Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 15
Orthonormality

Now, extending this Cauchy Schwarz inequality for functions. So, we have the inner product of 2 functions f t and g t over the interval a b the square of this is basically less than or equal to the integral from a to b x square t d t times integral from a to b g square t d t.

Now, we have derived this Cauchy Schwarz inequality and then the what we did for vectors we can say when this function it, when for these two functions the Cauchy Schwarz inequality achieves equality we can say that for g of t is alpha times f of t this holds. That means, we are scaling one function by the other function right this is for reals, but for complexes there is a trick which you will have to do and since I have assigned this as a homework you can see through the details.

Now, let us start with this notion of this inner product of x and y and considering the inner product of x and y can this be a measure of some relationship between x and y this is a question. So, to ask you know what is relationship between x and y in terms of inner product. So, let us consider the norm of x plus y square right. So, x and y are basically vectors in R n and they all have real entries right we start with this premise. So, the norm of x plus y square we expand this as the inner product of x with x plus inner product with y with y plus 2 times the inner product of x with y.

Now, if the inner product of x with y is 0 then we land up with the norm of x plus y square is basically the norm x square plus norm y square. Recall that inner product of x with x is basically norm x square and inner product with y with y is norm y square right this is just for your recall inner product the norm of x square is basically the inner product of x with x and the norm of y square is a inner product of y with y.

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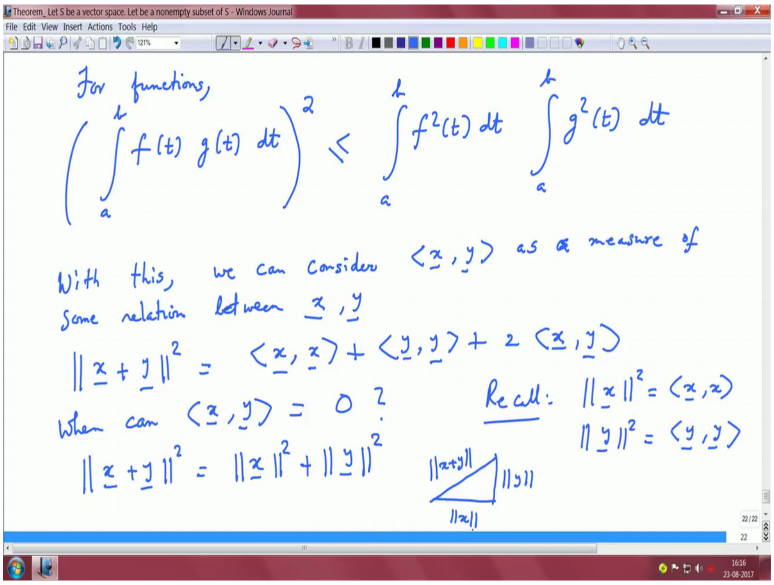

So, when the inner product of x with y is 0 then we land up with norm of x plus y square is basically norm x square plus norm y square and this is our familiar pythagoras theorem which indicates that x and y are at 90 degrees to each other.

So, this inner product has in some sense the notion of an angle embedded and that is what we are going to formally look into. So, this is the other quantity one if you think about vectors they have magnitude then there is also an angle between these vectors and we want to associate them.

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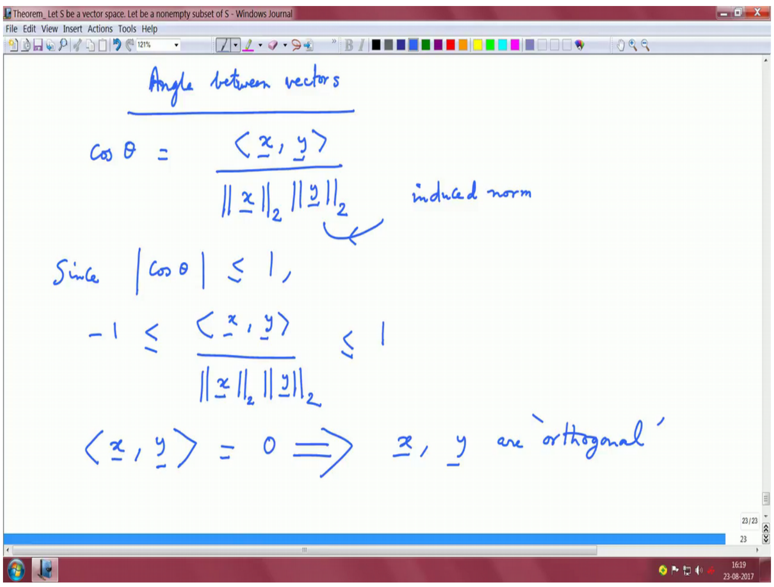

So, let us look into the notion of angle between vectors. Now, the cosine of the angle between vectors is basically the inner product of x with y divided by the norm L 2 norm of x and L 2 norm of y and this is basically the induced norm. And we know that cosine of this angle is between minus 1 and plus 1 right, so therefore, we can say minus one is less than or equal to this quantity is less than or equal to plus 1. Now, if the inner product of x y is 0, this implies x y or orthogonal that means, they are at 90 degrees to each other.

Now, can you think about this where x and y are essentially random variables and if you want to establish this notion of orthogonality what can you sort of extend this naturally to random variables. So, I think we have a sort of a framework now, with all these vectors I think we started off with the definitions of vector spaces, looking into subspaces, then we looked at what the norm is, their L p spaces right and then inner products right and an inner product space is essentially a vector space which is endowed with an inner product. Right not all vector spaces have to be endowed with inner product. If you endow a vector space with inner product it becomes an inner product space. And in signal processing we will need this very heavily.

And with this notion of now we have learned the notion of orthogonal vectors right and to think about orthogonal vectors we had to define this inner product without which things did not make any sense for us right. Now, with the notion of orthogonality we are ready to define what orthonormality is.

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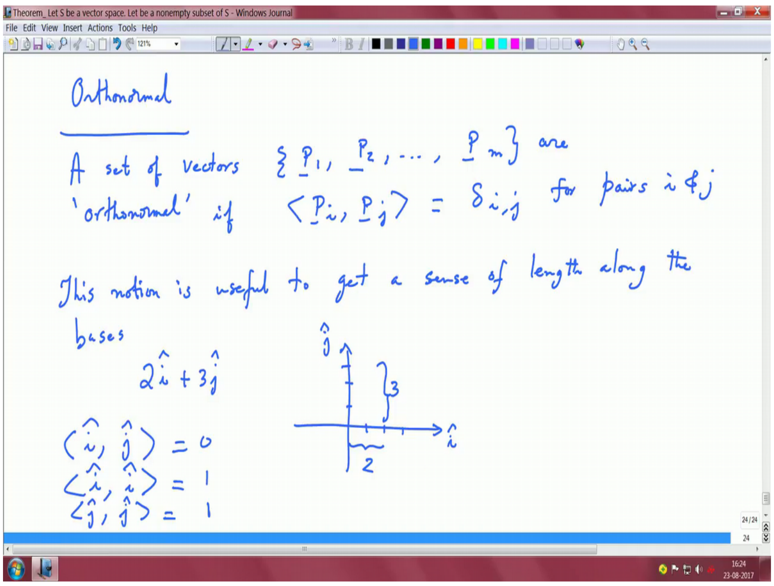

So, the idea is very simple basically if the norm of this, basically you look at vectors that are orthogonal to each other and then the norm of each of these vectors is 1 then; that means, they are orthonormal and you want them to have orthonormality because when you think about expanding a signal using the signal basis you want to go in the direction of the basis certain amount of coordinates, right. Therefore, if you want to go in the direction you want to set that to norm 1, otherwise you really not go in this direction right. If I say it is a vector 2 i plus 3 j I know the norm of i is 1. So, therefore, since the norm of i is 1 the unit vector i is 1 I can go 2 units along i, right. I mean 2 i is basically 2 units along i. So, this gives us this notion of orthonormality right.

So, to formally define things a set of vectors p 1, p 2, so on till p m are orthonormal if the inner product of P i with P j is delta ij for pairs i and j easy to verify right. If i is not equal to j basically this is evaluating to 0 condition for orthogonality and if i and j are the same this is basically delta i, i which is 1. So, you are having this measure. So, this is very useful this notion, is useful to get a sense of length along the bases. For example, if I think about as I told you before 2 i plus 3 j right; that means, in this plane since the norm of i with i is 1 and inner product of i with j is 0, inner product of i with i is 1 and j with j is 1. 2 i means basically you are translating 2 steps along i and you are indeed translating 3 steps along j right. And if it is not norm 1 then you would have trouble in defining things because that is the reason why you have to normalize.

So, I think these ideas seem intuitive. So, I think with this let me give you short exercise here, a small problem that you can examine.

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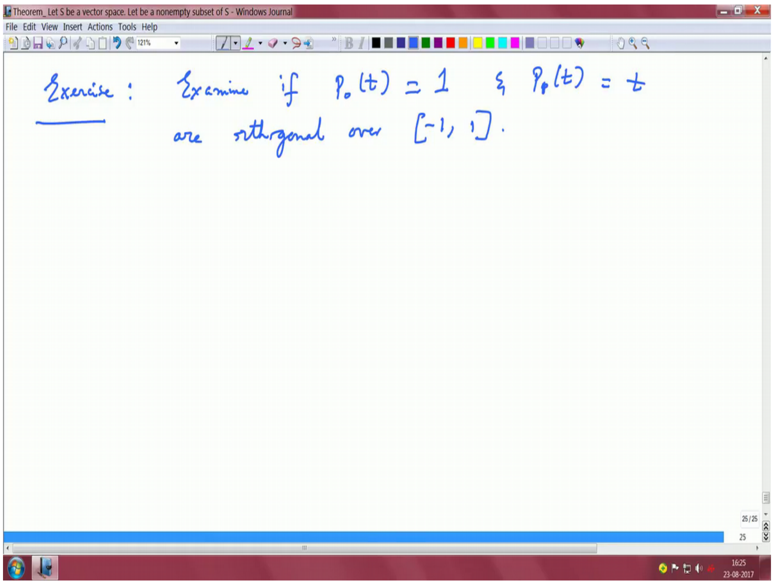

Examine if P naught of t equals 1 and P 1 of t equals t or orthogonal over the interval minus 1 to plus 1. Just check if this result is true I mean this is very straightforward just look at the inner product of these functions over the interval minus 1 to plus 1 and then cross check. So, that gives you sort of an idea how you can extend the notion of orthogonality for functions.