Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 12
Norms and inner product spaces

So, the next part of our study would be norms and norm vector spaces.

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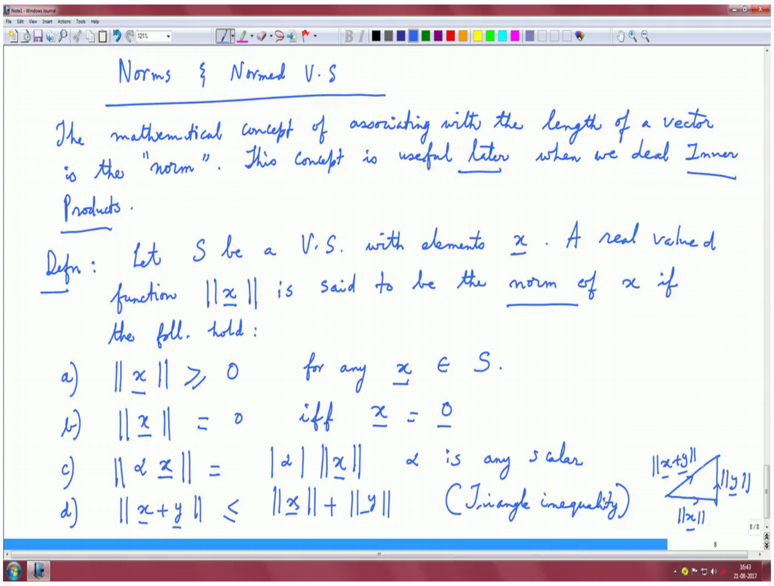

Now, the mathematical concept of associating the length of a vector is basically called a norm. I mean if think about a matrix for anything for example, if you think about the measure of your milk right, it is a liter or a gallon or some few cups whatever that cup measure is for you right there is a matrix, there is always a quantity that you associate with things. So, similarly we have the notion of associating some matrix to vectors.

So, the mathematical concept of associating with the length of a vector this is basically the norm and this is this concept is useful later when we deal with inner products and so on. So, will first come to the definition of what this norm is right. Let S be a vector space with elements x some, some x right it is just, so variable here.

A real valued function denoted within this, double bars is said to be the norm of x if the following conditions. So, they hold right. So, let us see what those conditions are. Norm of x is greater than or equal to 0 for any x. So, this is a vector. So, sometimes I miss this, these bars down, but I think you should be severe enough in the class to figure out that this is indeed a vector and not a scalar for any x belonging to the space. So, I think this should be making some sense for us right I mean this is a definition. There is no reason philosophically to wrote, learn this definitions at all because this come from we formulate things we understand we fix things and then we figure out a clear way of defining these quantities. So, one may think in from pedagogical reasons that you go with definitions of course, we go with definitions then we write our lamas and theorems and soon and so forth, but it could be also the other way around that you are interested in solving a problem you figure out what to solve the problem fix all the conditions towards the integrity of the solution and that then formulate your definitions what have to be done to define this things, this way.

So, I said length right length of a vector being negative makes no sense. It is like age of a individual is negative it makes no sense right, but maybe in a bank account your credit or debit right it can be positive negative that there is a notion of negativity and positivity. Here if I say length, length being negative 5 meters is observed right. So, therefore, the norm has to be greater than or equal to 0 right, 0 if it is. Basically we will come to that point here.

Norm of x is equal to 0 if and only if x is basically 0 it is a null, if it is a null then this length is basically 0 else it has some length. Then norm of alpha x is some alpha times norm of x and x can be a complex; that means, alpha can be a complex quantity as well. So, therefore, you have to bring in this notion of the absolute length and norm of x plus y is less than or equal to norm of x plus norm of y this is a triangle n equality that we know for vectors right and this is basically to prove this and you can invoke the pythagorean theorem this is length x, length y, this is length x plus y.

So, there are some interesting properties one can prove and we will do this is as a part of homeworks. The parallelogram property etcetera, polonaise theorem etcetera that you have studied in your high school they can be brought out to apply here for vectors. So, we will see this through a home works.

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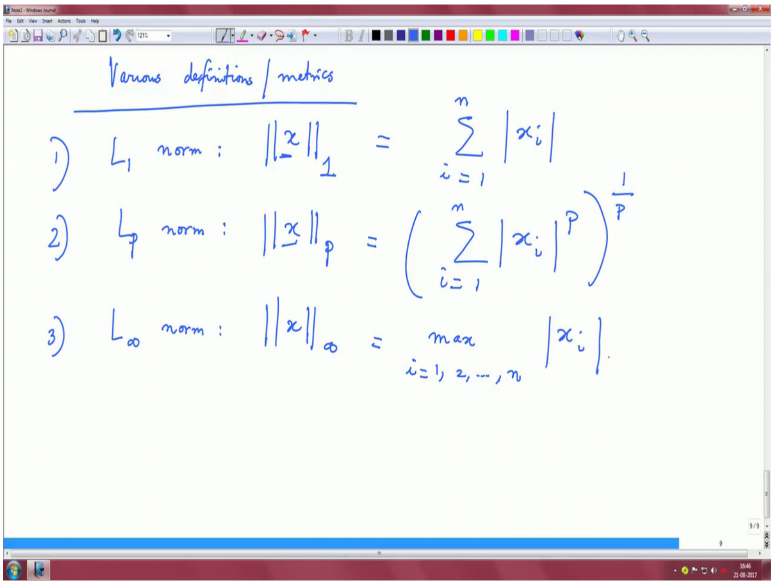

Now along with this notion of norm there are various other definitions or matrics for your normed norm spaces. And how do you define this norm? I said norm is some real valued function is what it is, what sort of real valued function is it these are questions which naturally come in to our mind, what sort of functions can be defined right and that gives us various definitions in matrix.

The first is the L 1 norm and this is denoted as this quantity here which is basically summa i equal to 1 to n assume that we have n such coordinates for this vector right. Again this is a vector here also underscore. So, take the absolute on each coordinate sum them up and this is what you get, this is the L 1 norm for L 1 distance. So, if you just analyze this we have the L p norm which is denoted as x p norm. So, which is basically summa I equals 1 to n absolute value of x i power p sum them up over all the n coordinates take the pth root of it. And you would have seen this in the context of the in the L 2 where you have to define this square root of x square plus y square right you would have see this quantity in somewhere or the other, and trivially for the L 2 norm p equals to 2 and then this is basically what we have. So, in general we can define L p norm which is which was what we have.

Now, we do this we get to the L infinity norm which is basically maximum where i equals 1, 2 dot dot dot till n absolute value of x i. So, these are the definitions right. Look at all the coordinates absolute value of that maximum over that is the L infinity norm.

Now, I think it is very important for us to think how these norms are related geometrically because if you think about the geometry it is very easy for you to visualize and picture, where this vector lies, what is the boundary, what are we trying to optimize and that picture becomes very very clear.

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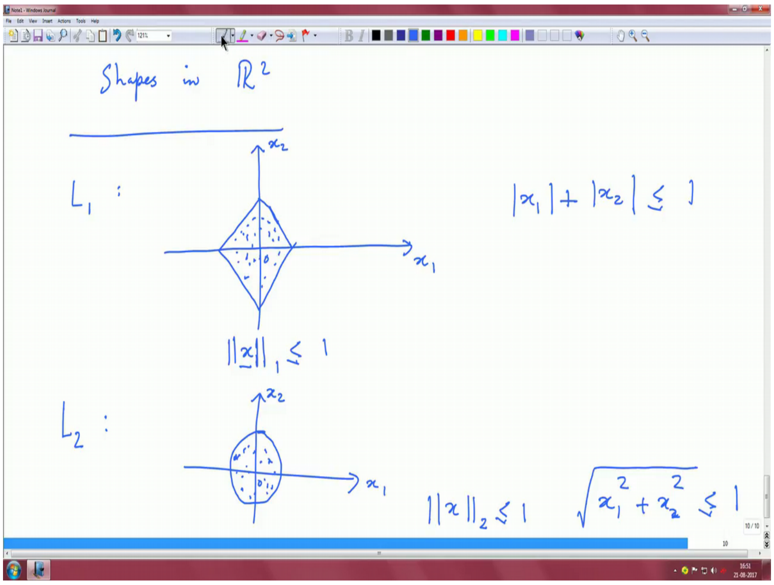

So, let us see the shapes in R 2; that means, we have all our vectors which are in R 2 that is 2 coordinates 2 dimensional vectors and we want to see where this norms reside. My rhombus is not exactly up to mark. So, if I look at the L 1 norm to be less than or equal to 1 right all the vectors should lie inside this rhombus right. So, this is basically modulus of x plus modulus this, this basically implies modulus of x 1 right, plus modulus of x 2 is less than or equal to 1 we need to satisfy this equation.

So, my rhombus is not perfectly rhombus. So, the way it looks here. So, I think you have to bear with me that this these angles are all 90 degrees right. This is modulus of x 1 plus modulus of x 2 is less than or equal to 1 now, you draw the boundaries for this and you will end up with this geometric shape. So, if the L 1 norm is less than or equal to 1 which means that you have all these 2 dimensional vectors which are lying inside this is rhombus.

So, let us see what L 2 does for us. Suppose I want the condition that my L 2 norm of x is less than or equal to 1 right, which means that when I take mod x 1 square it is basically x 1 square right plus x 2, square is less than or equal to I take the square root and that has to be less than 1 I mean I just could write a square root here right that is less than or equal to 1 square both sides you will have x 1 square plus x 2 square is less than or equal to 1 and we trivially know that x 1 square plus x 2 square equal to 1 is basically a circle right. Centered at the origin this is from our basic coordinate geometry. So, all our 2 dimensional vectors are lying inside this circle, clear, this is a geometry.

So, when we are saying at we are constrain now 2 set of vectors in L 1 and we want to optimize in L 1 or optimize in L 2 that means, we should think about the geometric this is where the vectors are lying inside this boundary and we are constraining it to this space and we are doing somethings on that right. And what norm should you chose for what problem it is up to you and up to the problem to decide. So, optimization typically is you know is it is many ways I mean it is subjective and really depends upon what you are objective is really.

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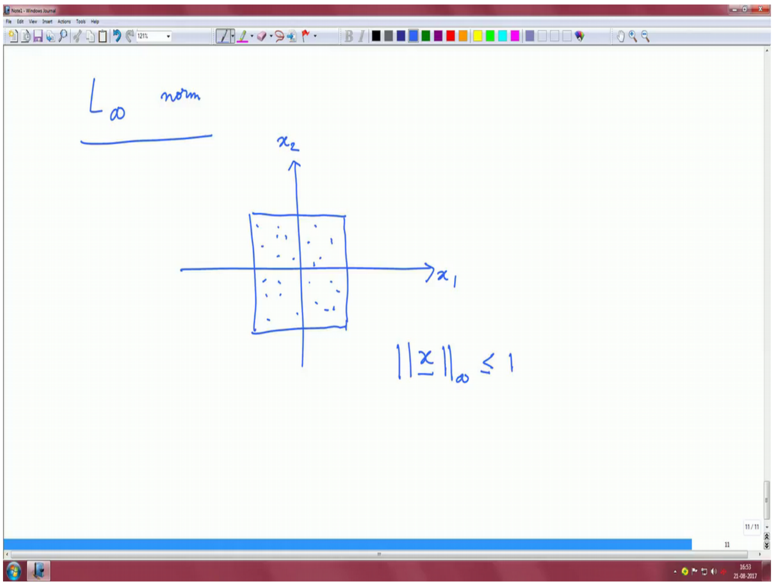

Now, similar to this L 1 and L 2 we also have the L infinity norm and this is again not so difficult, but use this definition that L infinity norm is maximum over all the coordinates the length of x i. So, basically it would just be if I say norm L infinity if less than or equal to 1, you can imagine that your maximum over absolute of x 1 or absolute of x 2 right. You cannot go above this wall here or go beyond this wall here so on and so forth right, everything is inside this clear. So, this gives you sort of a picture what the various norms are for vectors, clear.

The geometry is very important I think one is the math the equation the definition for that metric, but you have to get the geometry into your mind what is where exactly in what space in which geometric shape these vectors lie and that is very very important. And once you are able to translate it to that point it is easier for you to see the picture. Now, what we have for vectors we can do this for functions, signals are also functions of time right, we can do this for functions.

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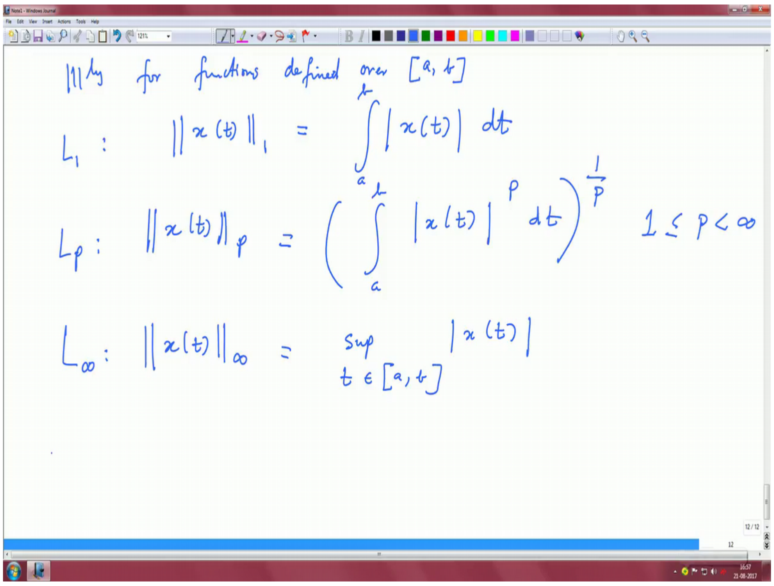

So, similarly for functions defined over this interval a b this is a closed interval we can define these norms as follows. L 1 is basically take the modulus of x of t integrated from a to b and that is that is your L 1 norm for this function. And I should now you can recall when you are when you learnt your Fourier conditions, so absolute sumability etcetera what is this, this definition here you see this in the sense.

For L p, this is basically pth root of this for every point in the signal basically you take the absolute value right you raise it to the power then you integrate it that is basically accumulate all these positive quantities and take the pth root of this and that is what the meaning is of this L p norm. And if p equals 2 you are looking at the square signal this is square integrability in some sense, clear.

We have not gone to signal case we are we are only talking about functions, but this is this is like a signal. So, we have the L infinity notion and this is sup, is superior value of the absolute value of x of t over this interval a to b right. So, you might not exactly get the maximum value and therefore, you have superior because in the you know it is the tightest upper bound that you can get to this quantity I mean you want to you want the maximum of this. You cannot exactly get maximum you have a seek sequence of bounds that you want and you want a tightest bound to x t, modulus of x t and that is a meaning of this superior and this is of course, of 1 less than or equal to p less than infinity.

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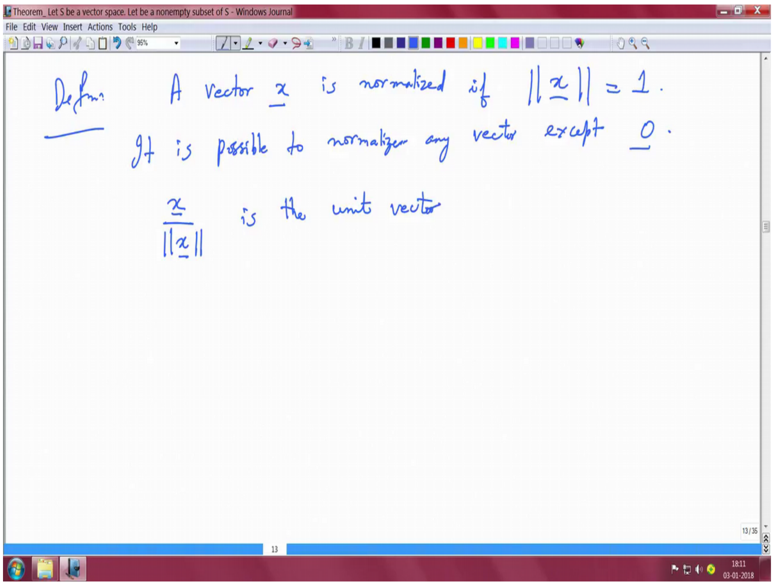

So, you have a definition a vector x is normalized if norm of x equal to 1. So, I think it is trivial to say that it is possible to normalize any vector except the 0 vector because if you know 0 vector you know you cannot normalize it, you cannot divide by 0 right. So, and x over norm of x is basically the unit is basically the unit vector because you take the norm of x over norm x the length is one trivially because norm x is the scalar you pull it out then you have norm x norm x over norm x cancels out right.

So, you have to be careful now and people just write norm straight forward you have to ask which norm in what sense at least that ability you should have to question in what sense is your definition in, what sense is your notion of length right and according to that you have to think whether it is working out or not. I mean we have to put this underscores because it is a vector.

So, with this we are ready to look into inner product, inner product spaces. So, I will define what an inner product spaces and then we can look into some of these results in the inner product spaces.

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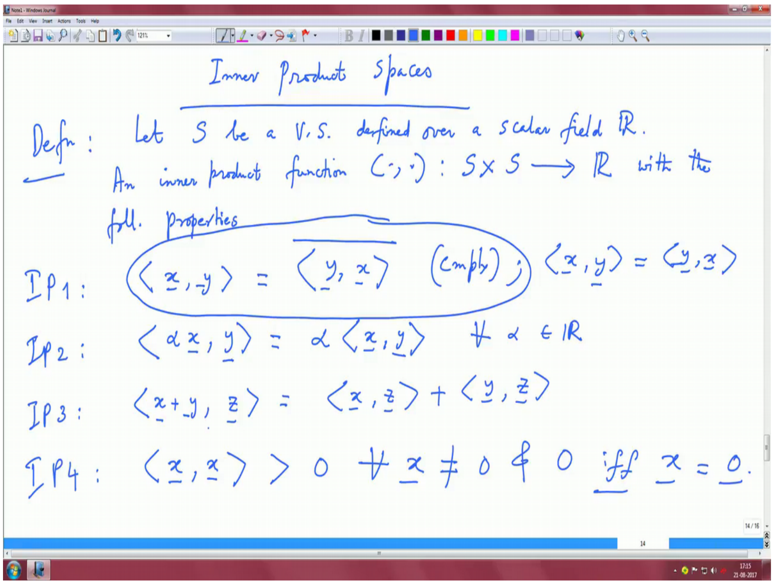

Let S be a vector space defined over is scalar field R an inner product function, is basically a function from S to S the Cartesian product of this vector space; that means, I should take 2 vectors right I means S to S means is a Cartesian product and from this it is a function from S to S to R with the following properties.

x y the inner product, you can have these vectors these are all reals or they can be having coordinates that are complex numbers right. So, inner product of x y is basically the inner product y x complement for the complex case. Hence you can say x y inner product is basically y x. We have the second property alpha x with y we can assume without much loss of generality we just consider the reals right for all the coordinates. So, if you take alpha x with y take the inner product it is basically alpha times the inner product with x and y for every alpha belonging to real numbers. So, there is the third property which is basically distributive you take x plus y and take the inner product with z this is basically x z, inner product of x z with and then in plus inner product with y and z right. This is basically distributive property.

And we have the last property which is the inner product of x with itself because now this is basically the Cartesian product of the vector space where I can pull another vector y to be x itself and I take the inner product of x with x and that is greater than 0 for all x not equal to 0 and 0 if and only if x is basically this null, null vector right. If all these conditions are satisfied then this is basically the inner, this is basically inner product function. So, it has to satisfy all these properties.

So, it has to be noted in this definition we are considering this inner product function as basically a function which is operating over the Cartesian product of this vector space to the set of all real numbers and when I gave this property IP 1 saying that this is the inner product of x y is basically you know y x conjugated and this is for complex I mean you think about this definition this is applicable to reals and this is where you have to focus upon x y the inner product of x y is basically the inner product with of y with x and this is the property which holds. I just mention complex just is in aside. So, if we define this inner product function to be a complex function that happens in some applications in the area, in quantum information or in various other applications then we may have to modify and tweak some of these definitions slightly differently.

Now, suppose I think about the inner product of x and y right, so I have x to be some x 1 so on till x n and some y is this vector y is basically coordinates y 1 till y n right.

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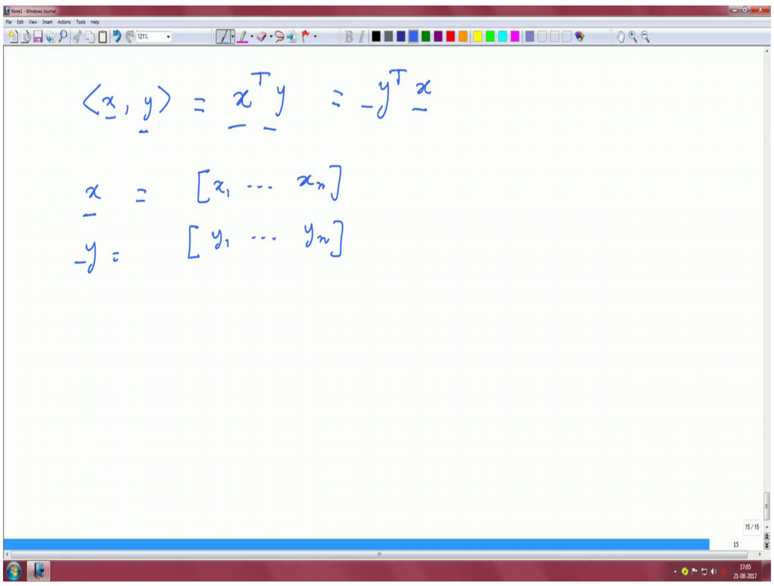

So, the inner product of x and y is basically x transpose y right. (Refer Time: 30:46), x 1 y 1 plus x 2 y 2 plus so on till x n y n and this is basically y transpose x. For vector source entries are all real numbers, if the complex entries there are some more trivialities which we will reserve it for homeworks.