Computer Aided Power System Analysis
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Lecture - 31
LU Decomposition

Hello, welcome to this lecture on the course on computer aided power system analysis. In the last lecture we have talked about the basic principle of the LU decomposition. So in this lecture we would be looking into the detail procedure of the LU decomposition, followed by an example. So let us start. So what we have is, we said that if we have got an matrix A, so we are talking about LU decomposition.

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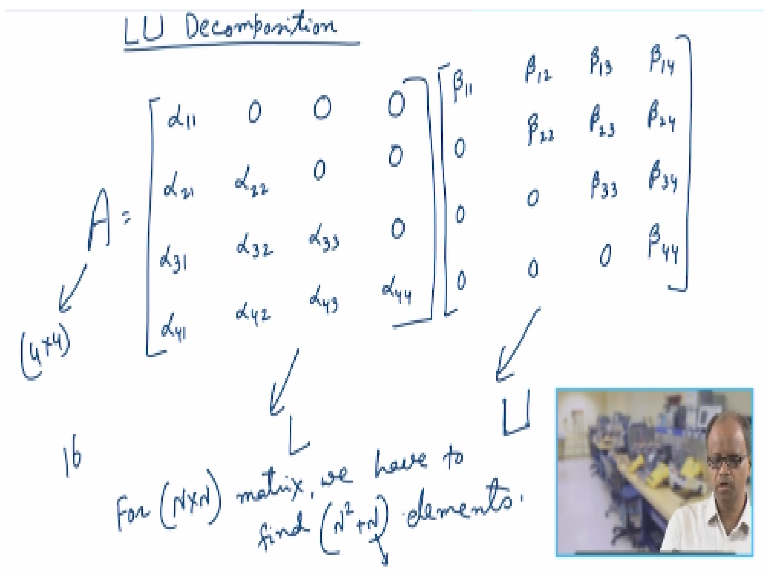

Let us say we have a matrix A 4 * 4 matrix that we would like to and lower triangular and an upper triangular matrix. Then we have got, so this is a standard lower triangular matrix and then we have an upper triangular matrix where we would have, then we will have 0, then beta then 0, 0, 0, 0, 0 beta 44. So I have given a matrix. We would like to decompose this given matrix into this two lower and upper triangular matrix.

Obviously, this is my L matrix, this is my U matrix. So then here because we have taken the example of 4 * 4 matrix, if it is 4 * 4 matrix so then therefore total number of elements is 16. But then here how many elements we have got? I mean we have actually have to compute, we have to compute here 1, 2, 3, 4, 5 10 + 10 that is 20. So then therefore for an N * N matrix, so for an N * N matrix we have to solve, find rather N 2 + N elements.

So this is pretty apparent from here because if you see we have already got this field. And then in this 6, we can put this 6 and then we are only remained with this, this, this and this. So then therefore this 10 plus this 6, 16 + 1, 2, 3, 4 20. So essentially what is happening? That this extra N terms is coming because we have got in both this matrices L and U we have got this elements at the main diagonal to be computed.

So now because I have got only 16 known quantities, 16 known quantities are coming from the 16 elements of this matrix A. So essentially in general we have got actually N square known quantities but then we have to compute N square + N unknown quantities. So then therefore we have to pre-specify N quantities. So usually what we do?

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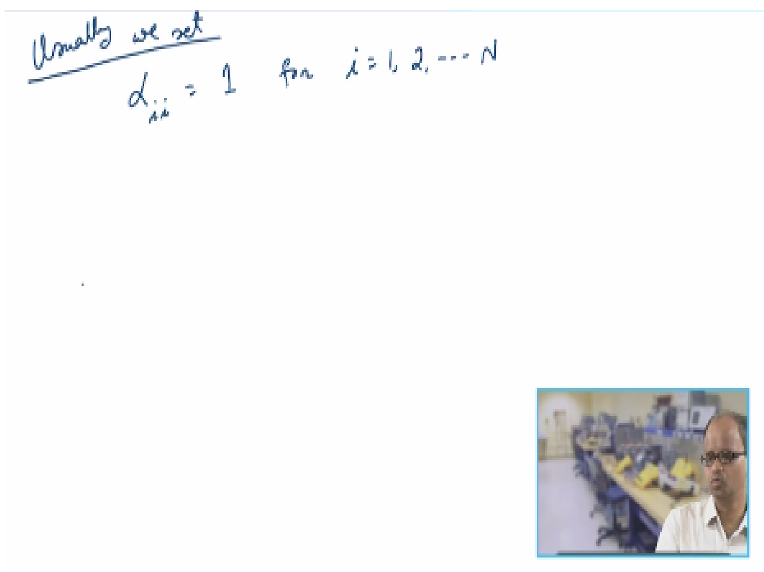

Usually we set alpha ii = 1 for i = 1, 2 to N. Usually we set alpha ii = 1 for i = 1 to N. So essentially what is meaning that we simply set all this diagonal elements of the lower triangular matrix to be equal to 1. And once we set all the diagonal elements of the lower triangular matrix equal to 1 so then therefore after that we are only left with N square unknown quantities and also we have got N square known quantity.

So then hopefully we would be able to solve this. Now to solve this, instead of trying to solve this by any mathematical process or rather by any kind of closed form expression what we usually follow is something called Crout's algorithm. So we now would simply like to look at that what this Crout's algorithm is. So this algorithm is a very beautiful algorithm and very easy to code.

So then for any N * N matrix if we would simply apply this Crout's algorithm so then we would be able to find out the L and U matrix for a given A matrix. So then now let us look at that what is Crout's algorithm.

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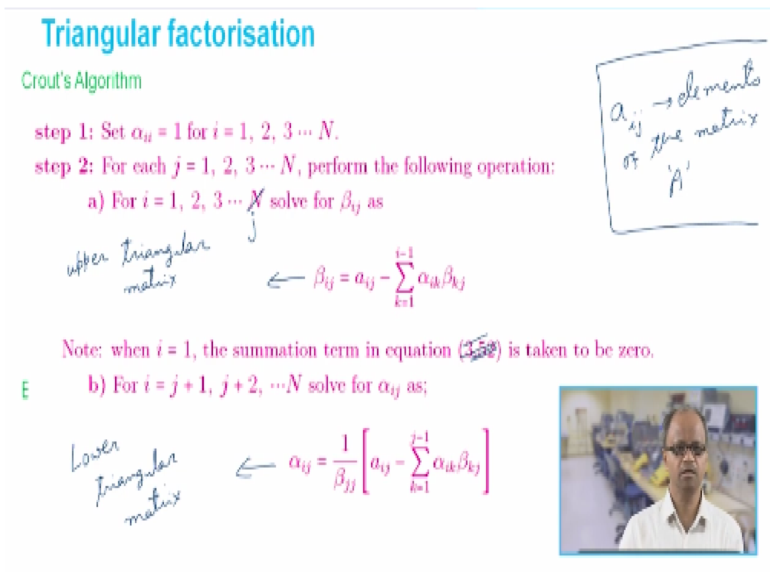

So this Crout's algorithm, so this is triangular factorization. Here we are actually talking about this LU factorization. So then in this Crout's algorithm what we do in the first step? So then essentially here, so in this algorithm essentially we are given this matrix A which is already given. So in the first step we set alpha A = 1 for i = 1, 2, 3 is equal to N. we have just now discussed the rationality behind them.

Then for each j 1, 2, 3 is equal to N perform the following operation. Now here in this there a little mistake. For i = 1, 2, 3 to N it should be actually j, it is not N, it should be actually j. So for i = 1, 2, 3, j solve for beta ij as by this expression. And then here note when i = 1 this particular summation in the equation I mean in this particular equation in this equation so basically 3.52 is essentially this equation.

And this equation is essentially taken to be 0. Because u see in this case what will happen? When i = 1 so then therefore this particular index will go from k = 1 to 0. Going from k = 1 to 0 does not have any meaning. So then therefore in that case this particular summation would be 0. So then in that case essentially beta ij would be equal to a ij. Please note that this a ij are the elements of the matrix A.

So this a ij are the elements of this matrix j and we are trying to solve for alpha and beta. And after that once we are done with this then after that i = j + 1 to j + 2 to N we solve for alpha ij. So then here first we are solving for this beta ij. Beta ij is nothing but the corresponding to this upper triangular matrix. So this corresponds to upper triangular matrix. So this is upper triangular matrix.

And this corresponds to lower triangular matrix. So after that what happens? That we simply keep on doing these two steps one by one till we reach j = 1, 2, 3 to N. Now from this particular algorithm nothing is really very much clear. So then let us look at an example to understand this algorithm in much more detail.

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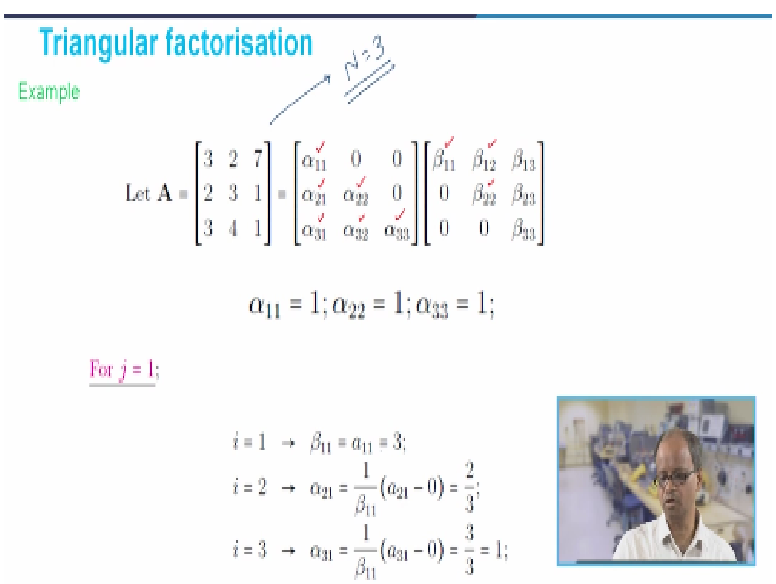

So let us take an example. So now just for the sake of convenience we have taken an matrix A which is basically 3 * 3 matrix. So this is the matrix 3, 2, 7; 2, 3, 1; 3, 4, 1 and we would like to decompose this matrix A into a matrix L and into a matrix U. So this is nothing but matrix L and this is nothing but matrix U. So in the first step as we have said that we would first alpha i = 1 for 1 = 1, 2, 3, N.

So here in this case N = 3. So in this case N = 3. So then therefore in the first case we simply set alpha 11 = 1; alpha 22 = 1; alpha 33 = 1. So by this algorithm alpha 11 = 1; 23 = 1. So now here for each j = 1, 2, 3 to N. So then therefore j will vary from 1, 2, and 3. So for each j 1, for each j 2, for each j = 3 we have to follow these two steps. First this step and secondly this step. So when j = 1 so what we will do? So now here j = 1.

So then i will now actually vary from 1, 2, 3, to j and now so then therefore j = 1. So then therefore when j = 1 i will only have one value that is 1. So then therefore it would be beta 11; i = 1, j = 1. So it would be beta 11. That would be = a 11 minus now because i = 1 and we have just now said that when i = 1 this term = 0. So then therefore beta 11 would be = a 11. So this is exactly the case. So beta 11 would be = a 11.

So this is a 11 and that is nothing but is = 3. So this is a 11, this is = 3. Now, so after we are finished with this, now from here i will vary from j + 1 to j + 2 to N. So then therefore because N = 3 and here j = 1 so then therefore i will now take 2 values; i will not take values of 2 and 3. So in this step i will not take values from 2 and 3. So the first case, so it would be alpha 2 i; i = 2. So when first i = 2.

So then alpha 2 1 I mean here i = 2 and j = 1. So alpha 21 would be = 1/beta 11. Please note that here in this case j = 1. So it is 1/beta 11. We have just now computed beta 11 from here. It would be = a 21 because i = 2 and j = 1; a 21 minus now here j = 1. So then here also this index k will vary from 1 to 0 which does not have any meaning. So then therefore this particular summation would be = 0.

So then therefore alpha 21 would be = 1/beta 11 * a 21. So it would be, so basically that is what it is. Alpha 21 = 1/beta 11 * a 21. Beta 11 = 3 you have just now calculated; a 21 is 2 so it is 2/3. Come here, so after this i now will take a value of 3. So when i will now take a value of 3 so then therefore this would be alpha 31 because i = 3 and j = 1. So alpha 31 that is would be = 1/beta 11 because j is still 1 * a. a is nothing but the element of the given matrix A.

This is a 31 minus now here still j = 1. So then therefore this summation does not have any meaning because this index will go from 1 to 0. So this summation does not have any meaning. So then it would be 1/beta 11 * a 31. So that is what precisely it is. So alpha 31 would be = 1/beta 11 * a 31 and a 31 is 3 and beta 11 is 3. So then it is = 1. So then when we finish j = 1 we have computed this quantity beta 11.

And then after that we have computed a 21 and a 31. Please note that alpha 11, alpha 22 and alpha 33 they are already calculated from here and in the first step we have calculated this and this and this. After this, so now so then we have finished the computation for j = 1. So when we have finished this computation for j = 1 after that j will take a value of 2. Now when j will take a value of 2 now i will vary from 1 to 2 because i will vary from 1, 2, 3 to 2.

So then i will take now 2 values 1 and 2. So now, so in the first case what we will have i = 1 and j = 2. So what we will have here? Beta 12 because i = 1and j = 2. So it is beta 12 = a 12 minus because i = 1 so then therefore again this particular summation does not have any meaning. So then therefore beta 12 would be = a 12.

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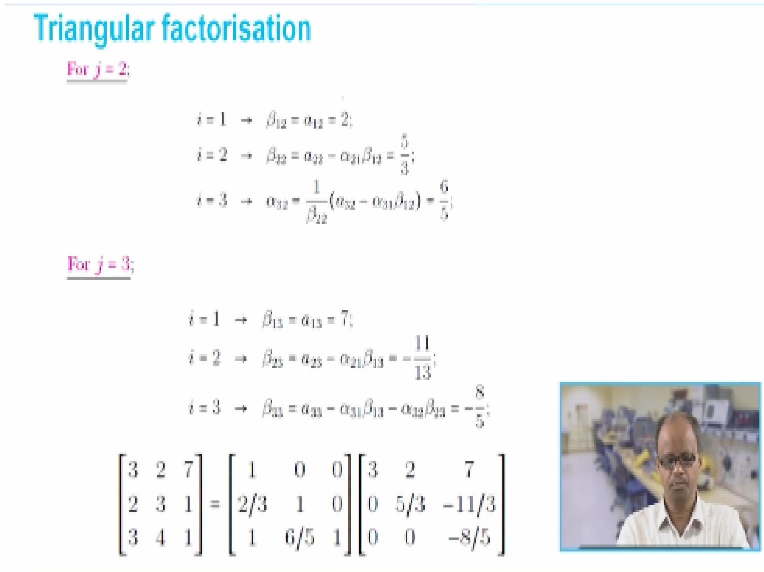

So when j = 2 so beta 12 would be = a 12 and a 12 if we look at this a 12 is 2 so then therefore beta 12 = a 12 that is = 2 fine. After this what happens? Now here j will, here i will take a value of 2. Please note that when j = 1, we will first finish this part then we will finish this part, right? so when j = 2 i will have to take 2 values 1 and 2. So now here i is first we have considered when i is taking a value of 1.

So now we have to consider the case when i is taking a value of 2. So now when i = 2 and j = 2 what we have got beta 22 = a 22. Please note here i = 2 and j = 2. So beta 22 = a 22 minus now i = 2. So then therefore this index would vary from k = 1 to 1. So then therefore this index k will have only one value that is k. So it will be = alpha 21 * beta 12. Please note here in this case i = 2, j = 2.

So then therefore a 22 – alpha 21 * beta 12. So let us see. So beta 22 would be = a 22 – alpha 21 * beta 12 and we have already computed beta 12 here a 22 is already known. And alpha 21 we have calculated in the last step. So if we put them alpha 21 = 2/3 and beta 12 = 2 and a 22 is 3. So if we put all these things then we get it is 5/3, right? So it would be 5/3. So a 22 is 3 and it is 2. So it is 5/3. So 3 – 4/3 so it is 5/3.

So now when i has taken now the values of 1, 2, 2. Now we come to this step. So when we come to this step, i is now would be taking a value from j + 1 to N. Now N is 3 and j

= 2. So then i will vary from 3, 2, 3. So then or in other words i will only have one value that is 3. So when i will have only one value that is = 3 so then in that case what will happen is so now we will have alpha 32. Please note that alpha sorry i = 3.

Here i = 3 and j = 2. So alpha 32 would be = 1/beta 22. Please note that we are talking about here j = 2. So it would be 1/beta 22 * a 32; i = 3 and j = 2 * a 32 minus now j varies now j = 2. Now because j = 2 so then therefore k will now vary from here 1, 2, 1; j – 1 is 1. So k would be varying from 1, 2, 1. So it would be alpha 31 * beta 12. So here what will happen? Ultimately, finally so then what is the expression?

Alpha i = 3, so i = 3, j = 2. So alpha 32 = 1/beta 22 * a 32 – alpha 31 * beta 12. So this is what let us see. So alpha 32 = 1/beta 22 * a 32 – alpha 31 * beta 12. Beta 22 we have just now calculated; a 32 is known; alpha 31 is known from here and beta 12 is known from here. So all these values are known. So we substitute all these values and we get a value of, numerical value of 6/5. So now because N = 3 so now j takes a value of 3.

So when j takes a value of 3 so i varies from 1, 2 and 3, right? now first we have calculated this, then this, then this. In the second step we have calculated beta 11 this, then alpha 21 this then alpha 31 this. Then when j = 2 we calculate beta 12 and beta 22. So we calculate beta 12 and beta 22. And also we calculate alpha 32. So L matrix is already done. So we have to now calculate this column.

So if we look at this, we actually here in this case we are actually going from column wise. So first in the case of L matrix, first we make all this diagonal elements = 1 and then after we calculate the first element for the beta matrix sorry for the U matrix then we compute the first column elements. Then in the next step when j = 2 we solve for this column. And once we solve for this column then we solve for the remaining element in the second column, right?

So then therefore when j would be = 3 so then therefore we will now only solve for this column. So essentially it goes by column wise. So first so when we start we simply start this we simply set this, all these elements of this L matrix to be equal to one unity. Then we start with the first column of the U matrix. Then we come back to the elements of the first column of the L matrix. That is basically when j = 1.

First we compute the elements of the first column of the U matrix. Then we compute the elements of the first column of the L matrix which have not been computed. When j = 2, first we compute the elements of the second column of the U matrix and then we calculate the uncalculated elements of the second column of the L matrix.

So when j = 3 we will calculate the elements of the third column of the U matrix and then here we will come back again to the third column of the L matrix but in the third column of the L matrix which is already known so there is nothing to be calculated. So then by this we can easily simply, so I mean this is basically the case of visualization of the entire process.

So first let us compute this calculation then we will try to I mean make it more general. So let us go back what we were discussing. So when j = 3 so then i would be taking the values of 1, 2, and 3. So first when i = 1 so it would be beta 13. That would be = a 13 – i = 1. So then therefore this particular summation does not have any meaning. So then therefore beta 13 would be = a 13. So let us look at, so beta 13 = a 13 that is = 7.

So a 13 = 7 that is = 7. Then i takes the value of 2. So it would be beta 23 = a 23 – now i = 2. So then therefore k will now have a value of 1 only. So because k would be only varying from 1, 2, 1. So then therefore what we will do? So then what we will have here? Alpha 21 * beta 13. So it would be, so then what would be the entire expression? Beta 23 = a 23 – alpha 21 * beta 13.

So this is what exactly is beta 23 = a 23 – alpha 21 * beta 13. So here we have already solved for everything; a 23 is known. Alpha 21 we have just now calculated. Alpha 21 we have calculated in the first step. So alpha 21 is calculated. Beta 13 we have just now calculated. So if we just put this we get this value. And in the last case when i = 3 so it would be beta 33 = a 33. Now k would now have 2 values 1 and 2.

Because i = 3 so this is 3 – 1 is 2. So now k would now take 2 values 1 and 2. So it would be a 33 – alpha 31 * beta 13 – alpha 32 * beta 23. So then it would be a 33 – alpha 31 * beta 13 – alpha 32 * beta 23. Now here alpha 31 we have already solved. Alpha 31 we have already solved from here. Beta 13 we have solved from here. Alpha 32 we have solved from here. Beta 23 we have solved from here. So we get this.

So all the elements are calculated. So then therefore my so then this is the, this is basically the result. So this is L matrix, this is the U matrix. So then therefore if we do this I mean generalization. So I mean how do I now basically visualize this process?

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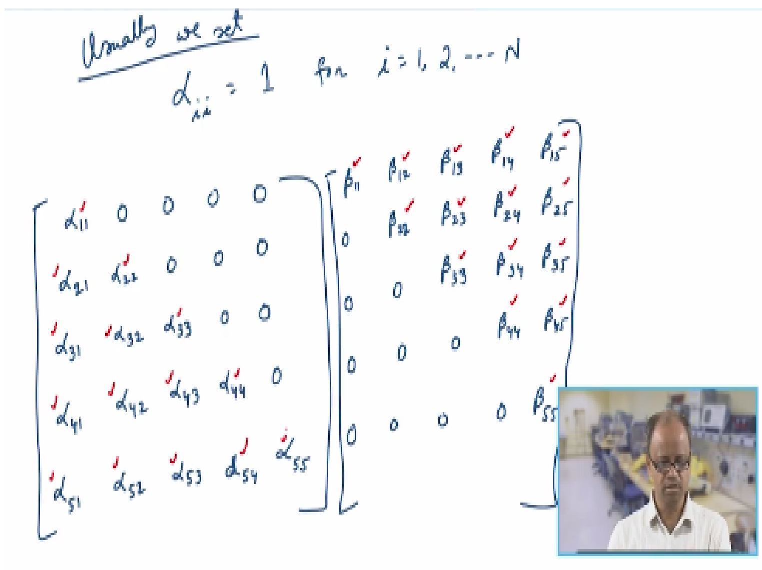

Let us say that I do have just for the sake of example let us say that I am taking a 5 * 5 just for a sake of example. So if we have a 5 * 5 so how would it go? Alpha 41, 42, 43, 44 0. Alpha 51, alpha 52, 53, 54 55. So this is the L matrix we have compute and multiplied by this beta matrix. So it is beta 11 25 sorry beta 22. Then beta sorry it will be 0, 0, 0, beta 44, beta 45. It would be 0, 0, 0, 0, beta 55.

So this is the U matrix. So now by the Crout's algorithm how it would do? We will first set this, all this alpha 1, alpha 22, alpha 33, alpha 44, alpha 55 to be = 1. So we will first set this to be = 1. Then what we will do? Then we will take j = 1. So when we will take j = 1we will first compute this one.

Once we compute this one then in the second step of this Crout's algorithm that is for j = 1 please note that for each and every value of j there are actually 2 steps. So in the first step of corresponding to j = 1 we first calculate this value and in the second step we calculate this values. Then when j = 2, so in the first step we calculate these values and then in the second step corresponding to j = 2 we calculate these values.

When j = 3 in the first step of corresponding to j = 3 we calculate this column because these are 0. So they do not have to be calculated and we calculate these values. So we are only basically moving column to column. First the column of this U matrix followed by the column of the L matrix. Then again column of the U matrix followed by the column of the L matrix.

Then again column of the U matrix followed by the column of the L matrix. Similarly, for j = 4 we calculate these values. And then calculate these values. And for corresponding j = 5 we calculate this column and then and this is already calculated. So we move from basically column wise. So first this column, then this, then this, then this, then this, then this.

So we can see that this is a very elegant algorithm and it has got it's the very simple pattern and if we and when we have looked into this algorithm it is very easy to code. So then, by this given any matrix A so we can by applying Crout's algorithm we can simply decompose them into L and U and then we have seen that how to calculate the final solution utilizing this L and U matrices.

So in this process of triangular factorization we really do not have to do any explicit inverse of the matrix A. So by this we finish our discussion on the numerical methods for solving a set of linear equations. From the next lecture onwards we will start looking at some other aspects. Thank you.