Basic Electrical Technology

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Lecture - 10
Transfer Function & Pole-Zero Domain

Hello everybody, in the last few sessions we have been discussing about modelling electric circuits and also something about their analysis in MATLAB and also on paper. In modelling the electric circuits we went through the process of obtaining the state equation and then we use the state equation model to input to MATLAB and then see the results the outputs in three domains: which is the time domain, the frequency domain and the pole-zero domain.

In this session let us have a closer look at the pole-zero domain and a method of modelling called the transfer function modelling is again a special case of the state equation. As I told you earlier the state equation will give you the overall total dynamic behaviour of the system it contains in it the information about the steady-state model and also transfer function model. We saw in the last session how we go about modelling for sinusoidal steady-state conditions by replacing d by dt with j omega.

In this session we shall look at what is called as a transfer function modelling. I will explain what transfer function is later on as the class progresses. But we will see that it has lot of useful features in studying the system in the pole-zero domain and also to some extent in the frequency domain. So today's session of course is going to be focussing on transfer functions. What are transfer functions; how they will be obtained this will be the focus of discussion in this session.

If we look at the state equation of any general circuit or system it is of the form dx by dt equals Ax plus Bu and the output equation Y is equal to Cx plus Du. This is the standard form of the state equation, this is the state equation.

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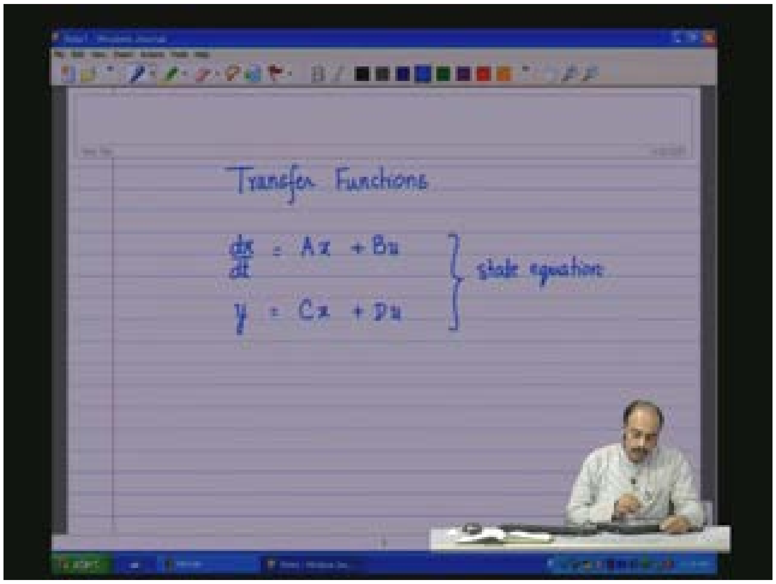

What is going to differ from system to system, circuit to circuit is the A matrix which is the parameter matrix for the system which defines the characteristics of the particular circuit, the B matrix or the input matrix input scaling matrix, the C matrix which is the output scaling matrix and the D matrix for the feedforward matrix. This is the output equation and the first one is the dynamic equation.

If there is more than one state X itself will be a state vector xs' and if there are more than one input u will be the input vector, A will be a matrix, B will be a matrix so also C and D. Now this is the state equation that we know how to obtain for any given circuit. Now if we look at the block schematic of this one the state equation is basically composed of many first-order linear differential equations.

Now if you look at the block diagram of this particular system let us consider this here the dynamic equation. So you have the input, it is scaled by a B matrix or B variable so it is given to a summer where one adds Ax. So let us say that we have the parameter matrix of the system A here and we are obtaining X from somewhere so you have here Bu and you have here Ax and therefore on summing it up you have dx by dt. This on integrating will give you x and that x is fed back to obtain dx by dt and if this x is fed through a scaling matrix C one obtains Cx and then if we have a summer here plus and a plus and from this input u let us pass it through a D matrix one obtains y.

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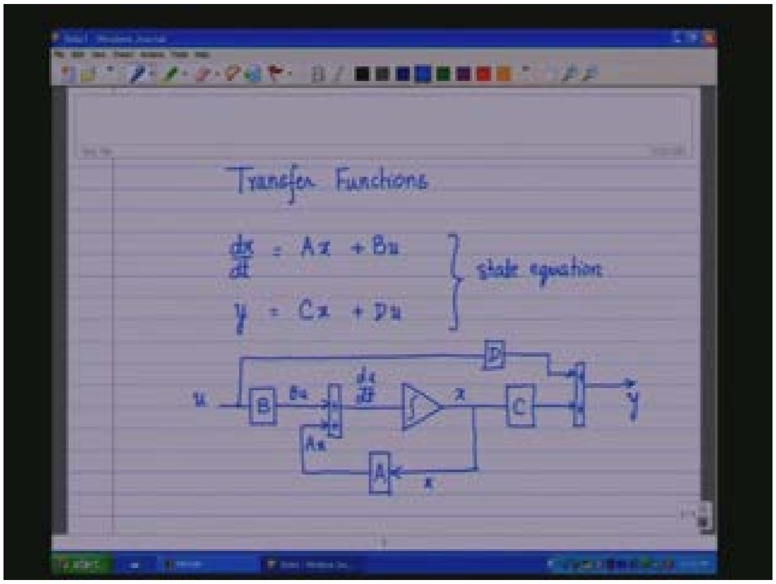

So, starting from u here the input variable we get the variable Y intermediate this is all what happens. This is a common block diagram for any system because we are working with the general state equation. Now what is important here to be noticed is that there is one integration (Refer Slide Time: 8:13) and two summing blocks. The integration is fundamental to any physical system any physical system as they can be represented as state equation and the state equation has one integrator per energy storing element.

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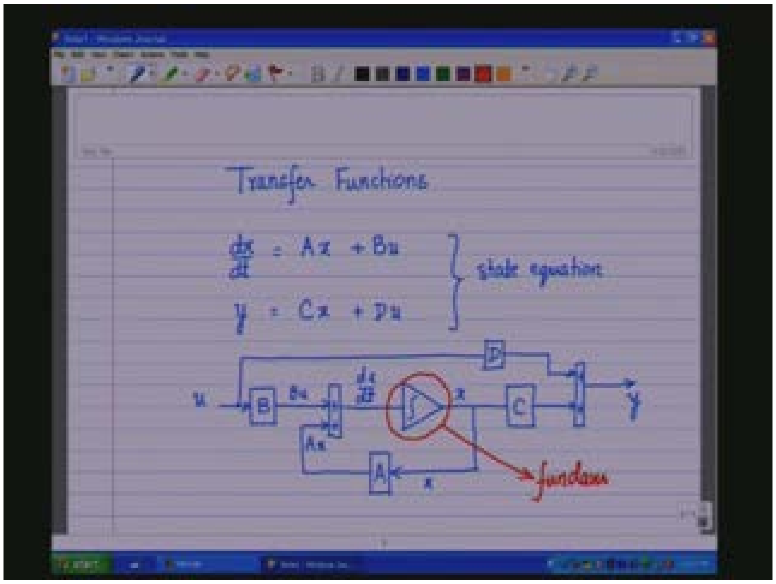

So the integrator can be used to model any physical system and also for simulating any physical system and thereby for analysing. So integrator becomes a very useful component in the physical system and physical block system or block diagram where you can analyse and simulate the system.

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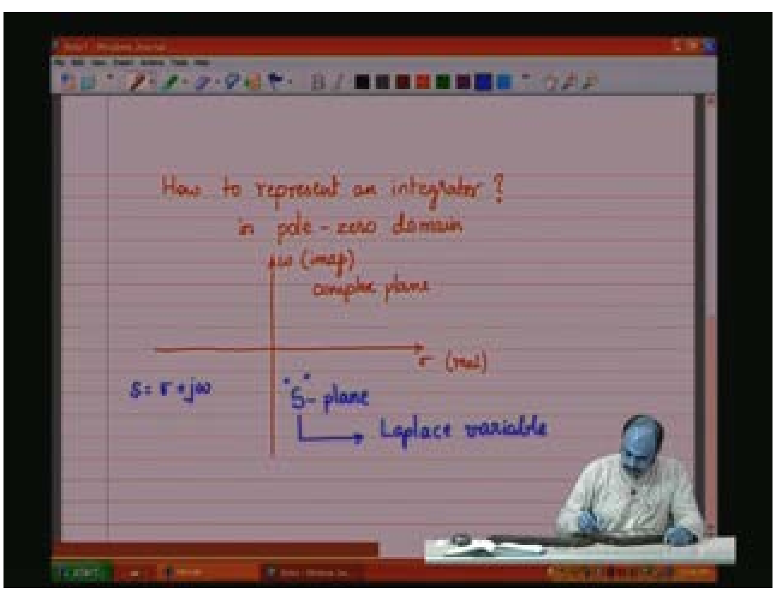

How do we how to represent an integrator where in pole-zero domain in the pole-zero domain? So let us look at this first. This will give us an idea for using it for the other components. In fact, integrator being the basic fundamental building block we will use this to model the other components like L and C's and then thereby getting the model for the other complex circuits. In the pole-zero domain we had two orthogonal axes like that; you have the sigma axis which is the real axis and we have the omega axis or the imaginary axis and this complex plane is called the S-plane; this is called the S-plane. Note this variable s. This variable s is the same as the Laplace variable, the Laplace variable in Laplace transforms. So this is also called the S-plane where the s variable s is equal sigma plus j omega. This is how the pole-zero domain is represented we saw that earlier. Now we will see how to get the poles and the zeros from the electric circuits.

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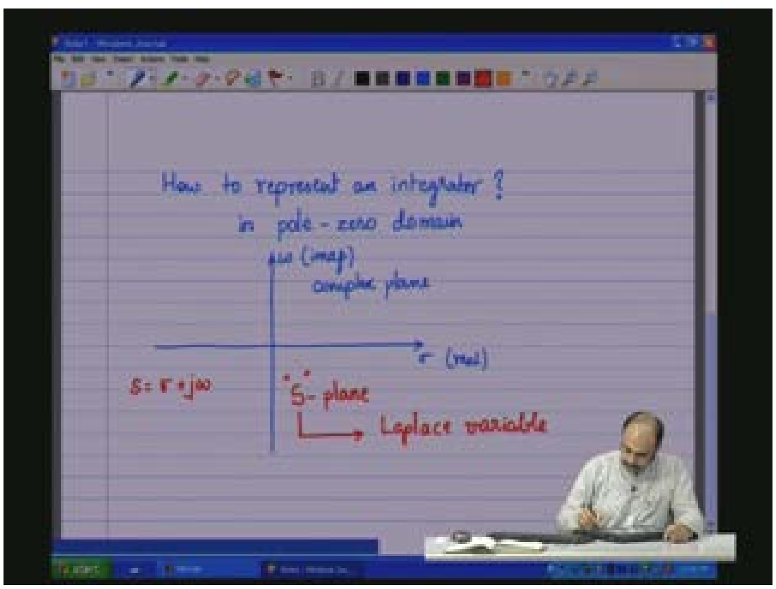

Now consider that we have an integrator, a block I am just representing it as a block, it performs the integral action, there is an input and there is an output and let us call the input as u and the output as y. Now, to represent this system in the pole zero domain we have to convert it into variables which are functions of s rather than functions of time. So this whole domain this is in time domain, we have to obtain a system in the pole-zero domain which are functions of the s variable or the Laplace variable because the pole-zero domain is in terms of the s variable.

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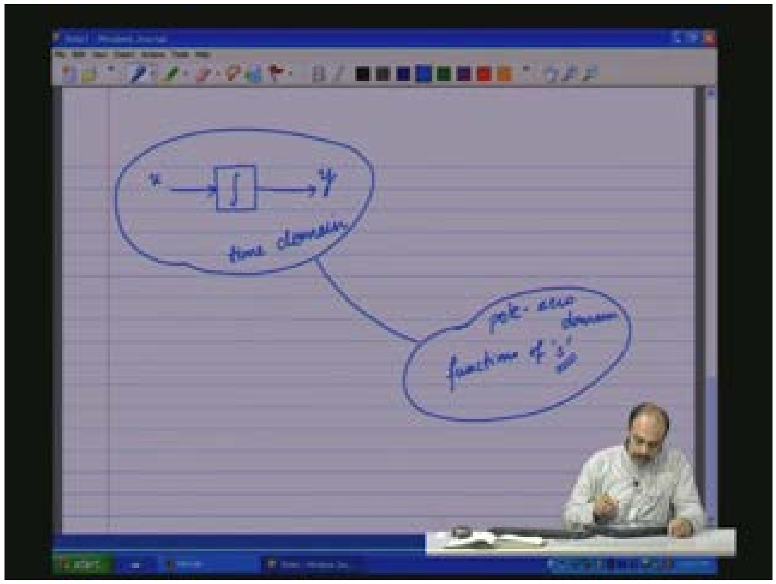

So how do we go about getting this, how do we go about mapping this which means we have to transit from a system u which is equal to the in which is the input in the time domain, why which is the output in the time domain and there is an integrator represented in the time domain and this should be representable or represented as an integrator in the s domain with a input so u should be now a function of s, the output y should become a function of s and this also should become a function of s. Then we say that this is in the S-domain or Laplace domain or the pole-zero domain; S-domain or the pole-zero domain like that.

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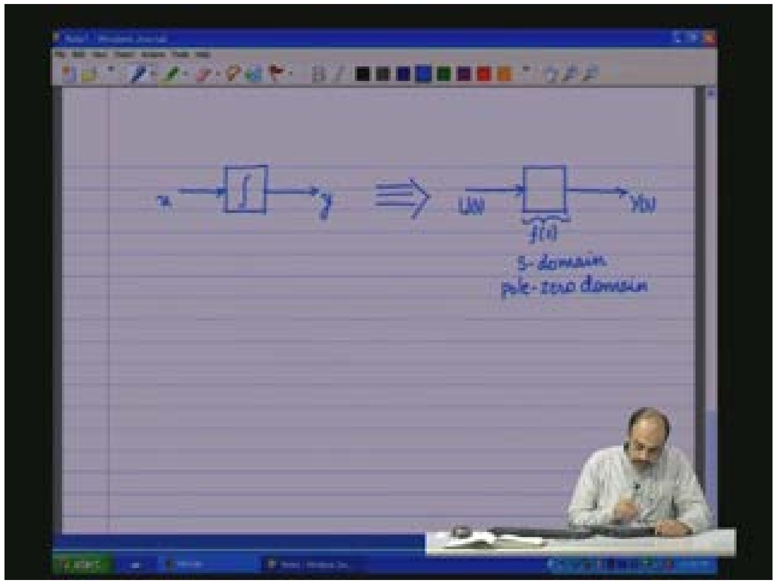

Notice that I have replaced the lower case u by upper case U, lower case y by upper case Y it is just a convention. All the variables in the S-domain or the Laplace domain or the pole-zero domain we represent it with upper case. We now would like to see what is Y(s) by U(s) equal to; this is give the input and output relationship in the S-domain.

The integrator by nature, if you look at the nature of the integrator if you give a disturbance it will accumulate basically it will accumulate……… let us say I have here I have here (Refer Slide Time: 15:51) a waveform wave shape like that as a function of time; when this is passed through the integrator the output of the integrator is going to be the same, so the output of the integrator will be like that, it starts accumulating the area, then the area is calculated and then it keeps constant, the output never decays. Once the output has some value there is no possibility for it to decay or grow which means whenever the input is zero whenever the input is zero the output will just stay at its previous value. So one of the main characters characteristic feature of the integrator is neither there is growth nor there is decay of the output; this is the special characteristic feature of the integrator.

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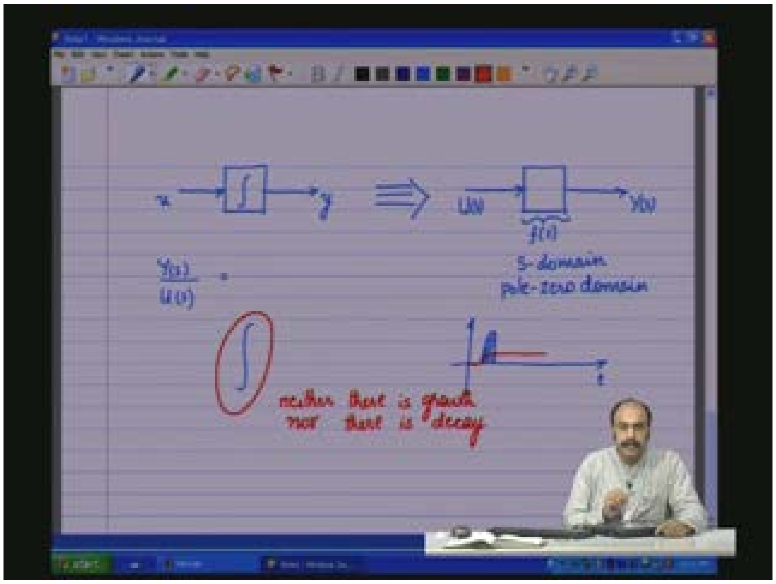

So if we go back to the S-plane let us say I will write the S-plane, you have the sigma and the omega and this is the S-plane and let us say I have a pole there exactly at S equal 0 plus j 0 which means which means at this point if I look at the time evolution which we say is exponential of e to the power of minus 0 plus j 0 into t which is equal to 1 so it is just a constant one, no growth nodes.

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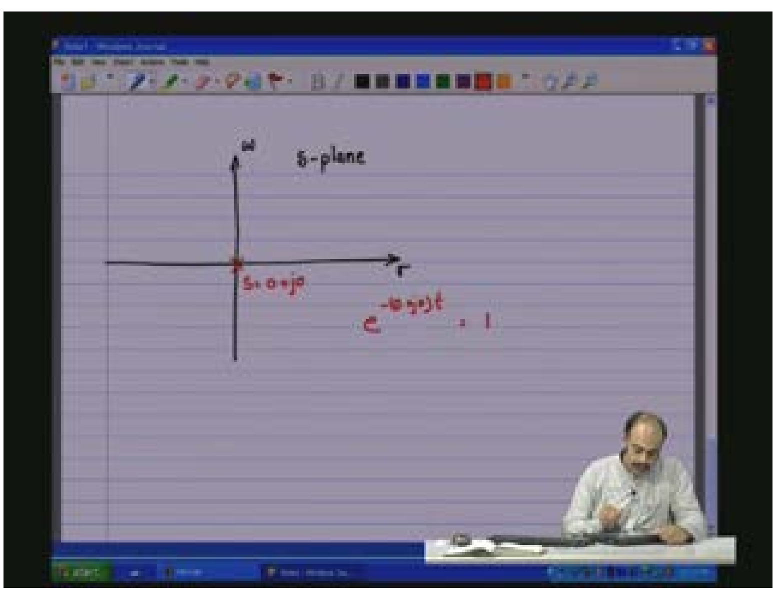

If it had been if there had been a pole at this point minus a then here it would have been e to the power of minus at a being some number other than zero and therefore here you will see that there is an exponential decay; starting from 1 this will decay at a rate as defined by a. Or if there is a value here b, now this is e to the power of bt where b is positive which will lead to exponential growth and it will start growing as time progresses. So a pole on the right of the S-plane is going to lead you to exponential growth which is an unstable system and a pole on the left of the S-plane is going to lead to exponential decay a stable system, a pole which is on the j of the omega axis will on the omega axis will neither allow you, will neither cause decay nor grow and a pole exactly at zero is an integrator.

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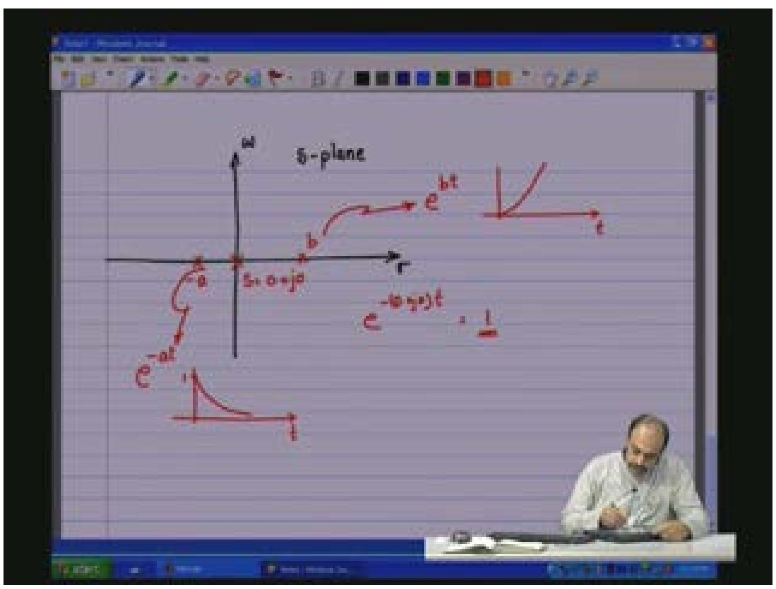

In fact, if you take a variable which is a time function let us say; let us say integral of x dt this when I take the Laplace transform of this variable will result in X(s) by s or it is nothing but (1 by s) X(s) that is 1 by s on the Laplace transform that variable. This represents an integrator and this is at s equal to 0 which is what we have put the pole here. the denominator The roots of the denominator or poles, roots of the numerator of any function which is function of s is the zero. Of course here the numerator is just only one, the denominator has a s value there and therefore it can be zero that is this function has a root at the denominator equal to s is equal to 0 and that is a pole. So at pole at the pole value when s is equal to 0 when you substitute s as 0 the gain is infinite; the gain of the function here becomes infinite between the input and the output.

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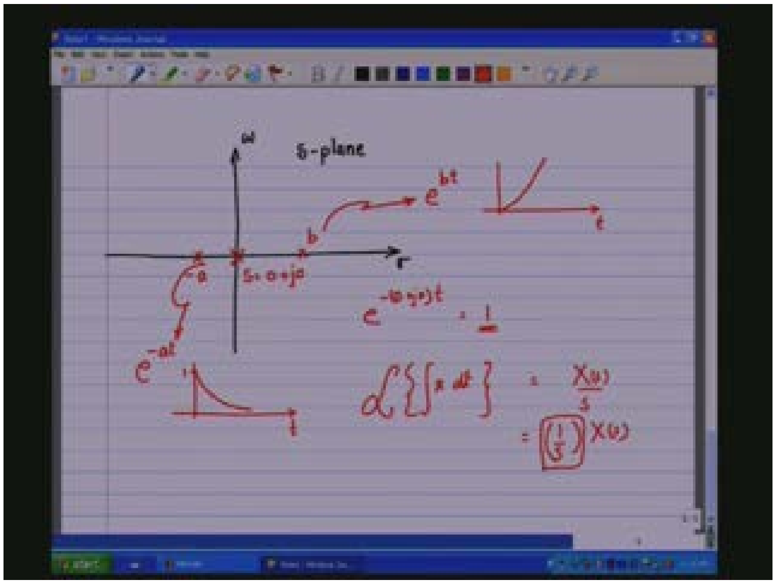

We will discuss the concept of the pole and zero, the physical concept of the pole and zero just a bit later but now we have come to one important conclusion that an integrator can be represented as 1 by s in the S-plane or the pole-zero plane. So this is time domain and this is the pole-zero domain or the S-plane.

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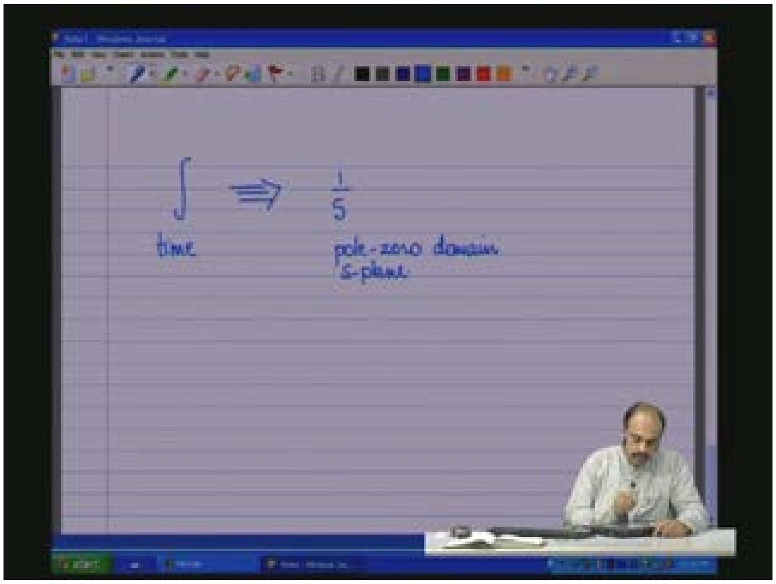

So now the integrator as such with an input u and output y can be represented in the pole-zero domain as………… now the input is a function of s U(s), output is the function of s Y(s) and the integrator can be represented as 1 by s or Y by s the output by the input is 1 by s. In fact, this we call it as a transfer function of the integrator.

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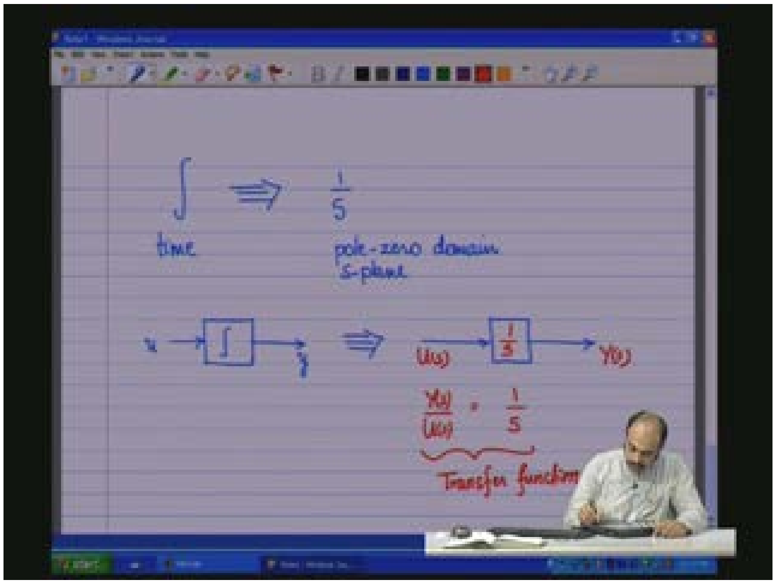

Transfer function is defined as transfer function is defined as the Laplace transform of the output variable by the Laplace transform of the input variable under zero initial conditions under zero initial conditions. This is how a transfer function is defined. And what we got as the transfer function of the integrator which is Y(s) by U(s) is 1 by s under the condition with the integrator initial condition is zero.

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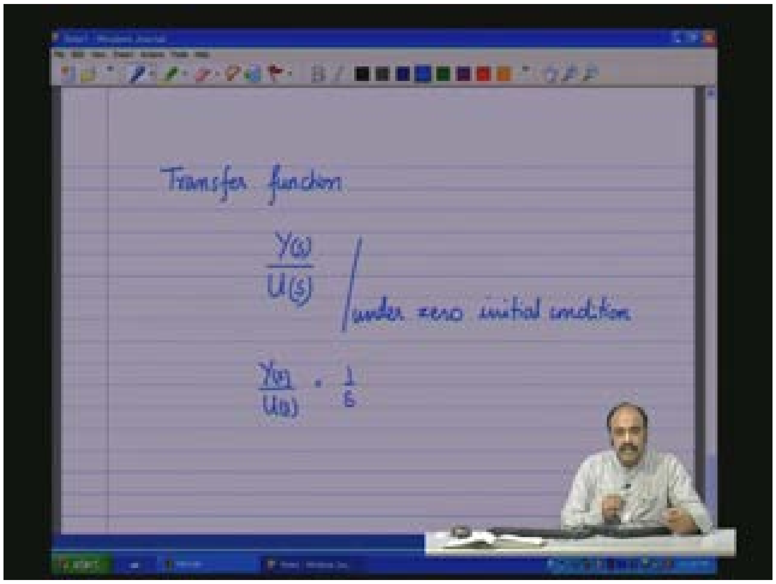

Now we should be able to obtain the transfer function of any circuit and the transfer function of any circuit will be of this general form: Y(s) output variable Laplace transform, input variable Laplace transform equals, it will have a numerator which is a function of a numerator which is a function of s a numerator polynomial a denominator which is a function of s a denominator polynomial.

Now the roots of the denominator or pole locations in S-plane, the roots of the numerator or pole or zero locations there be zero locations in the S-plane. So that is basically the important concept in the case of the transfer function. It basically takes the time domain variables into the pole-zero domain variable or the Laplace domain variable which is a function of s and that results in a numerator polynomial and the denominator polynomial the roots of which form the zeros and the poles; the roots of the numerator polynomial at the zero locations in the S-plane or the pole-zero plane and the roots of the denominator polynomial form the positions or the locations of the poles of the circuit.

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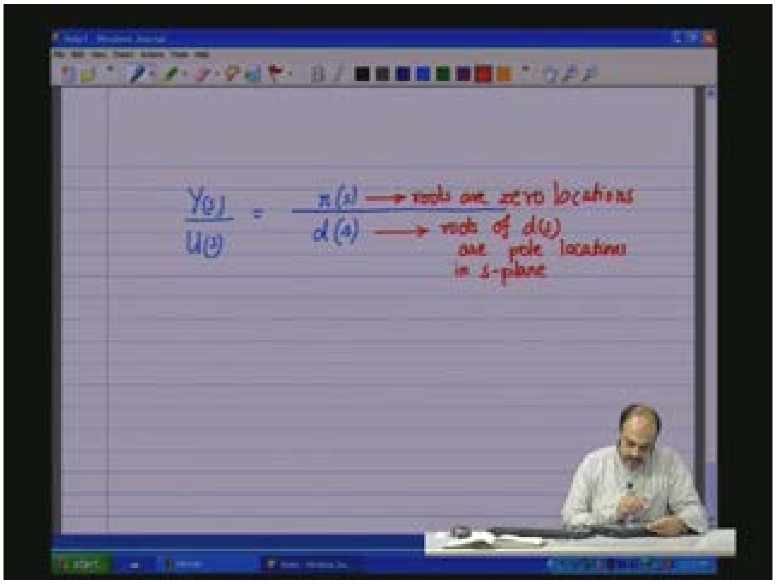

Now consider a capacitance, how do we represent the capacitance in the pole-zero domain or in the Laplace domain?

We have a capacitance, we have a capacitance current features flowing and there is a voltage across the capacitance which is the state variable which is V c, the dynamic equation is d V c by dt is equal to i c. Or rewriting it you have V c equals 1 by c integral of i c dt that is now converted into integral.

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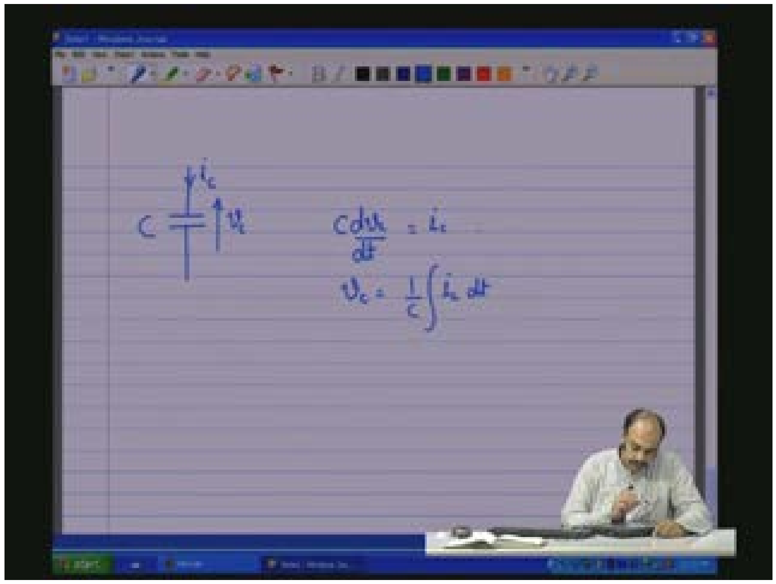

Now how to represent it in the Laplace domain; how do we go about doing that?

Now if you look at this (Refer Slide Time: 27:27) this is the dependent variable and this is the independent variable so we have a system here wherein you have i c which gets multiplied by 1 by c, passes through……… because c is a constant that passes through an integrator and what you obtain is V c.

Now to convert it into the Laplace domain to Laplace domain what do you do; we change the case of the variables to upper case; now instead of being a function of t it should become a function of s; a constant still remains a constant even in the Laplace domain and this is followed by an integrator in the Laplace domain, it becomes 1 by s and this gives you an output which is the uppercase V c as a function of s. So this would be the mapping of the time domain block or the time domain equation into the Laplace domain equation or the Laplace domain block.

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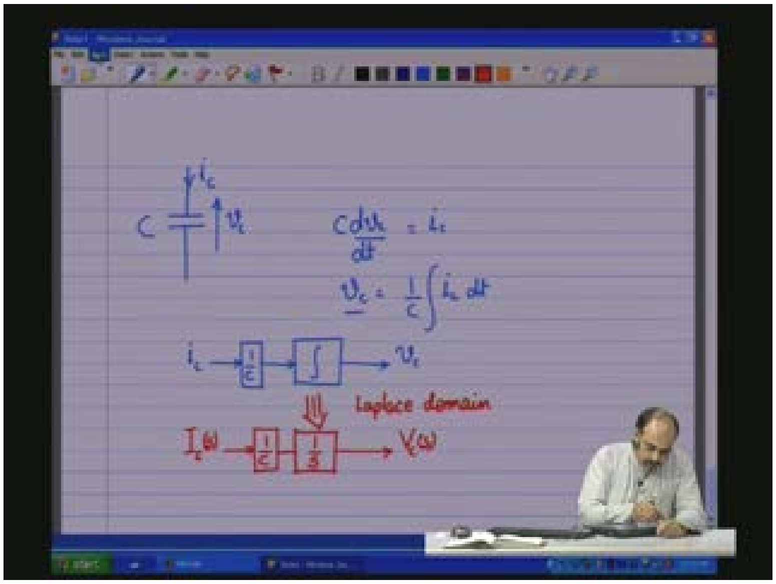

Now let us see that from this we have I c function of (s) 1 by C into 1 by S equals V c which is a function of (s). Or V c by I c (s) function of (s) is given by 1 by sC. Compare this compare with V c by I c functions of time which were equal to 1 by j omega C. This is for the sinusoidal steady-state condition. This we have obtained for the sinusoidal steady-state and this is Laplace domain or the pole-zero domain.

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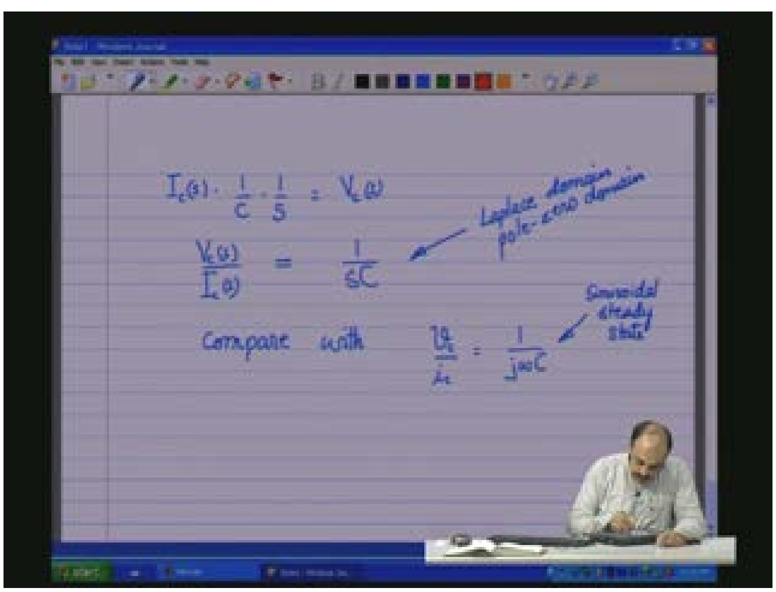

Now notice the equivalence notice the equivalence here (Refer Slide Time: 31:18). So, for the capacitive reactance the j omega of the sinusoidal steady-state is replaced with s to obtain the capacitive reactance in the Laplace domain.

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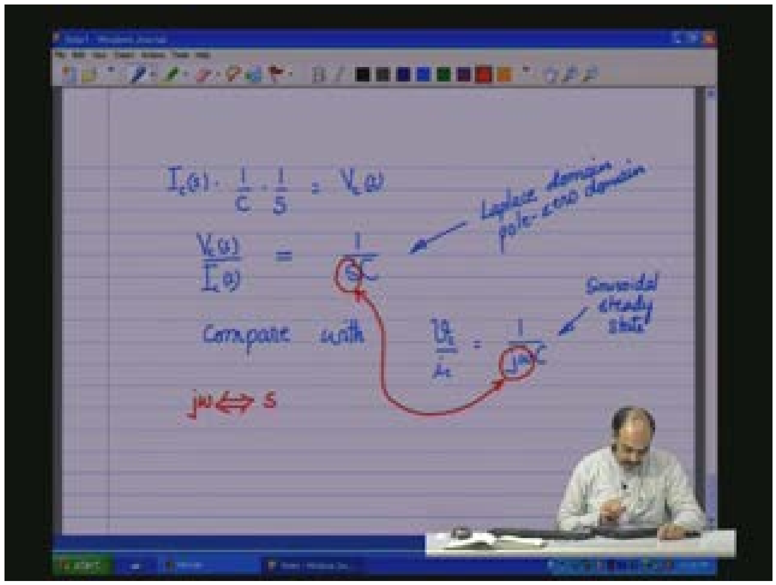

Likewise for the inductor also we have the inductor L, there is a state variable which is the current which is flowing through L, there is the voltage across L V L. Now the dynamic equation for this is L di L by dt is equal to the voltage across L which is V L. Now here the independent variable is the voltage across L and the dependent variable is i L and i L which is equal to 1 by L integral of V L dt. So we can rewrite this above equation this differential equation in this form so that we bring in integral part there and this can be written as you have a V L input of an independent variable which is scaled by L a constant and this is passed through an integrator which is going to give you i L.

Therefore, this time domain process has to be reflected as Laplace domain process by the following mapping. So we call it as V L as a function of (s), the constant parameters remain the same in all the domains, this is followed by an integrator an integrator in the Laplace domain represented by 1 by s and this results in a current in the Laplace domain denoted by I L a function of (s).

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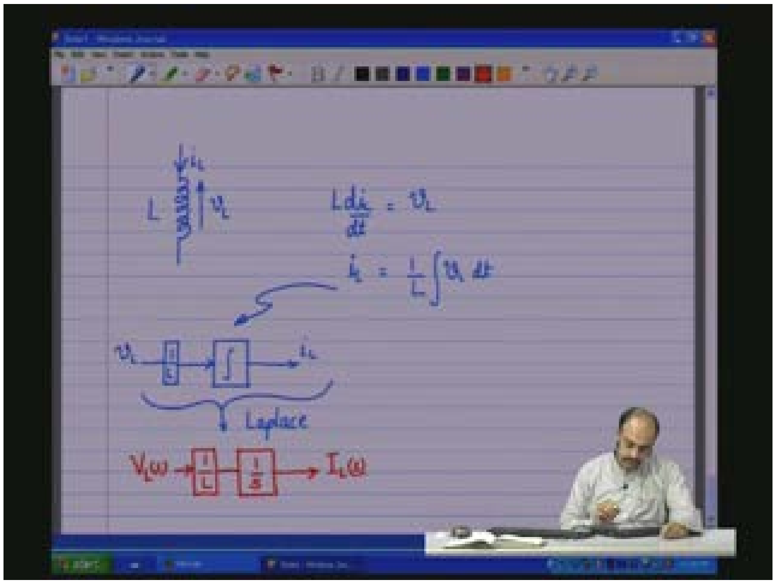

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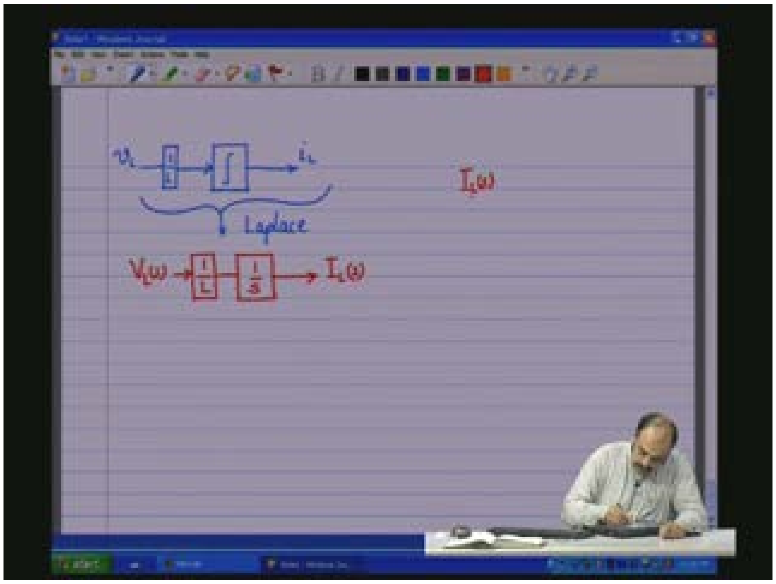

Now here let us say I L a function of (s) by V L which is also a function of (s) equals what; we want to obtain this. So we know that VL function of s into 1 by L here into 1 by s is equal to I

L(s). So this gives V L a function of (s) by I L a function of (s) which is equal to sL or if you put it in this form here: I L(s) by V L(s) equals 1 by sL. Compare this with compare i L by V L which is equal 1 by j omega L the reactance. Look at a very similar equivalence (Refer Slide Time: 35:46) that arises j omega if it is replaced by s j omega which is in the sinusoidal steady-state model if it is replaced by s then one obtains the Laplace model.

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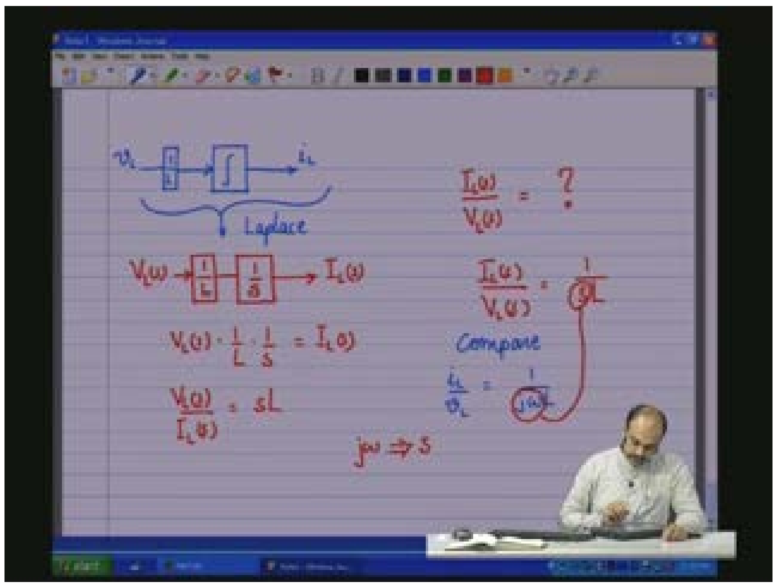

Let us have a look at a simple circuit. So if you have an RC circuit like this; this is an R C circuit (Refer Slide Time: 36:13) with V i R and C which has a V c here. Now if we want to transform it to the Laplace domain or the S-domain or the pole-zero domain the circuit topology remains similar, we have the R C so we have V i which is now a function of (s); lower case v has been converted to upper case V, R remains as R, C will result in……. here it was a reactance which is 1 by j omega C and here it will result in 1 by sC and the voltage across C is now cap V c a function of (s).

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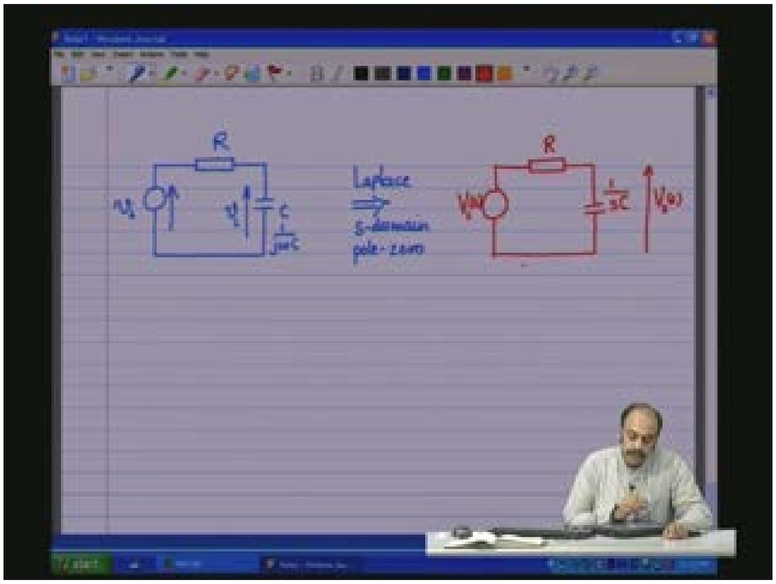

Now this is the circuit which is now transformed into the Laplace domain. And if we perform the analysis on this we will get the transfer function; for example, if we need to know let us say V 0 or V c in this case V c(s) by V i(s) this will give the relationship between the output V c and the input V i in the Laplace domain or the S-domain. So the analysis is performed in a manner similar to what is done with resistive network or in the case of sinusoidal steady-state network, so you have 1 by sC divide by R plus 1 by sC. So the voltage the voltage across the capacitance is this impedance divided by the total series impedance in the Laplace domain. Or if we simplify this you get 1 multiplying throughout by sC in the numerator and the denominator you have sRC plus 1.

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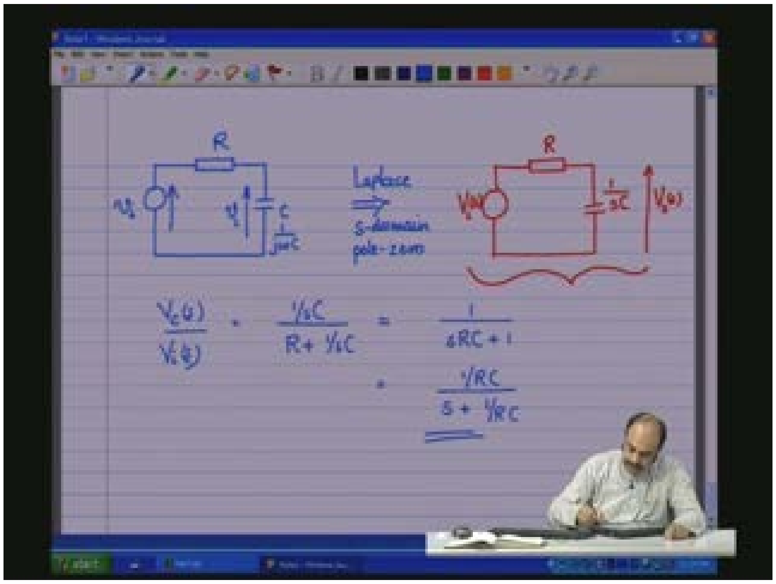

Still further simplifying we can write it as: 1 by RC divided by s plus 1 by RC. So this is the transfer function for this network, for this RC network. What does this mean or what does this imply?

We have V c a function of (s) divided by V i a function of (s) which is equal to 1 by RC divided by S plus 1 by RC. So this is the transfer function between the input and the output that is the transfer function of the output variable divided by the input variable is this. It has a numerator polynomial, there is no S term in the numerator polynomial so there are no zeros; there is in the denominator polynomial an S term that is there is S to the power of 1, this has a route what is the route? At S equal to minus 1 by RC at S equal to minus 1 by RC if we substitute here this will go to infinity so the gain goes to infinity and that is a pole because the routes of all the denominator terms will be poles, the routes of the numerator terms will be zeros there is no s term in the numerator so no zeros here for this circuit and you have the only one pole which is located at S is equal to minus 1 by RC which means if I now draw the S-plane so I have S-plane, you have the sigma, you have the omega so this is the S-plane.

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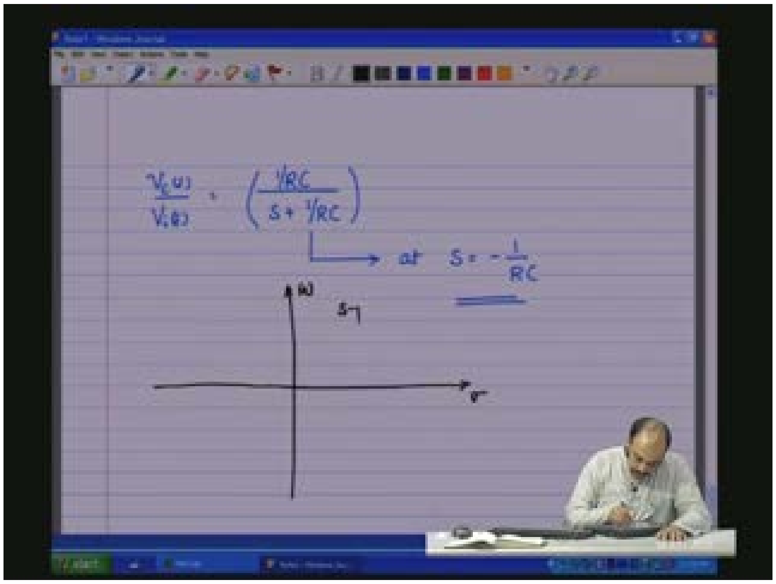

Now S is equal to minus 1 by RC therefore this is 0, I take the negative real axis somewhere here (Refer Slide Time: 41:39) let us say we have minus 1 by RC so that is a pole location. So this will be the pole-zero picture of the RC network. So what does it mean; it means that when I have any disturbance or when I have any particular excitation and that excitation is removed that excitation is going to decay at minus 1 by RC into t or e to the power of minus t by RC. So it is going to decay with a time constant of RC that is what it means when you talk of a pole; it means when I say I have a pole located at minus 1 by RC it means that it is going to decay with the time constant of RC.

If this pole starts coming closer to zero then what is happening to RC; RC is becoming higher and higher till when it starts becoming close to infinity this 1 by RC is at 0 which means that as it starts coming to zero the pole the decaying action becomes slower and slower because RC time constant becomes larger and larger, time becoming larger means it becomes slower and slower until at this point there is no decay which is integral action so that is the integral. And as this pole starts moving away from the omega axis the RC becomes smaller and smaller which means that the time constant is fast which means that any excitation if after it is removed will decay very quickly so that is basically the meaning of the pole location there. There is also another physical notion of the pole which we will again discuss as we progress further in this modelling.

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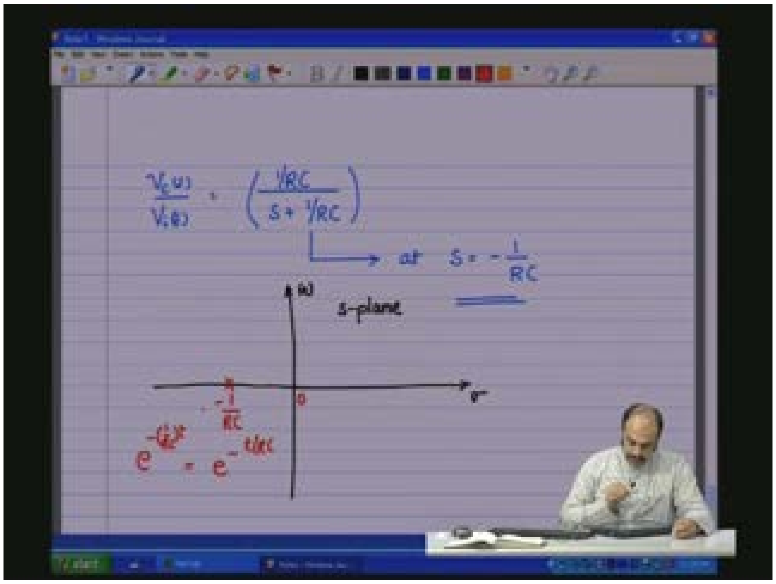

Likewise if we take the case of the inductor circuit let us say we have a source, we have a resistance R and an inductance L so we have an inductance L, a resistance R, we have a V i and then there is a state variable i L which is here.

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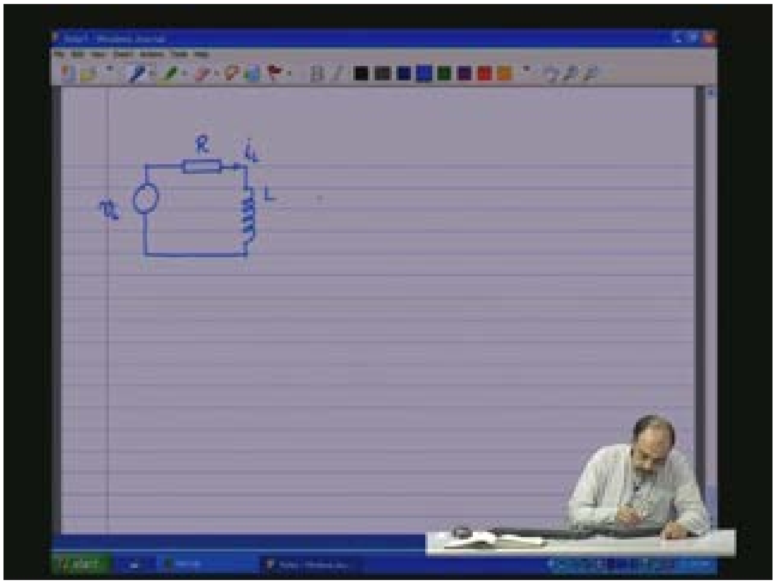

Now if this has to be shifted to the S-domain or the Laplace domain we will now have the circuit topologically very similar to what is there in the time domain; only what is going to change is the way you will represent the variables. So it is V i now it is S, R still remains as R, now the inductive reactance what was given here is normally j omega L in the case of a sinusoidal steady-state the inductive reactance and here it becomes sL.

Now if you want to find let us say the current through the circuit i L and you would like to see what is i L by V i(s) now that is nothing but voltage divided by whatever the impedance which is R plus sL so the voltage divided by…………. this is the system which gives you if I rearrange the terms 1 by L divided by S plus R by L.

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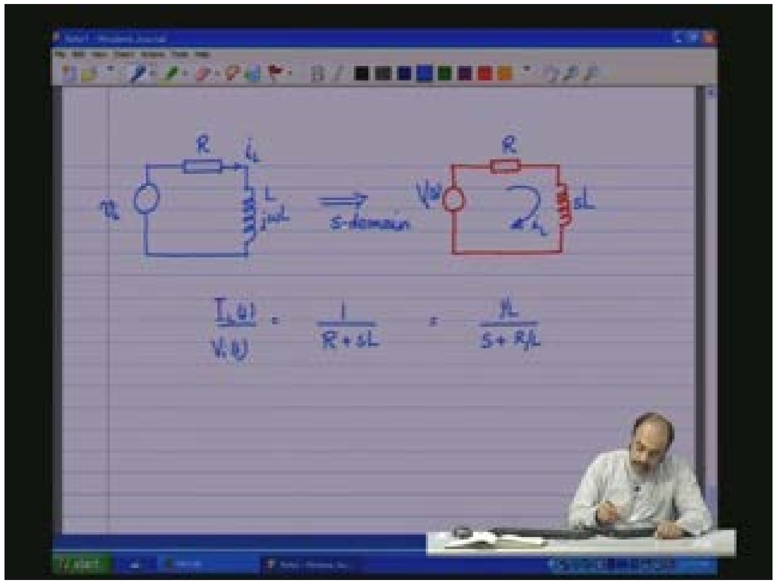

Here also notice that there is only a scaling factor in the numerator, there is no function of S in the numerator and therefore there are no zeros. In the denominator there is a pole because there is a first-order S there coming into the picture and the pole is at S equal minus R by L minus R by L. So what it basically means is that if I have the S-plane sigma axis, the omega axis this is the 0 S is equal to minus R by L again the minus part of the real axis, I have a pole at some point here with this minus R by L. This again means that any excitation after it is removed will decay at minus R by L t.

Or you take the time constant here which is t by L by R, L by R is the time constant with which this is going to decay in the case of the RL circuit. And here again as the pole starts moving closer to zero it means that either R is becoming 0 or L is becoming large in effect L by R is becoming large which means the time is expanding becoming large and takes much longer time to decay till the point at this point zero where either L is infinite or R is 0 this point there is no decay and as you start going away from omega L the L by R time constant starts decreasing and its starts decaying faster and faster so that is basically the meaning of that.

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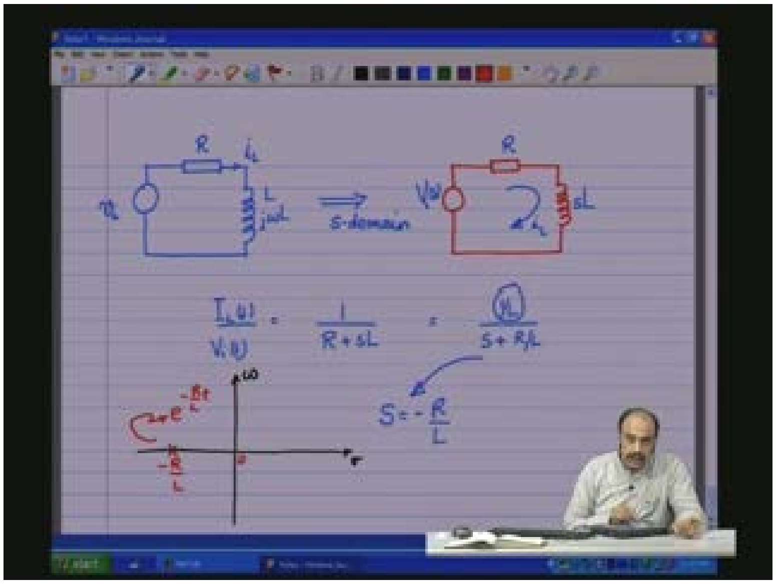

So if you take any arbitrary circuit, let us take the RLC circuit; you have an R, you have a L followed by let us say a C followed by a C. So you have V i so R, a L and a C and you have V c. Now this you have to transform to the S-domain or the pole-zero domain and to do that we write this circuit again in the same fashion, we have a C and this is……….. so the input V i becomes V i(s) a function of (s), the output is V c a function of (s) this is the input, R remains as R, the inductive reactance in the Laplace domain is sL, the capacitive reactance in the Laplace domain is sC and you have your circuit in the Laplace domain. So total let us say the input output relationship that is the gain between the output and the input which is V c by V i(s) the Laplace transform of the output V c by the Laplace transform of the input V i which is going to give you the transfer function which is here 1 by sC and that is V i into 1 by sC divided by all the elements in Cs 1 by sC this is like a normal analysis that you would do with resistive network or sinusoidal steady-state quantity network and this is going to result in if I make a slight simplification 1 divided by s square LC plus sRC plus 1.

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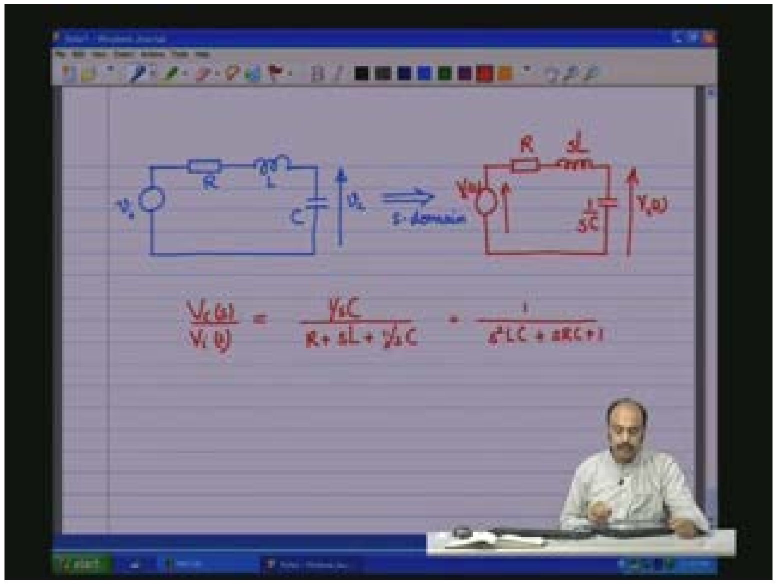

So here you see again no zeros. But in the denominator the s is of order 2 so this is a polynomial of order 2 in s therefore this will have two roots, you will have roots which means two poles, there are no zeros but two poles. So the order of s will indicate the number of roots and therefore number of poles for the system and if you write the S-plane omega so there can be two poles but of course the location of the poles can vary, it can be two poles on the real axis or it could be two complex poles. But if there is a pole on the complex pole which is not on the real axis there should always be a mirror image of the pole there to indicate that there are L and C components which are 180 degrees out of phase. So this will give you a location of, let us say a general location of a pole.

Now here (Refer Slide Time: 52:45) this projection on the real axis will give you the decaying action, will give you an idea about decaying action provided there is some excitation and the excitation is removed and this will give you a notion of the frequency at which it will oscillate. So let us say there is an excitation or disturbance which is given and taken out, this will start oscillating but the oscillations will decay. So this decay of these oscillations is governed by this projection value here and these oscillation frequencies are governed by……… this oscillation frequency is governed by this projection on the vertical axis of the S-plane. So this is a typical this is a typical waveform nature of the RLC circuit and this is some this is called an under damped circuit. Depending upon the positions of the poles there one can get fully damped that is no oscillations to very large damp very large under damped oscillations based on the locations of the poles here which of course we will see in a bit later.

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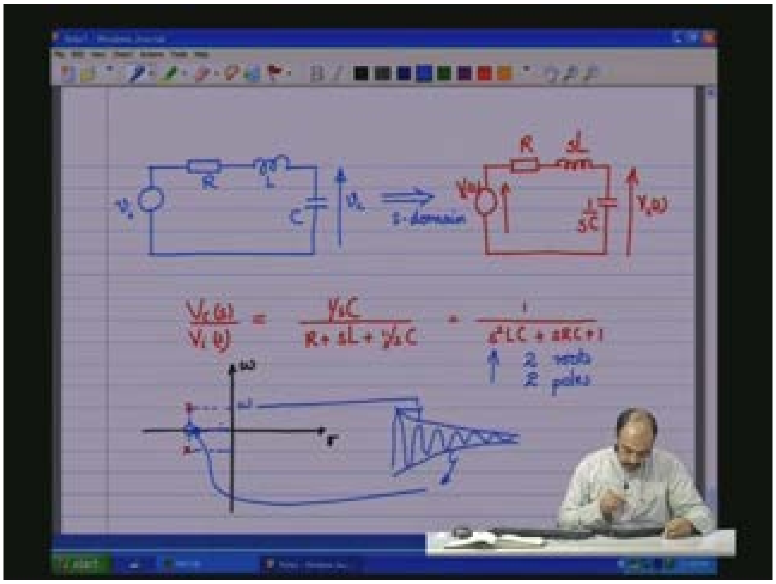

Now what is important to note at this point here is that state equation is a complete description of the system along with the dynamic behaviour and out of which under some constraints we have shown that when d by dt is replaced by j omega, if the inputs of sinusoidal then we can get the sinusoidal steady-state equation from the state equation. Likewise for the transfer function because transfer function is under the constraint that there are no initial conditions because state space has no such restriction, so a transfer function should also be should also be obtainable from the basic state equation.

So the state equation in the general form you see that it is dx by dt plus sorry equals Ax plus Bu this is dynamic equation and the output equation is given by Cx plus Du. So if there are more than one state x is a vector, A is the matrix, Y can also be a vector and u can also be a vector for multiple inputs. So this is valid for multiple inputs, multiple outputs and for any number of state of the system and for the case of the transfer function it is valid only for single input single output.

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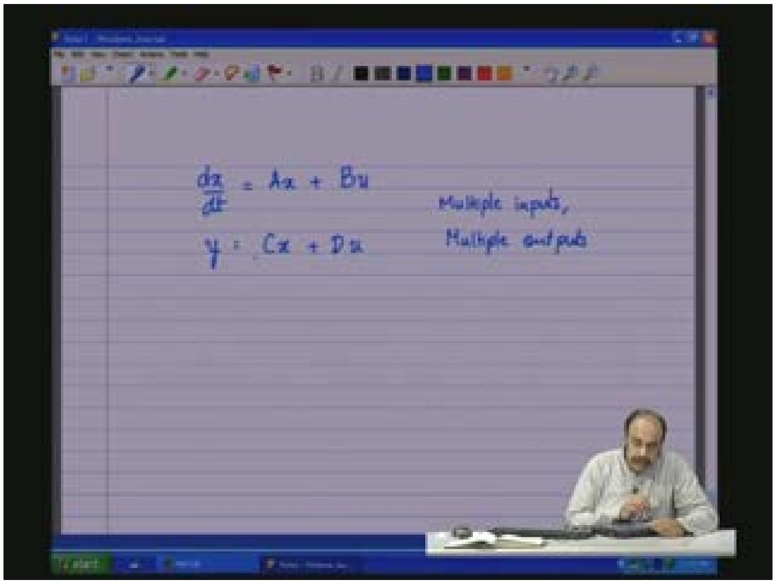

Now how do we obtain the state equation for the case of from the for the case of any circuit where you have the equations given like this. So the transfer function is obtained in the following manner.

Taking this dynamic equation we have s X(s) because when we say integral is replaceable by 1 by s d by dt is replaceable by s when the initial conditions are zero. So you can say: s X(s) is equal to A x(s) plus B u(s) or sI minus A if it is matrix form X(s) is equal to B u(s) or X(s) will be SI minus A inverse B u(s). Now if you consider the output equation: Y(s) equals C for X(s) we will substitute this that is sI minus A inverse B u(s) plus D u(s) and we want to find the transfer function of the output by the input the output the Laplace transform of the output by the Laplace transform of the input which is C sI minus A inverse B plus D. So this is the transfer function (Refer Slide Time: 58:21) obtainable from a state equation which has matrix A B C D.

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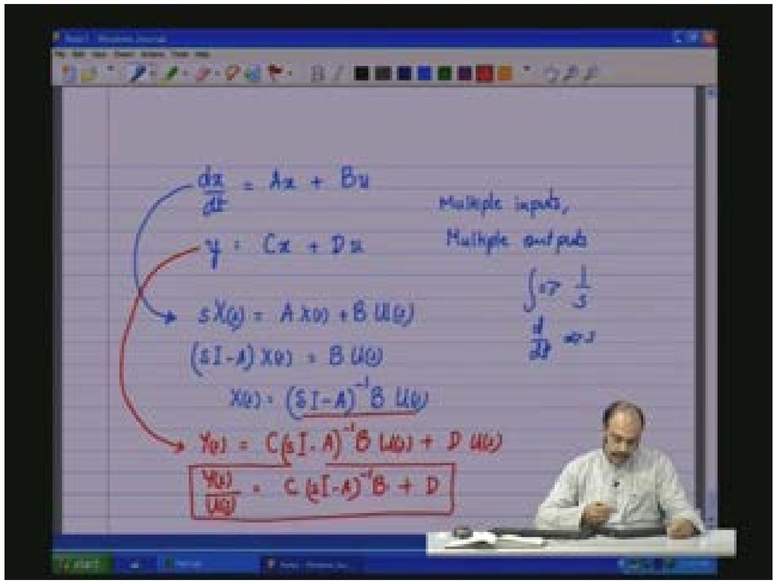

So given A B C D matrices sure can obtain the transfer function also from the state equation and once the transfer function is known, the roots of the denominator polynomial or the pole locations in the S-plane, the roots of the numerator polynomial or the zero locations in the S-plane and they give an idea about the damping properties, the decay properties of the system for any excitations and disturbances.

You will of course be using the transfer function and the state equations both together interchangeably at various points of time in the discussion of the analysis of the circuits in future also. So there will be lot of usage of these transfer functions in the future discussions.