Mathematical Methods and Techniques in Signal Processing - I

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Lecture – 10
Unique representation theorem

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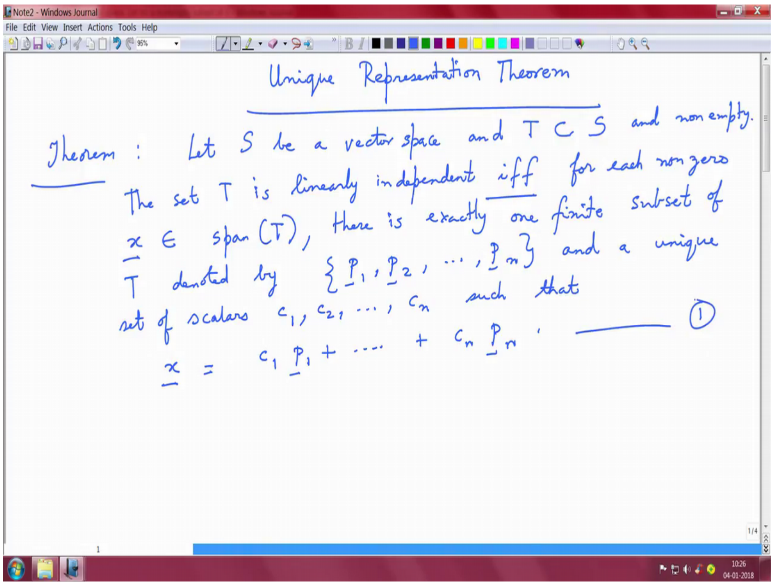

So, let us prove an important result in linear algebra which is the unique representation theorem. So, I will state the theorem let S be a vector space and T be a subset of this vector space and nonempty. The set T is linearly independent if and only if, I mean there is a forward statement as well as a reverse statement if and only if for each nonzero x belonging to the span of this space T, there is exactly one finite subset of T denoted by p 1 this collection of vectors p 1 p 2 so on till p suffix n and a unique set of scalars c 1 c 2 so on till c n such that x can be written as linear combination of all these vectors p 1 to p n through these scalars c 1 through c n.

So, before we begin we have to understand what we need to prove. So, there is a forward statement which says linear independence should imply unique representation and then there is a river statement which says if it is unique representation then it is linearly independent and that is this if and only if statement. So, let us try to prove both these parts.

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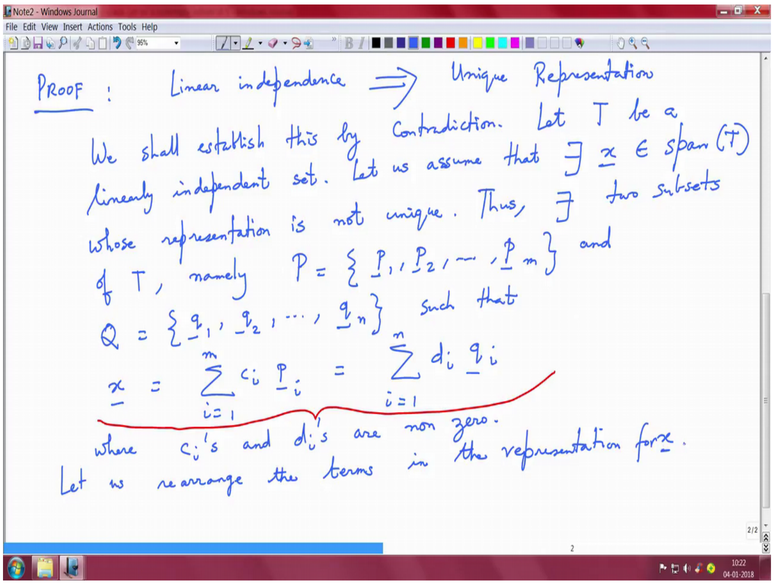

So, to prove this part we shall establish this by contradiction. Let T be a linearly independent set. Let us assume that there exist x is a nonzero vector belonging to the span of T whose representation is not unique and this is where we have to establish this result. Thus, there exist 2 subsets of T namely a set P which is comprising of vectors p 1 p 2 so on till ps of x m, the p is span T, Q having q 1 q 2 so on q suffix n. The Q is span T see the difference between m and n here such that x can be written in 2 forms one using the set P and other using set Q. So, x is basically a linear combination of vectors from the set P and we said that this representation is not unique so therefore, we are using another combination using vectors q i and associated scalars ds of i, where I think I have to make a point here c is and d is are nonzero.

Now, let us rearrange the terms in representation of x that is whatever we have written down here. So, let us rearrange the terms in the representation for x.

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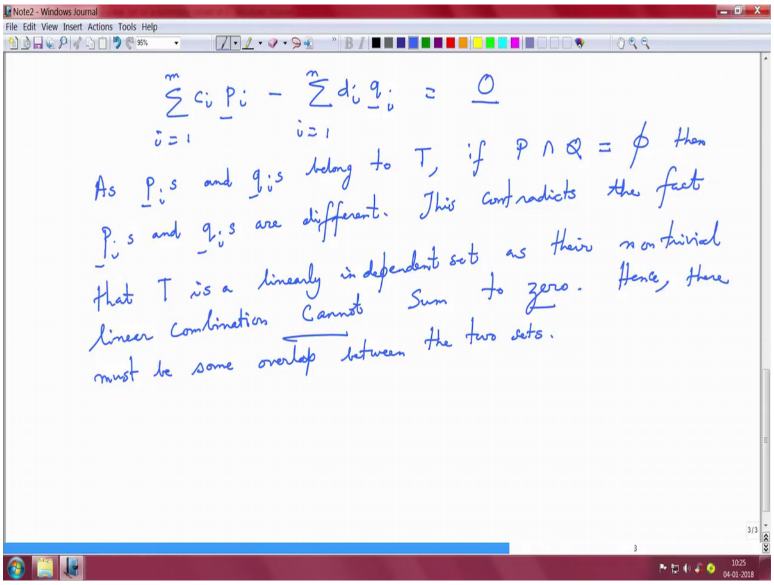

So, what we get is sigma i equals q to m cs of i p i minus summation i going from 1 to n ds of xi qs of xi this is basically the 0 vector. Now, as p is and q is belong to T right if P intersection Q is a null set then p is and q is are different. So, this contradicts the fact that T is linearly independent set as their non trivial linear combination cannot sum to 0. Hence there must be some overlap between the 2 sets right.

As q is and pi's belong to T if the intersection of p and q is a null set then pi's and q is are different this contradicts the fact that T is a linearly independent set as their nontrivial combination cannot sum to 0. So, therefore, there must be some overlap between these sets. So, let us proceed further.

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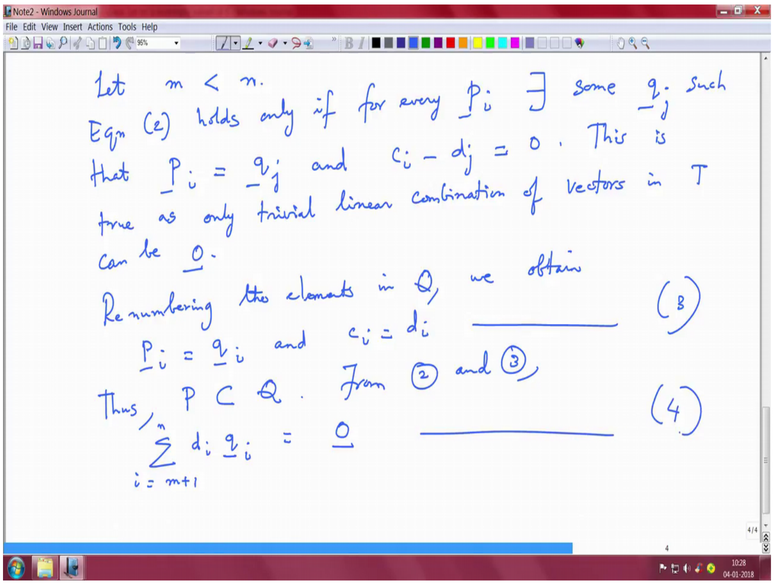

Let m be less than n right because there must be reasons there is some overlap and m be less than n without loss of generality. Now, call this equation before equation 2 and possibly the first equation that we have in the theorem we can just call this equation one. So, equation 2 holds only if for every ps of x i there exists some qs of xj such that p's of x i equals qs of xj and ci minus dj vanishes. This is true as only trivial linear combination of vectors in T can be 0.

Now remembering the elements in q we obtain say ps of x i is qs of x i we can just label this and then c i. So, scalar c i equals d i thus the set p is part of the set q let us just call this, this equality here as equation 3. So, from 2 and 3 we land up with summation I going from n plus 1 to n, d i qs of x i is basically this 0 vector. So, let us call this equation number 4.

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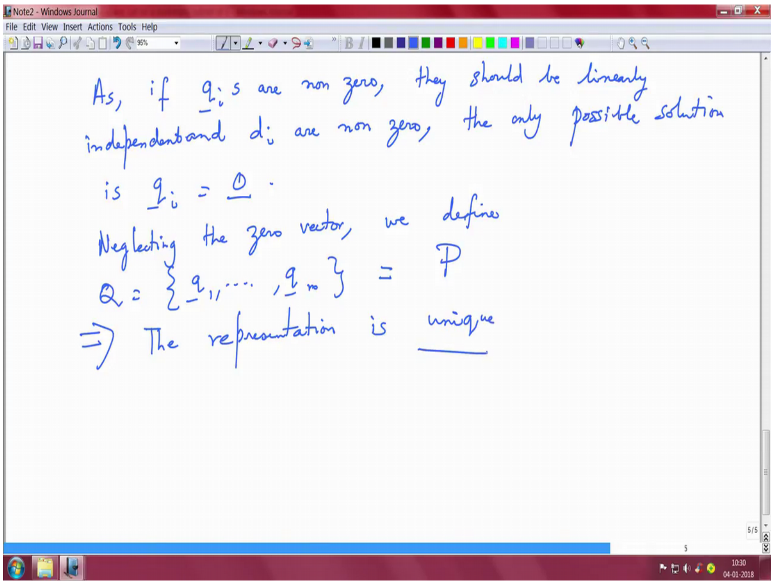

So, as if q is are nonzero they should be linearly independent and d i or nonzero, therefore, the only possible solution is q i being this 0 vector. So, therefore, if we neglect the 0 vector we define the set q is basically q 1 so on till q m which is basically this set p which phase arrived at because p i equals q i for i equals 1 to m and therefore, the representation is unique.

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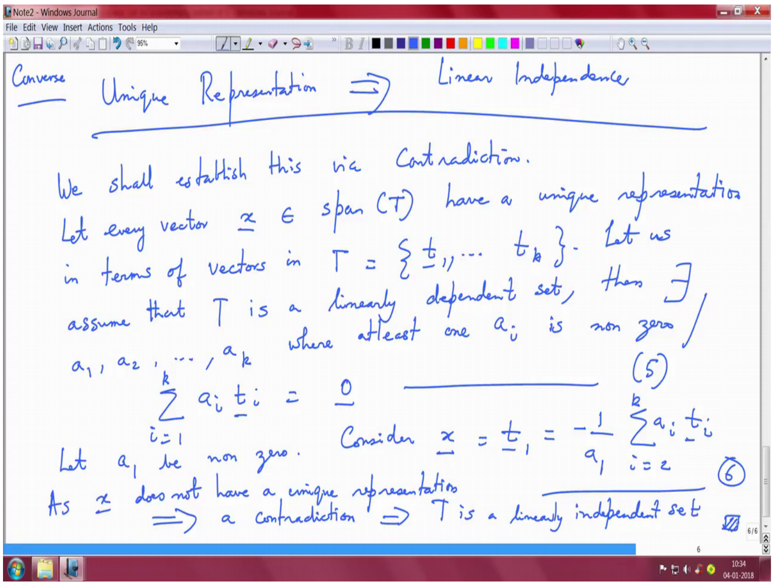

So, let us get to the other part which is, if we have a unique representation does it imply linear independence. So, again we shall establish this through contradiction. Let every vector x belonging to span of this set T have a unique representation in terms of vectors in the set T comprising of say t 1 to some t k. Let us assume that T is a linearly dependent set then there exists a 1 a 2 so on till a k, where at least one a i is nonzero such that this linear combination is a 0 vector.

Let a 1 be nonzero consider and call this equation 5. Consider x which is equal to t 1 say suppose and this is given by minus 1 upon a 1 summation i equals to 2 k, a i t i. Now, as x does not have a unique representation this implies a contradiction because we assume that it has a unique representation so therefore, T is a linearly independent set and this completes the proof of this theorem. We will stop here.