Mathematical Methods and Techniques in Signal Processing - I

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Lecture – 14
Cauchy Schwartz inequality

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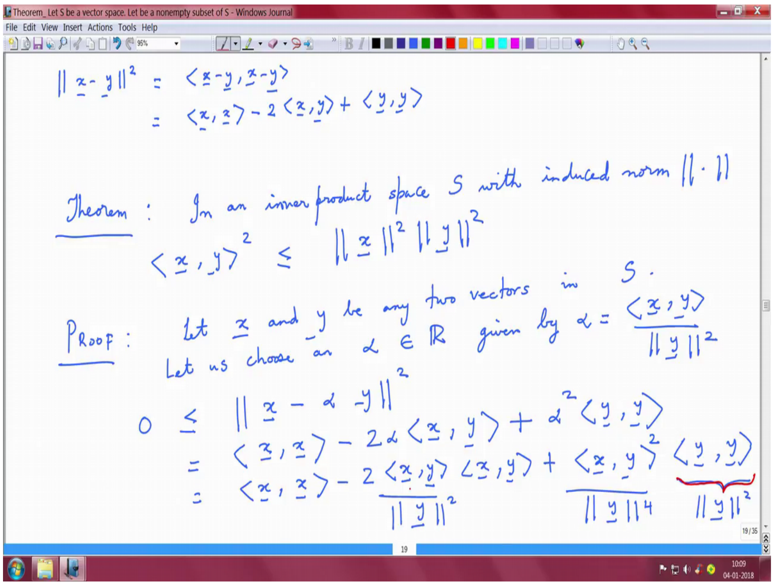

Let us start with a theorem. The statement of the theorem is as follows in an inner product space S with induced norm the inner product of x y of 2 vectors x y square is less than or equal to the norm of x square times the norm of y square. So, this is the statement of the theorem. So, let us try to prove this result.

So, let x and y be any 2 vectors in this space S. Let us choose an alpha that belongs to the set of real numbers given by alpha equals the inner product of x with y divided by norm of y square. Let us choose this quantity we will see how this is connected. Now, we know from our basics that the norm is 0 or it is positive, therefore, 0 is less than or equal to norm of x minus alpha times vector y square. So, let us expand this norm. So, this is basically written as the inner product of x with x minus 2 times alpha times inner product of x with y plus alpha square times the inner product of y with y right.

So, this we will plug in this quantity alpha here, this is basically the inner product of x with x minus 2 times, this is inner product of x with y substitute alpha put this as a alpha is positive quantity. So, even if I put a mod here I mean I can just put a square here because it is anyway a positive quantity almost y square times inner product of y with y and this is an alpha square here, so this has to be power 4. So, let us simplify this and this is basically norm of y square, this quantity is basically norm of y square and just put this in red. So, this would just cancel you have a norm of y square in the denominator and you have inner product of x y square that appears in the numerator. So, let us just take this forward. So, therefore, we have 0 less than or equal to. So, this is.

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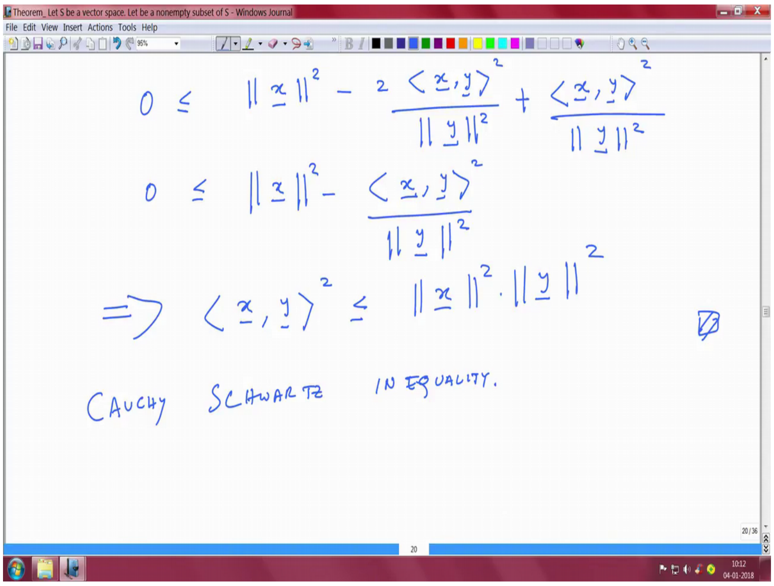

Student: We will cut the square sir.

Ok.

Student: (Refer Time: 05:58).

So, what we have to do.

Student: (Refer Time: 06:00) come to one page.

Yeah. So, what we do is we have a norm of x with x. I mean if you take the inner product of x with x. So, this can be written as norm of x square this is 2 times the inner product of x with y square upon norm of y square plus the inner product of x with y square upon norm of y square. So, what we do, slightly now 0 is less than or equal to norm of x square minus norm of inner product of x with y inner product of x with y square upon norm of y square. So, I think this is what we have to establish. So, this implies that the inner product of x with y square is less than or equal to norm of x square times norm of y square and this holds with equality if vector x is alpha times vector y.

And this is a very important inequality, this is a very important inequality and this is called Cauchy Schwarz inequality. And you will see the consequences of applying Cauchy Schwarz inequality when you have to deal with match filters and so on and so forth in signal processing. So, this is a very important tool or a very important and useful inequality.

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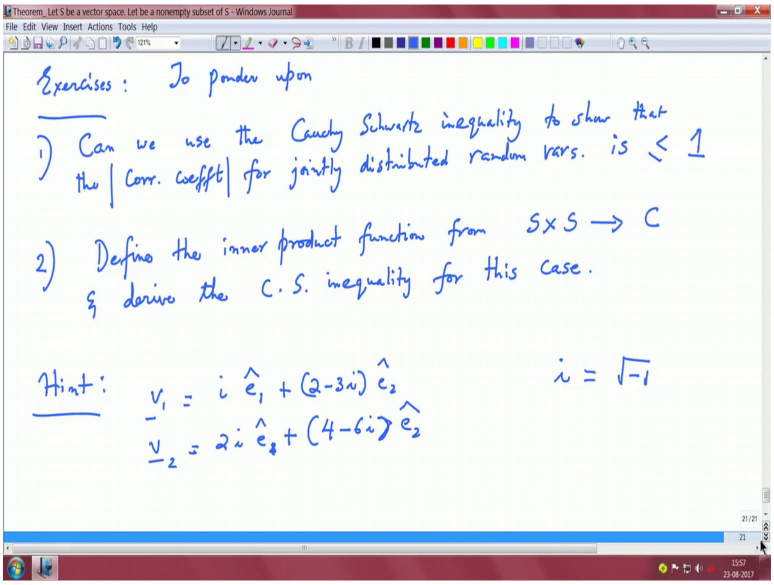

Questions or examples to ponder upon. First is as follows, can we use the Cauchy Schwarz inequality to show that the correlation coefficient for jointly distributed random variables is basically less than or equal to 1. I would say the absolute value here the absolute value of the correlation coefficient for jointly distributed random variables is less than or equal to 1. So, correlation coefficient can be minus 1 or plus 1 on the 2 extrema and then it has to be between minus 1 and plus 1 right.

So, now, I want you to use this trick that we just derived basically start we did not, we started off with induced norms and just from our notion of induced norms and the notion of our metric we just arrived at this derived this equation for this inequality for Cauchy Schwarz. And I want you to use that to show that the magnitude of the correlation coefficient is going to be between minus 1 and plus 1 right. Our absolute value of correlation coefficient is less than or equal to 1.

And second define the inner product function from the Cartesian product of 2 vector spaces to a complex number, to a complex number I mean the value that this function takes it can be complex numbers. So, define the inner product function from the Cartesian product of the electro spaces to a complex number and derive the Cauchy Schwarz inequality for this case. So, I just give you a small hint why and how you can have these inner products of complex vectors that can be potentially complex right. For example, if I look at i times let us take the vector in the natural basis, i is basically square root of minus 1 right; i in even direction and I will take say 2 i, 2 time 2 minus 3 i. We can choose whatever we like you know this is in e 2 direction. Let us say this vector is v 1 and I have another vector v 2 which is say 2 times i in the e 2 direction in e 1 direction plus some quantity 4 minus 6 i in the e 2 direction right.

So, if you take the inner product you will land up with some complex quantities here, potentially. So, what can you how would you fix your function and what is this notion of inner product for vectors that can have complex entries. And with this definition I would want you to derive the Cauchy Schwarz inequality. Now, this motion of inner products can be sort of extended to functions right and I think I would like to state this result for functions.

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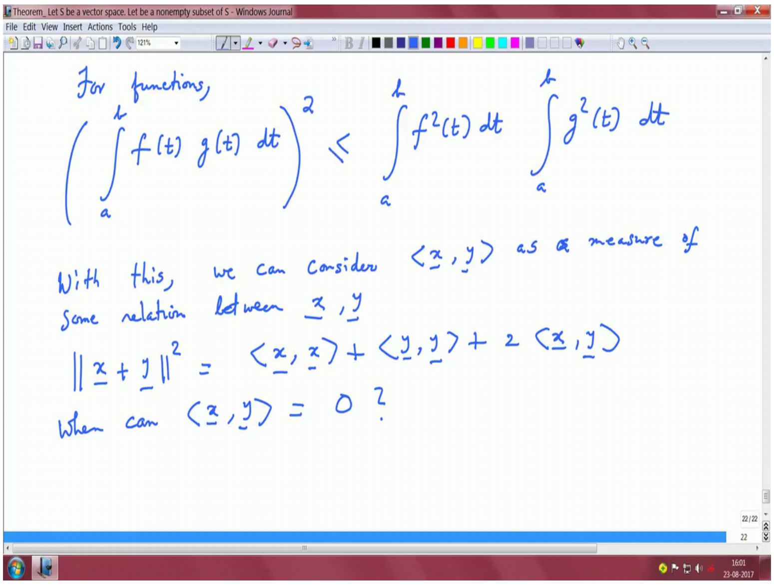

Suppose I give you functions f of t and g of t defined over the interval a to b right this is the notion of the inner product of these two functions and you extend this Cauchy

Schwarz inequality to functions we have the inner product of f of t with g of t. So, this basically bounded by the individual square norms or the integral norms of the functions. So, this is another property.

So, with this we can consider this inner product as a measure of some relation between vectors x and y right. To give you a hint you know let us look at x plus y square right and we expand it out as the inner product of x, x with x then the inner product of y with y plus 2 times inner product of x with y. So, if you ask this question when can vectors x and y the inner product of this be 0 right. This is a question I mean we just basically think about this as induce norm it started start off with induce norm we expand this induce norm in this form. Now, this is the general equation.

Now, if I think of the inner product of x and y to be 0 can I land up with this equation that the norm square of x plus y is basically inner product of x with x and inner product of y with y which is basically the familiar pythagoras theorem that we have. And intuition that we get with pythagoras theorem is if you have a right angle triangle the sum of the squares of the adjacent sides is basically the square on the hypotenuse, now what is this x and y. If x and y are adjacent to each other and the angle between them is 0 then this holds as basically equivalent to the pythagoras theorem right. So, the question now is do we have a notion for the angle between vectors right and that is what we will consider next.

We can stop here.