Mathematical Methods and Techniques in Signal Processing - I
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Lecture – 11
Basis and cardinality of basis

So, with this notion of linear independence of vectors and a collection of vectors that can span the space we are ready to now define what is called the basis right. We will just define what the basis is. But I am sure most of you would have taken for granted what the basis is from your undergraduate signal processing right. When you say Fourier basis you would have just taken it for granted what the basis is now we are formally coming to the definition of what the basis is.

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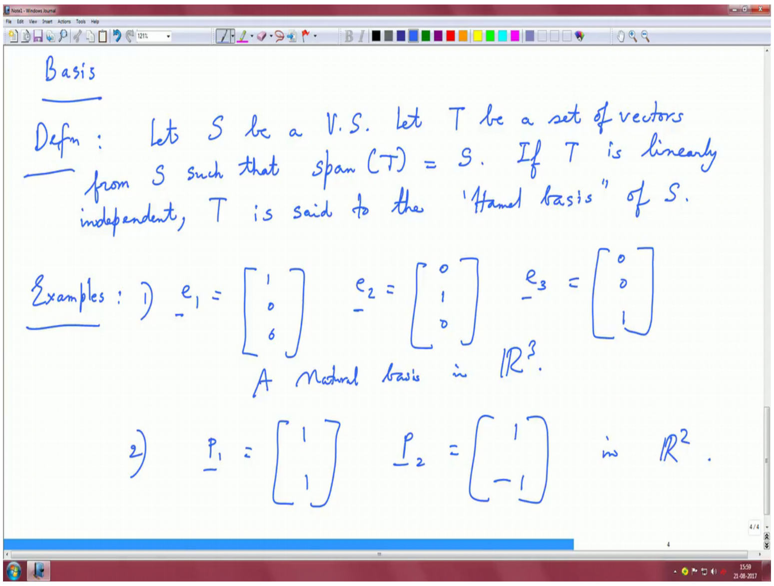

So, let us start with this definition. Let S be a vector space, let T be a set vectors from S such that span of T equals S if T is linearly independent, T is said to be the Hamel basis of S. You could even ignore the word Hamel if you would want to just its a basis right there are plenty of good examples for this for this basis. So, if you think of the natural basis for example, 1 0 0, 0 1 0 and then 0 0 1 this is basically a natural basis in R 3. And you can have 1 1 and 1 minus 1 as basis in R 2 you can also have 0 1 and 1 0 you could, but you know I am writing it slightly differently, but we will see through these connections you can have possibly infinite set of such basis.

So, with this notion of what a basis is we are ready to prove a theorem which says that if T 1 and T 2 are the Hamel basis for a vector space S then T 1 and T 2 have the same cardinality. So, we will prove this result and this has important consequences.

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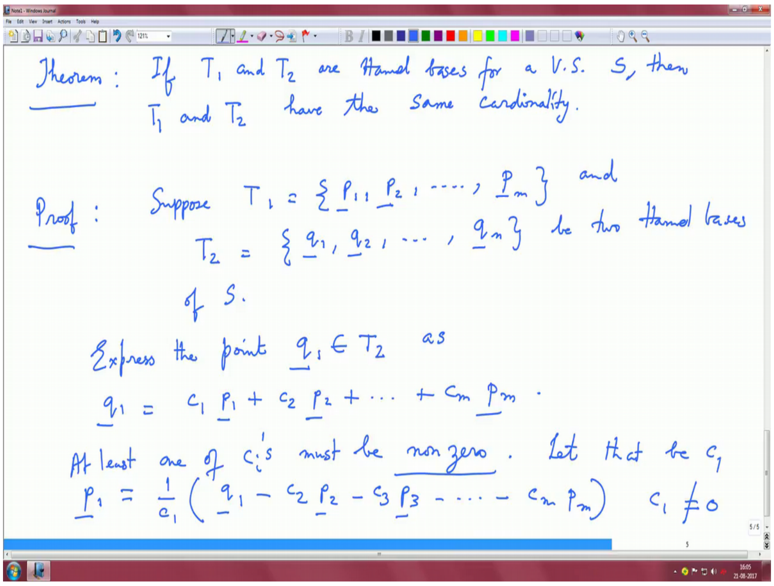

If T 1 and T 2 or Hamel basis we can make this a plural for a vector space S then T 1 and T 2 have the same cardinality and from intuition you should get this picture right. If I say it is a 2 dimensional space and my basis is say 0 1 and 1 0 or 1 1 and 1 my you know one minus one then likely I may not have 1 1 1 minus 1 and some x y which is going to be the basis for that and that is what we need to prove it in this result.

So, we will start with the proof. Suppose T 1 is this collection p 1, p 2, so on till p m and T 2 is the set which is q 1, q 2, so on till q n and let this be to handle bases of this vector spaces. So, the first step what we do is we express the point q 1 belonging to T 2 as false. So, we pick this vector q 1 in T 2 and we express it using the vectors in T 1. So, let us suppose q 1 is c 1 p 1 plus c 2 p 2 plus so on till c m pm. So, of course, we must say here that at least 1 of c i’s must be nonzero right because it is a definition of a basis and let this the c i be is say some c 1, at least one of them to be nonzero and let this be c 1. So, now, p 1 is basically 1 over c 1 times q 1 basically we are rearranging this in a different form minus c 2 p 2 minus c 3 p 3 dot dot minus c m pm. So, we pick a point q 1 belonging to T 2 express it in this form and at least one of the c i is must be nonzero and let that be this element c 1 and then p 1 is basically read it in this form just rewriting rearranging this equation slightly differently.

Now since we have written it in this form we can say that we can eliminate p 1 as a basis in T 1 and we can instead replace that with q 1 right. You see the idea how we are progressing right. We pick one in T 2 express it in terms of T 1 and then we say just we can replace that vector that basis vector in T 1 with this new element.

Let us that see intuition or that is the idea and we will see how we progress on that.

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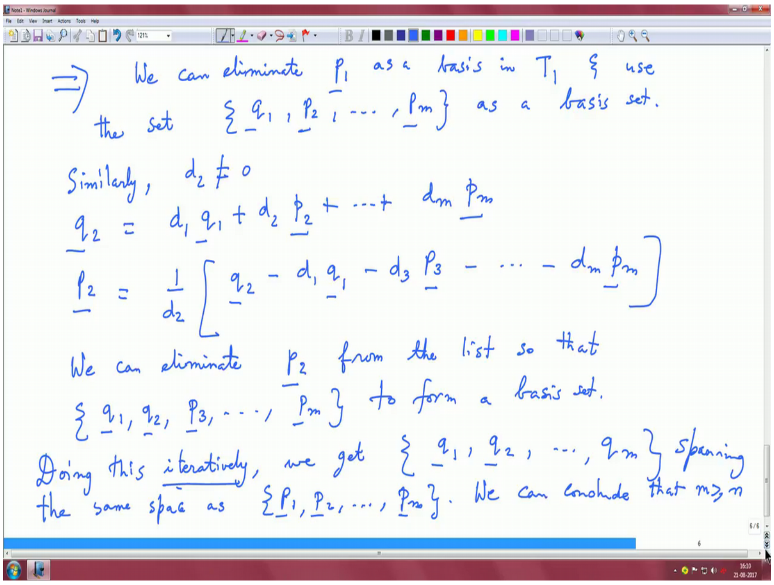

So, this implies we can eliminate p 1 as a basis in T 1 and use this set q 1 p 2 so on till ps to m as a basis. So, now we have the intuition into the proof of this result. So, we will try this again. So, similarly we say some d 2 is not equal to 0 I pick q 2 which is written as d one q 1 plus d 2 p 2 just observe this change q 1 then p 2 plus dot dot dot ds of xm p m. So, this is how we wrote the representation of q 2 in terms of our newly formed set of basis right.

So, now, we do this elimination again we say p 2 is 1 over d 2 times q 2 minus d 1 q 1 minus d 3 p 3 minus dot dot dot minus ds of xm ps of xm. So, now, we have sort of gotten the trick towards doing this right. So, now, I expressed p 2 in terms of the rest. So, we can eliminate p 2 from the list so that we have a new set q 1 q 2 p 3 dot dot dot p m to form a new basis set. So, we do this iteratively and we land up with q 1 q 2 so on till qs of xm spanning the same space as p 1, p 2, so on till ps of xm. So, we can also conclude that m is greater than or equal to n, in the process of this replacement we can say that m is greater than or equal to n.

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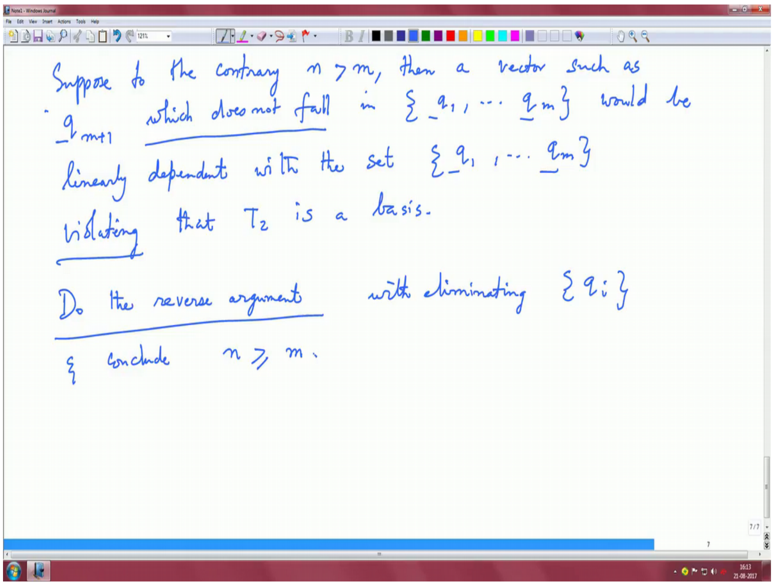

Now, suppose to the contrary n is greater than m right. Suppose to the contrary n is greater than m then a vector such as q of m plus 1 which does not fall in this set this is very important which does not fall in this set q 1 to q m would be linearly dependent with the set q 1 to q m. I did all the replacement on the contrary I assume n is greater than m then a vector such as q m plus 1 which does not fall in this set q 1 to q m would be linearly dependent with this set.

Now, this violates that T 2 is a basis because we had T 1 and T 2 both to be the basis and now we have a contradiction. So, we do this in the reverse argument. So, we started off with some element in T 2 and expressing it in terms of T 1 and then replacing that element in T 1. So, we do this, we can do this in the reverse argument with eliminating q i and then we can conclude n is greater than or equal to m. In one side we said m is greater than or equal to n and in the other side we are saying that n should be greater than or equal to m and basically this implies is only possible if m equals n and this shows that the cardinality has to be the same.

So, the idea is as follows you have some m some n. So, once you assume that m is greater than or equal to n for which you fix some m fix some n start replacing all these ns with what you have in the earlier set and then you conclude that basically m is greater than or equal to n because if otherwise if n was greater than m you would have this contradiction here. And in the second case you will follow the reverse argument by eliminating all the q i's and then conclude that n should be greater than or equal to m, but with these two statements they have to be the same which would imply m equals n.