Computer Aided Power System Analysis
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Lecture - 29
Gaussian Elimination and Optimal Ordering

Hello, welcome to this lecture on the course computer aided power system analysis. In the last lecture we have looked into one example of Gaussian elimination method for solving a set of linear simultaneous equations. We have seen that, that this method is quite easy to program and is also very powerful and it also does not need any kind of explicit matrix inversion.

So then therefore this method is quite computationally handy as we do not have to inverse a matrix. However, in this process of Gaussian elimination the sparsity of this matrix actually gets lost. Now when we have started our discussion regarding the solution of our power flow equation involving the Jacobian matrices unknown quantities.

And the mismatch vectors, we have noted that this Jacobian matrix is actually quite sparse and then we said that because we have to invert this Jacobian matrix which is quite sparse so it require a lot of computational work and also as the inverse of a sparse matrix is not necessarily a sparse matrix itself so then therefore we are actually losing the advantage of handling a sparse matrix.

So as a result we have started looking into the alternative methods of solving a set of linear equations and in that course of action we have looked into this Gaussian elimination method which really does not require any explicit matrix inversion. But in this process as we will just today we will discuss that in this process we are often bound to lose the advantage of the sparsity. So now let us look at this that why it is so.

Now if we look at this, so now let us look at one initial matrix. So in this matrix at the left hand side this crosses are nothing but this nonzero quantities.

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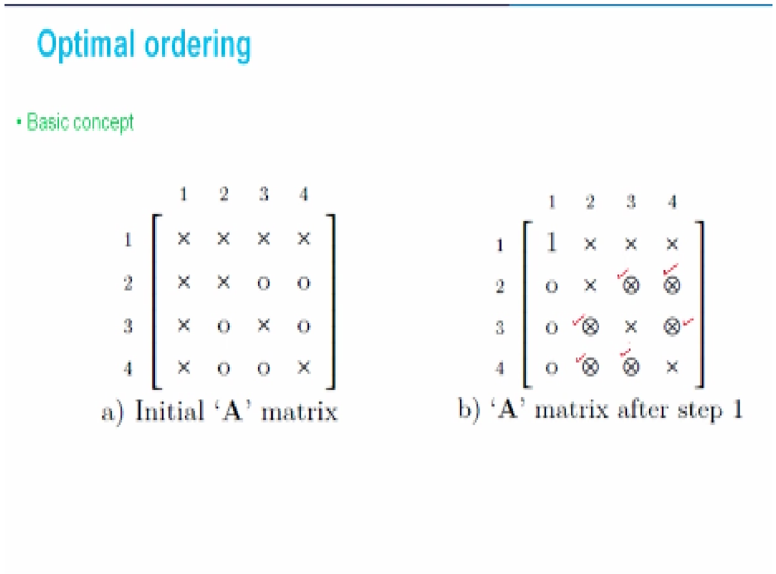

And the small circles are basically nothing but the 0. So then therefore if this matrix we are looking into so then therefore what we get is, is something like this.

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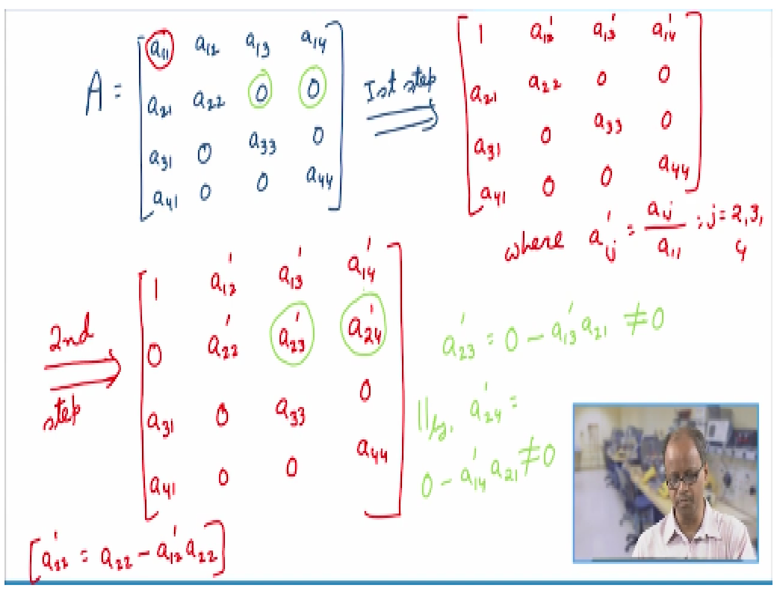

So we have got matrix A, A is given by a 11, a 12, a 13, a 14 and then we will have a 21, a 22. Then we will have a 21, a 22. These two are 0. Then we have a 31 and a 33. Then we have a 31, 0, a 33, 0. And then we have a 41, a 44. And we have a 41, 0, 0, a 44. So this is the original matrix. Now if we wish to do the Gaussian elimination. So our first step is, so our first step would be, first step. So after first step what we do?

We will simply divide this first row by a 1. So we will take this a 11 as the pivot. So we take a 11 as pivot and simply divide the first row by this pivot. So after this first step what we get is, we get something like 1. Then let us say a 12 dash, a 13 dash, a 14 dash. And this becomes a 21 and these two are the remaining rows in the first step remain unchanged. So remaining rows in the first step remain unchanged.

Where a 1j dash is actually a 1j/a 11 for j = 2, 3, 4. So now if we do the second step, so in the second step what we do? In the second step what we do? We will do multiply this first row by a 21 and then simply subtract this row from this row. So then therefore what we get is in the second step we get 1, a 12 dash, a 13 dash, a 14 dash and then this becomes 0 because it would be simply this minus this.

Then this becomes a 22 dash and where a 22 dash would be where a 22 dash is nothing but a 22 minus a 12 dash * a 22, right? So this is a 22 dash. This would be a 23 dash and this would be a 24 dash and the rest of the let us say we are not considering now this elimination of this matrices sorry these elements. So this let us keep this as it is 0, 0, a 44. Now what is a 22 dash and now actually what is a 23 dash and a 24 dash?

A 23 dash would be a 23 dash would be 0 – a 13 dash * a 21. So those would be a nonzero quantity. Similarly, a 24 dash would be 0 – a 14 dash * a 21. It is also a nonzero quantity. So then what does it mean? It means that in the original matrix, that is in the originally given matrix, these two terms are 0. However, while doing this Gaussian elimination these same elements becomes nonzero.

So as a result so what happens that although our original matrix was to some extent sparse but after the first step or rather after the second step this advantage of sparsity gets destroyed. So because of this fact that some original 0 element becomes now nonzero. So this phenomena we call as fill-in problem.

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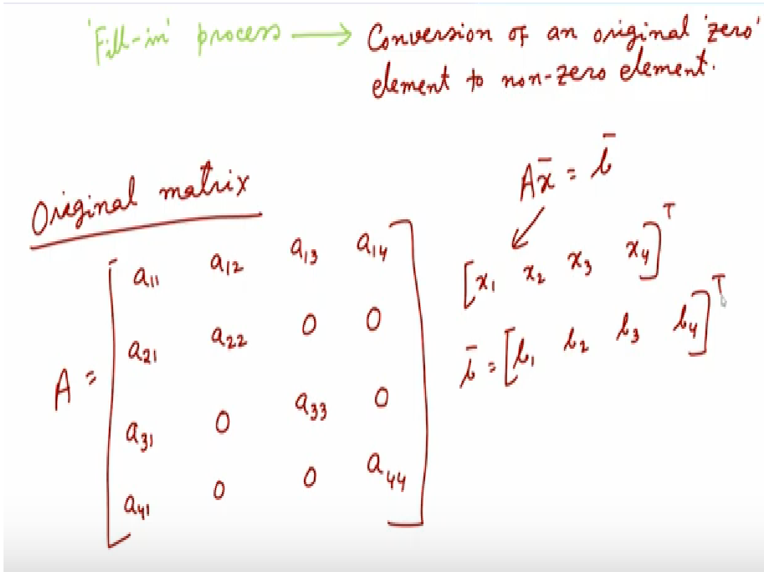

So it is fill–in process. This is called as fill-in process and this fill-in process is basically a conversion of an original 0 element to nonzero element. Now obviously when some 0 element are getting converted to nonzero element so then therefore the advantage of sparsity of this matrix gets destroyed. So here if we look at this, so then what happens is that we are just now saying that I mean this is a problem of fill-in.

So here this is a problem of fill-in. this is also a problem of fill-in. In this case also, here also we will have a fill-in problem by the same logic because this particular term would be 0 – 1 * this. So then therefore here also we will have a fill-in problem. Here also we will have a fill-in problem. Here also we will have a fill-in problem. Here also we will have a fill-in problem.

So then therefore what happens after all process over now if we just look into originally in this matrix there were six 0 elements; 1, 2, 3, 4, 5, 6. Originally, there were six 0 elements. But after we are finished with the first step, in fact it is not first step in our parlance but let us say that after we are finished with the first step, instead of original six 0 elements we are only left with 3 0 elements.

And all the original 0 elements they are being converted to nonzero elements because of this fill-in problem. So then therefore we can see that if we apply this Gaussian elimination method as it is without any such modification we will have the fill-in problem which will ultimately destroy the sparsity of the matrix. Now the question is that is there any way that we can really maintain the sparsity of this matrix while continuing with our Gaussian elimination method. Yes, that is possible.

That is possible by actually tackling these equations by an proper order. Now to understand that let us look the original matrix is this. So we had this original matrix. Again, let us say that we had this original matrix A. So let us look at original matrix is A. So original matrix A was let us again write a 11, a 12, a 13, a 14 and a 21, a 22, 0, 0; a 31, 0, a 33, 0 and a 41, 0, 0, a 44. So this is the original matrix.

Now obviously our equation is basically A x = b. So if it is A x = b, so if this matrix belongs to some equation A x = b where x is a vector, [x 1, x 2, x 3, x 4] transpose and b is a vector [b 1, b 2, b 3, b 4] transpose so then therefore obviously you can say that a 11 x 1 + a 12 x 2 + a 13 x 3 + a 14 x 4 = b 1 and so on and so forth. Now if we rewrite these equations in a certain order, we get a very interesting structure.

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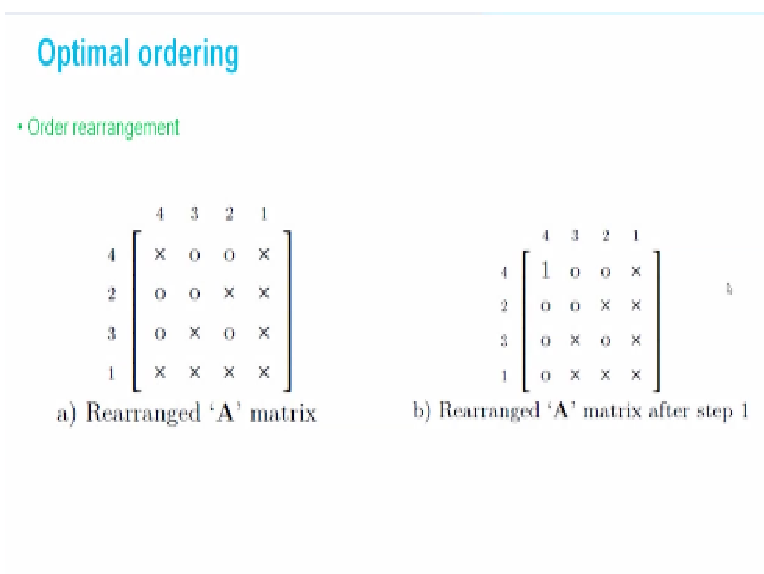

So let us look at that what would be this order. So that order would be something like this. So essentially if we look at this, so then if we just write down this equation.

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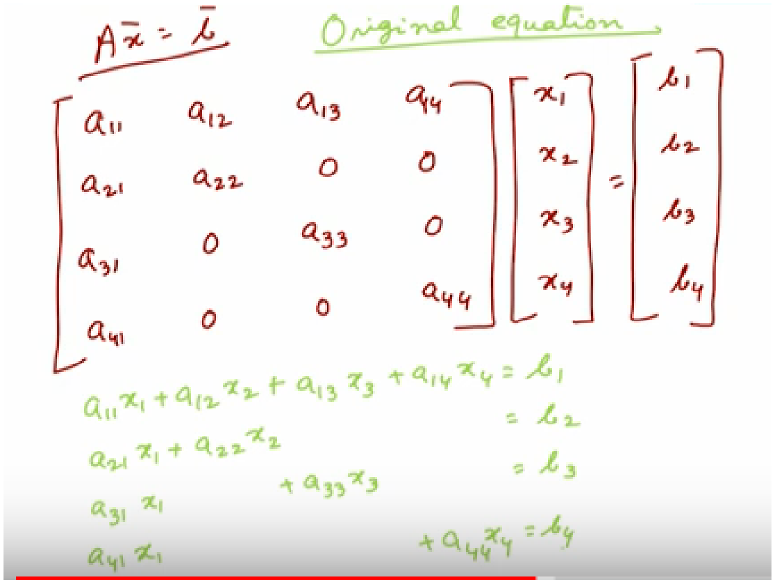

So it is A x = b. So to understand this in a more detailed manner again let us just write it very quickly. Then it will be clear in a moment. So [a 31, 0; a 33, 0; a 41, 0, 0, a 44] and then so then it is [x 1, x 2, x 3, x 4] and this is [b 1, b 2, b 3, b 4] right? So this is the original equation. So this we write as the original equation. Now what we do is this. Now we are trying to simply I mean rearrange this equations in a certain order.

So how do we rearrange this? First we rearrange them. So this rearrangement, so rearranged equation.

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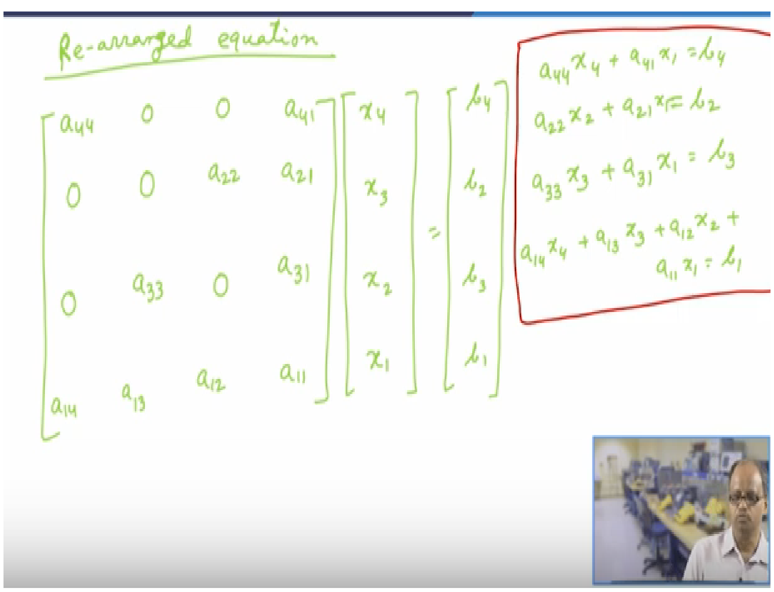

So let us look at rearranged equation. So there is some matrix. That same matrix we would be writing them in this rearrangement. Then here also we would have some rearrangement and here what we do is we have b 4, b 3, b 2, b 1. So b 4, b 3, b 2, b 1. Now because this is b 4, so now what happens? Now here we write x 4, x 3, x 2, x 1. So then therefore how this elements will now look like?

So then because this is so now if we look at this here, equations are a 11 x 1 + a 12 x 2 + a 13 x 3 + a 14 x 4 = b 1. Then a 21 x 1 + a 22 x 2 = b 2; a 31 x 1 + a 33 x 3 = b 3 and it is a 41 x 1 + a 44 x 4 = b 4. So then therefore if we now rearrange this so it would be, it should be a 44, 0, 0, a 41. So here also by this a 44 x 4 + a 41 x 1 = b 4 exactly identically the same thing. There is a little mistake, it should be actually x 4, x 2, x 3.

Now in the second case if this is case so then it would be a 14, a 13, a 12, a 11; x 1 = b 1. For x 2 for b 2; for b 2 we will have a 21, a 22. So then therefore b 2 so now let us make some other this thing. So we say that it is this is this we change, this we change. So it is x 4, x 2, x 3, b 2, b 3. So now this 0, 0, x, x. So now how it would look like? For b 2 what we have is x 3 and x 4 are 0.

So then essentially the coefficients corresponding to x 3 and x 4 would be 0. And the coefficients corresponding to x 1 and x 2 would be nonzero. So then therefore coefficients corresponding to x 4 and x 3 would be 0. So then we again it is x 3, x 2. So it is 0, 0 and then it is a 22 x 2 and a 21 x 1. So it is a 22 x 2, a 21 x 1. So that is = b 2. So we get this 0, 0, a 22 x 2, a 21 x 1. That we get, right?

And lastly for b 3 equation what you have? For b 3 only the term corresponding to x 1 and x 3 are nonzero and the terms corresponding to x 2 and x 4 are 0. So then terms corresponding to x 2 and x 4 are 0. So then x 4 is 0, x 2 is 0. Corresponding to x 3 it is a 33 and corresponding to x 32 is a 31. So what we have done is we have simply changed the order of the equations.

So once we have changed this order of this equations this equations are identically the same. So if we write this equations we get a 44 x 4 + a 41 x 1 = b 4. Please compare this; a 44 x 4 = b 4. Then from this a 22 x 2 + a 21 x 1 = b 2. From this a 33 x 3 + a 31 sorry this is x 1, should be x 1. A 31 x 1 = b 3. So let us check. A 33 x 3 + a 31 x 1 = b 3. And the last one is a 14 x 4 + a 13 x 3 + a 12 x 2 + a 11 x 1 = b 1. So please crosscheck it.

So it is a 14 x 4 + a 13 x 3 + a 12 x 2 + a 1 x1 = b 1. So then basically the equations are remaining the same. Only thing is that what we have done is we have simply rewritten this equations in a certain order. Now, so then therefore basically this structure we have here represented here. So these are all nonzero elements. So now if we do this Gaussian elimination on this, so then what do I get?

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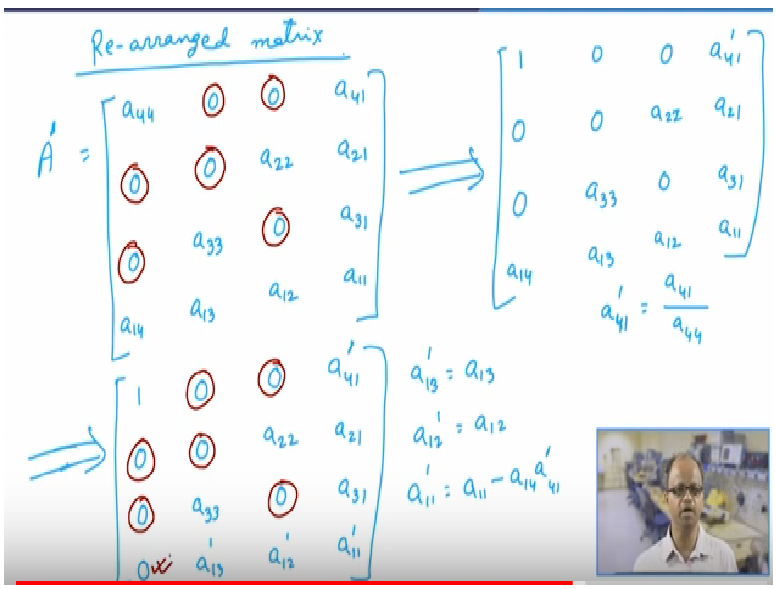

So now our, so rearranged matrix is let us say rearranged matrix is A star A dash. So this is rearranged matrix. We will just do this Gaussian elimination on this and we will see. A 44, 0, 0, a 41 so it is a 44, 0, 0, a 41 and then 0, 0, a 22, a 21. Then 0, 0, a 22, a 21. Then 0, a 33, 0, a 31. And 14, 13, 12, 11. And a 14, a 13, a 12, a 11. So this is rearranged matrix. So first what we get? We do is that 1, 0, 0, a 41 dash and this rest become as it is.

Then 0, a 33, 0, a 31 and then a 14, a 13, a 12, a 11 where a 41 dash is nothing but a 41, a 41. It is a 41/a 44. After this what we do? After this, now this element is already 0, this element is already 0. So then therefore we do not have to eliminate anything here. After this what we do? We simply multiply this row by 14 and then subtract this multiplied row from this row. So then therefore we will get 0 here.

So then what we get is 1, 0, 0, a 41 dash. Then 0, 0, a 22, a 21. Then 0, a 33, 0, a 31 and this becomes 0 and this becomes a 13 dash, a 12 dash, a 11 dash. So what would be a 13 dash? A 13 dash would be a 13 only. Similarly, a 12 dash would be a 12 only and a 11 dash would be a 11 – a 14 * a 41 dash, right? So this would be the case. So now what happens?

Now if you look at this, initially in our rearranged matrix how many zeros was there. There has to 6; 1, 2, 3, 4, 5, 6. And after we finish up to this operation how many zeros we have got? 1, 2, 3, 4, 5, 6, 7. So then therefore we not only returned those original 6 zero positions, please note that this original 0 positions are this, this, this, this, this, and this. So these are the original 0 positions. And the new 0 position we have created is this.

So then therefore we not only have returned the 0 positions, but we have also created another 0 position. So then therefore we have not only preserved the sparsity but we have actually increased the sparsity, right? So then therefore what we can see is that just by ordering this original set of matrix or rather ordering this original set of sparse equations in a certain optimal fashion, we can not only return our sparsity of this original matrix but also we can possibly even increase the sparsity of the resulting matrices obtained after Gaussian elimination method.

So this is precisely what we are showing here. So here these are the original 0 positions and after we have done this entire exercise as we have shown there, so we have got this resulting matrix what we have shown. So then therefore from this we can see that if we order the original set of equations in a certain fashion we can simply preserve the sparsity and also we can possibly increase the sparsity.

So then therefore the question baits that what should be this original order. Are there as such rather than there is as such no scientific I mean there is as such no foolproof mathematical method exists but there are a certain strategies available. So here we will be discussing those strategies very briefly.

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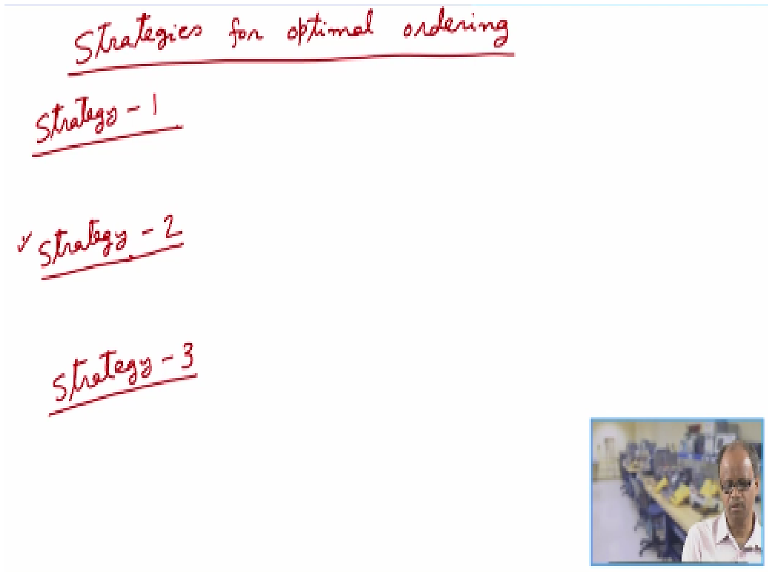

So one such strategy is, so then strategies for optimal ordering. So there are different strategies. Here we would be discussing only 3 strategies very briefly. So strategy 1, strategy 1 is that we simply look at the original matrix and then simply look at that which row have got this minimum number of off-diagonal nonzero element, right? So then basically we would simply order the rows vis-a-vis I mean we simply order the rows in an increasing order with respect to their number of nonzero off-diagonal matrices.

So then therefore what we do is that we simply operate on the first row that is which have got this least number of nonzero element, off-diagonal nonzero element and then we operate on the second row which have got this second least number of nonzero off-diagonal element and so on and so forth. So this is the simplest one. In strategy 2 what we do is that we first start with the row which has got this least number of off-diagonal nonzero element.

Then we complete one iteration or rather the one complete step of this Gaussian elimination. That is we simply take that particular element as a pivot or rather we simply take the corresponding element at that particular row to be pivot and then simply try to make the other corresponding elements in that same column to be 0. So then after we do that so then we will get a modified matrix.

Then we simply look at that modified matrix and then we try to again look at that which row in that modified matrix again has got this least number of nonzero off-diagonal elements and then again we operate on that particular row only. So then therefore the difference between this and this is, so the difference between strategy 1 and strategy 2 is that in strategy 1 we simply fix our order of operation a priori and it remains fixed throughout the Gaussian elimination method.

In strategy 2, the operation is changed in each and every step. That is at the first step, we first look at the row which has got this minimum number of off-diagonal nonzero elements. Then we do one iteration or rather then we complete one step of this Gaussian elimination. After that we get the modified rows. Again in that in those modified rows we simply look at that which row has got the minimum number of off-diagonal nonzero element and again we do operate on them.

Whereas in the strategy 1 we first simply decide that we will operate from let us say first on row 3 then followed by row 5 then followed by row 7 then followed by row 2 etc. irrespective of the change which is taking place in these rows after each and every step of this Gaussian elimination. In strategy 3 what we do is in strategy 3 what we do is, in strategy 3 we try to predict that if I take essentially this particular row as a pivot then what would be the effect of the then what would be the effect on the resulting matrix.

So then therefore what we do is that we first take a row then we simply try to predict that if we do the Gaussian elimination by taking this row as the operating row, then what would be the number of nonzero off-diagonal elements in all the other rows after first step, right? So if this operation gives me a satisfactory operation then and only then we go through this.

Otherwise, we simply choose another row and simply repeat the process. So obviously this strategy essentially requires a lot of simulation. That is we have to first do a kind of offline simulation and to see that what would be the effect of this. So then therefore this particular strategy is actually computationally intensive. So then therefore for practical implementation this strategy 3 is not really very much popular.

Rather we usually try to go for strategy 1 or strategy 2. In between strategy 1 and strategy 2, strategy 2 looks to be very promising because it always tries to operate on that row which has got this minimum number of nonzero off-diagonal elements.

And as we have already seen that if we start our operation with a row which has got this minimum number of nonzero off-diagonal elements so then therefore the effect of that operation on all the other rows of first is the minimum because as the maximum elements all these rows are 0.

So then therefore the operation with 0 always, so then therefore if we operate with this row which has got maximum number of zeros, so then therefore all the other rows will not be really affected because whatever we add, subtract or multiply and add with 0 those elements essentially remains same. So out of these 3, strategy 2 is actually basically most popular. So we finish today this lecture. In the next lecture we will start looking into some other strategies of solving a set of linear equations. Thank you.