact, if we think about complexes then w has to be hermitical matrix, but we will think about this in the contact context of real, just real entries for this matrix w right.

Let us assume that we are thinking about this. We will not define this in this face by this weighing matrix. The question is can this quantity x y the weighed inner product be used as a norm? It is a sort of a natural question that we get right, can this be used as a norm.

The answer to this is it cannot be used as a norm. Simple counter would be basically if you think about this as a induced weighted norm where you know y is same as x and then I have x transpose Wx is what I have right and if it has to be a norm this quantity x transpose Wx has to be strictly greater than 0 for your w for your matrix w right.

And trivially when x is 0, then we know that condition where it for x not equal to 0, for x not equal to 0 right x transpose Wx has to be strictly greater than 0 for all x and we cannot guarantee this unless this is a strictly positive definite matrix right and therefore, this is an important counter case where if you wanted to extend this notion of inner product or the induce norm or inner product as a weighted inner product using a weighing matrix you cannot think about this quantity as a norm . So, I will also leave this as a sort of an exercise, it's very straightforward you know to think about it.

Ah suppose I think about this vector x to be some alpha alpha and if I want to consider this quantity x transpose W x, where this w is given by this matrix a, b , c, d and if this has to be strictly greater than 0 for every x , x not equal to 0 . Is it true for all possibilities of parameters a, b, c and d. Now, you just expand x. So, and you take x to be some alpha alpha here and you multiply by this weighing matrix , if you did this quantity you will land up with alpha square a plus b plus c plus d , is what you would you would get right.

And this quantity can be potentially 0 even though x is not the all 0 vector. So, if a plus b plus c plus d is 0 or a combination of this is 0 or a plus b is 0 and c plus d is 0. You know individually in whatever way you want to simplify various combinations of a , b , c, d to sum to 0 irrespective of x which is not a null vector, we land up with the induced weight weighted norm to be 0, which is not what we wanted right.

So, therefore, this is another case to think about, why this weighted inner product cannot be considered as a norm and in detection problems when you have to deal with a statistical detection under a Gaussian noise assumptions sometimes you will land up with a quantity in the exponent there you will have to look at some weighted norm of some random vectors, you will you will encounter these problems are normal to delve into this class, but if you if you look at look into statistical decision theory there you will land up with such situations and in such cases this symmetric is really useful to think about , if it is going to be a norm or not.

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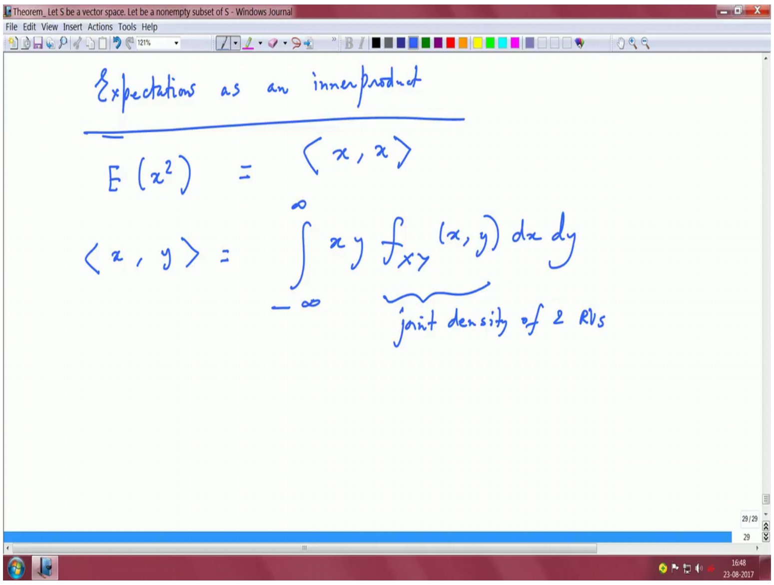

Now, just an aside, can we think about expectation of a random variable in terms of an inner product right. So, if, let us look at this quantity which is expected value of x square right it is ah. So, if the expected value of x is 0; that means, 0 mean, random variable basically expected value of x square x square is basically the variance right otherwise it is the normal second moment. So, can you think about this as the inner product of x with x under some measure which is your probability measure right, if you think about I mean you sort of think about x y variables and I say if I think if I define this to be x y times, this quantity right this is basically if you think about two random variables x and y and this is basically the joint density f x y of x y this quantity is basically the joint density of 2 random variables.

And what we are doing here is basically computing the mean, right this is computing the mean and can we think about this mean as sort of an inner product. If you if you under this measure right, I mean if you think about the standard inner product you would take the product of the 2 functions and then integrate them over this interval right, now if you bring in the notion of probability measure where you define a density function right, can you think about the function I mean of a basically the expectation of 2 random variables as some measure of inner product, I mean if x and y a y is x, then you have a measure of x with x and variances is basically under this measure, it is basically a sort of an interpretation of an inner product. Ok with these are some ideas to just ponder about.