# Optimal Control
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## Lecture – 09
## Optimum of Functions with Conditions

So, welcome friends to this class which is on optimum functions with condition. In the previous class we have discussed about the optimum of a function in the functional and the different cases we have seen.

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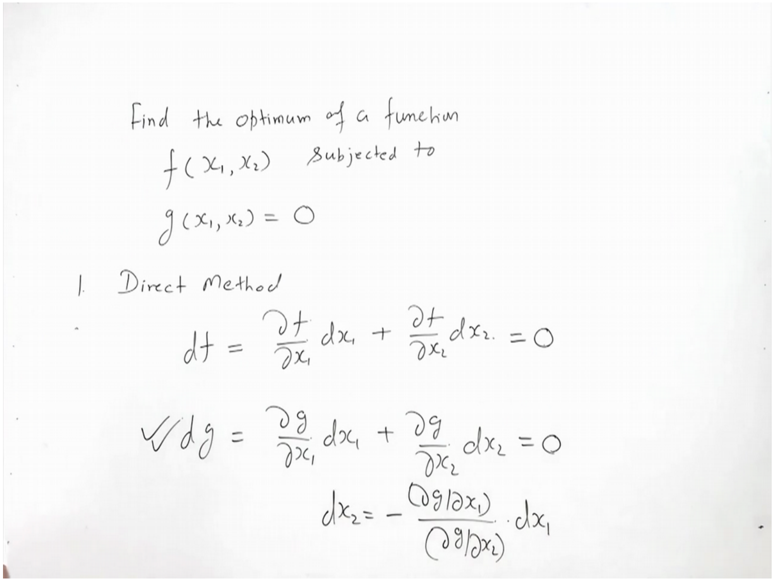


So, with optimum of a function with condition first we will formulate a problem, our problem is to find the optimum of a function f which is a function of the 2 variable x 1 and x 2 and this is subjected to the condition g x 1, x 2 equals to 0. So, my problem here is to find out the optimum value of the f which is subjected to the condition g of x 1 x 2 equal to 0.

So, this problem can be solved in the 2 approaches the first we call the direct approach and second is the Lagrangian multiplier approach. So, this problem can be solved as direct method and my second method is the Lagrangian multiplier method. So, one by one we will discuss both the methods. Let us first take the direct method. So, what we have to do if we have to find out the optimal value of a given function f which is a dependent on a single variable. So, simply we will say the del f by del x to 0 or as we have discussed previously the differential of the function must be equal to 0. So, this means if I will write df which is nothing but del f by del x 1, dx 1 plus del f by del x 2 dx 2 and normally we say that must be equal to 0.

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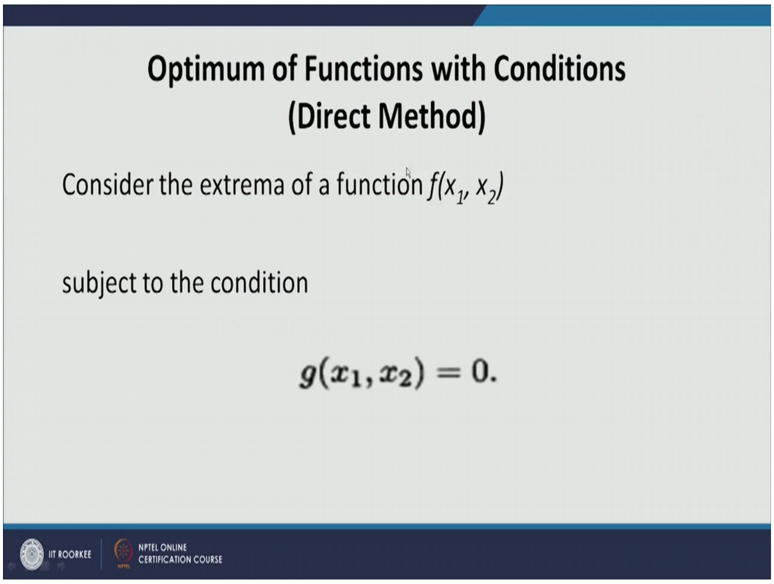


So, my problem is to find out the we are consider a function f x 1, x 2 which is subjected to the condition g x 1, x 2 equal to 0 and I objective to find out the optimum value of this function f x. So, my necessary condition for extremal will be del f by del x 1 dx 1, del f by del x 2 dx 2 that must be equal to 0.

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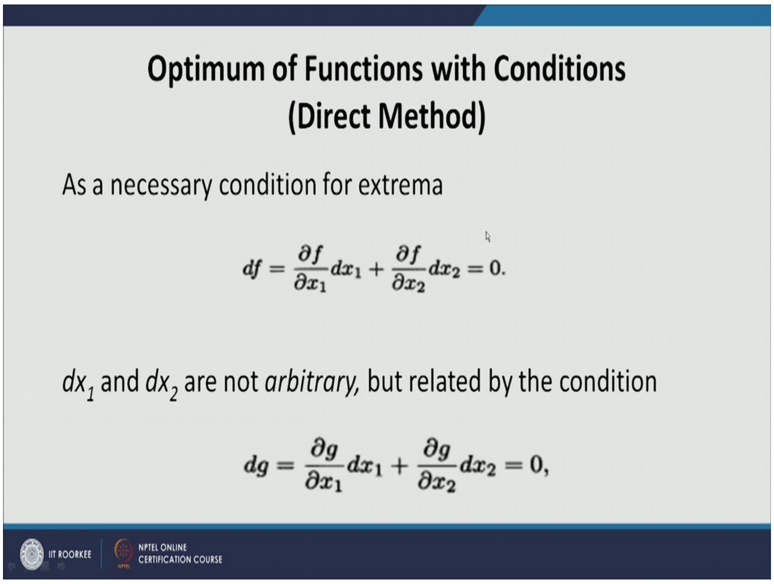

Now if these 2 variables are independent then we can say del f by del x 1 must be equal to 0 and del f by del x 2 must be equal to 0, but in this case we consider the x 1 and the x 2 are not independent. If I treat means the x 1 x 2 are related to each other they can have their value. So, independently we cannot say del f by del x 1 will be 0 and del f by del x 2 will be 0, but as a minimal condition I can say for this my df must be equal to 0.

So, this is my necessary condition, which we are saying here as a necessary condition for extrema my df should be 0. Similarly g is also a function of x 1 x 2, so if I will write dg which will be del g by del x 1 dx 1 plus del g by del x 2 dx 2 as g is already 0, so dg will also be 0. So, what we can say dx 1 and dx 2 are not arbitrary in this case, but they are related to the relation dg equal to del g by del x 1 del g by del x 2 dx 2 that is equal to 0.

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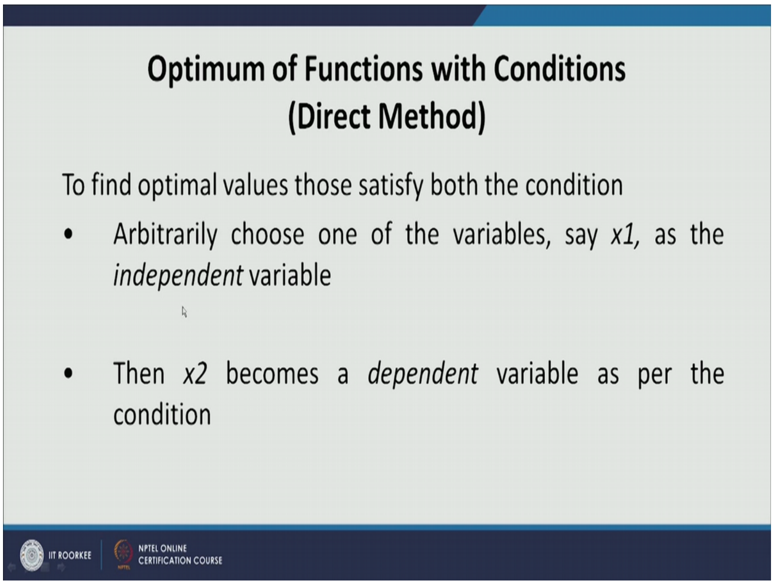


So, to find out the optimal value which satisfy both the condition we arbitrary choose one of the variable which is x 1 as the independent variable. So, naturally in that case x 2 will be the dependent variable as per our given condition. So, if x 1 is independent x 2 is dependent what we can do, we can find out the value of the dx 2 in terms of the dx 1.

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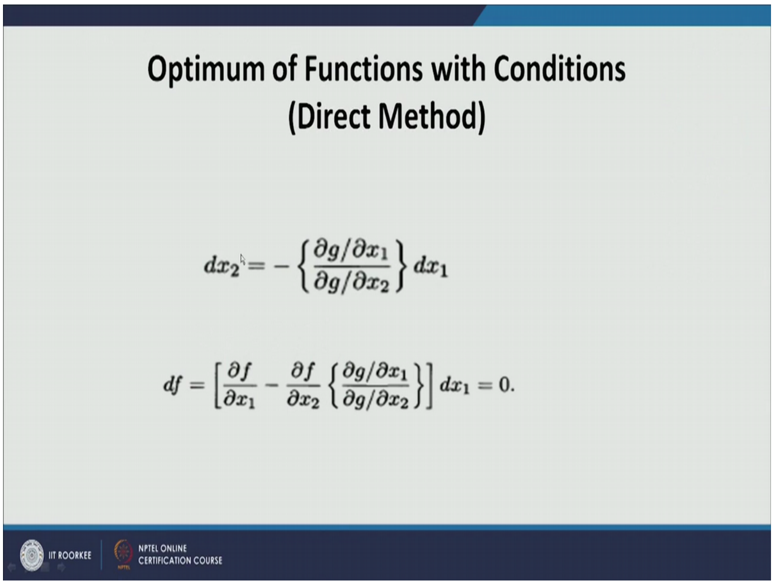


So, that we are finding from dg equation. So, by this equation if I am writing dx 2, so I can simply write it as minus del g by del x 1 divided by del g by del x 2 and multiplied with the dx 1. So, now, if I will place the value of the dx 2 in this, so this my whole expression will be with dx 1 value. So, in the next we will place this dx 2 in df in this expression. So, this means del f by del x 1 dx 1, due to this, this will be the del f by del x 2 multiplied negative sign we have considered here del g by del x 1 divided by del g by del x 2 that is sorry and this is also multiplied with dx 1 and this equals to 0.

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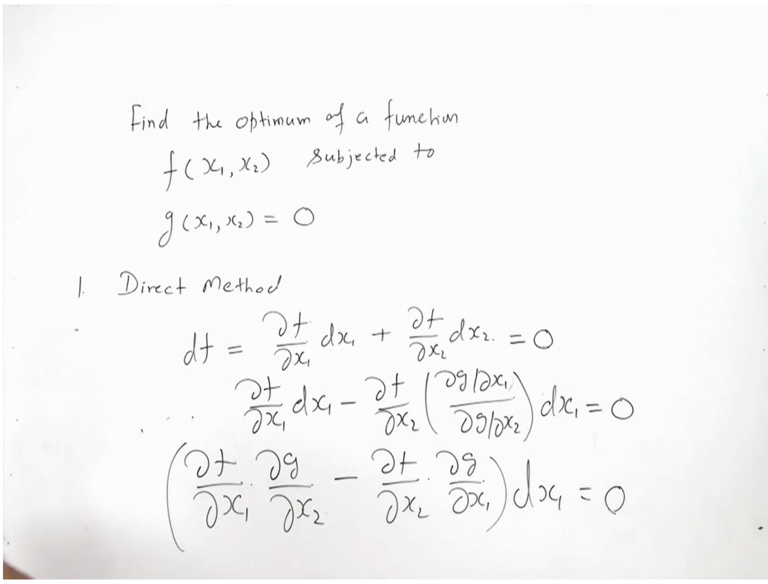

So, if I will simplify. So, this is nothing, but my del f by del x 1 in to del g by del x 2 minus del f by del x 2 into del g by del x 1 whole multiplied with dx 1 equal to 0. So, now, in this case my dx 1 is arbitrary, so the coefficient of related to dx 1 that will be 0.

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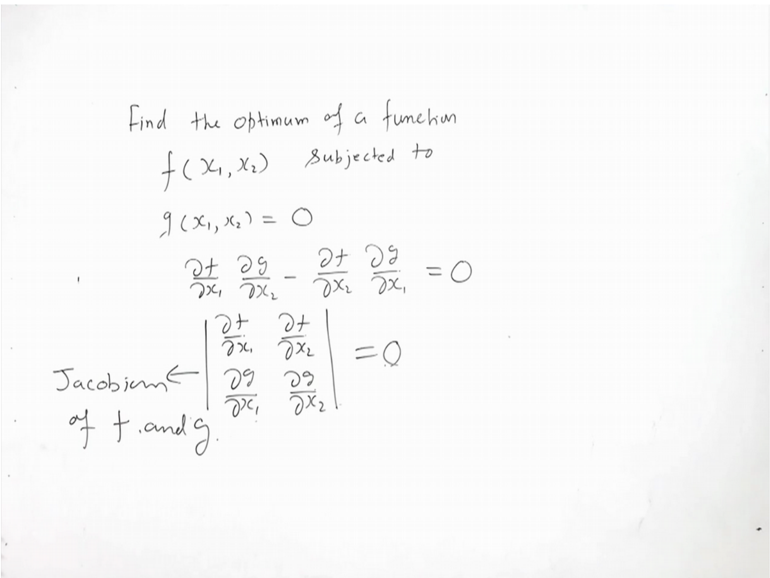


So, I can say my del f by del x 1, del g by del x 2 minus del f by del x 2 del g by del x 1 with must be equal to 0 or we can simply write as the determinant of del f by del x 1 del f by del x 2 del g by del x 1 del g by del x 2 that determinant must be equal to 0. So, for extreme of a functional my condition will be this must be equal to 0 or this must be equator 0 and this is nothing but we call the Jacobean of f and g.

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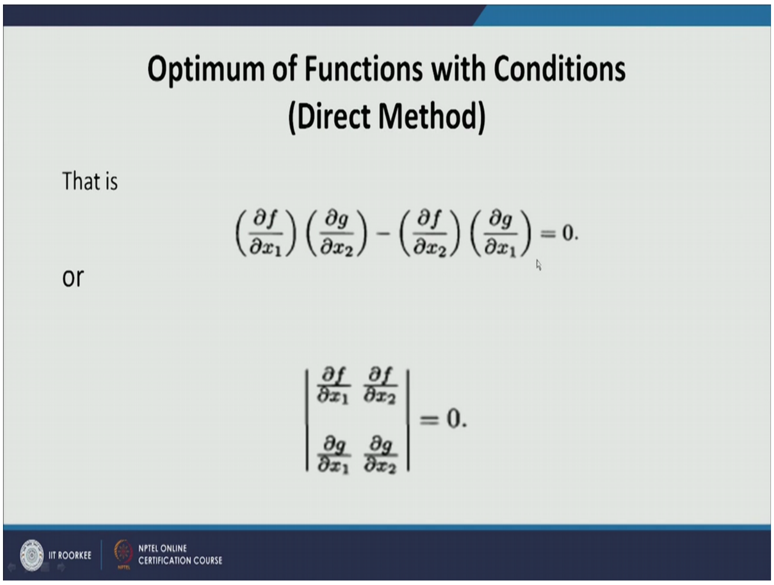


So, what we say my condition coming out to be del f by del x 1 del g by del x 2 minus del f by del x 2 del g by del x 2 must be equal to 0 or the determinant of del f by del x 1 del f by del x 2 del g by del x 1 del g by del x 2 that is going to be 0. So, what actually we have done, if a function is a function of the 2 dependent variable - one is the x 1 other is the x 2 what we can do we can eliminate one of the variable minimize it using the another variable and place the value because both the variable are related to each other. So, if we can find out the one variable we can find out the other variable also. That you will see in a very interesting problem.

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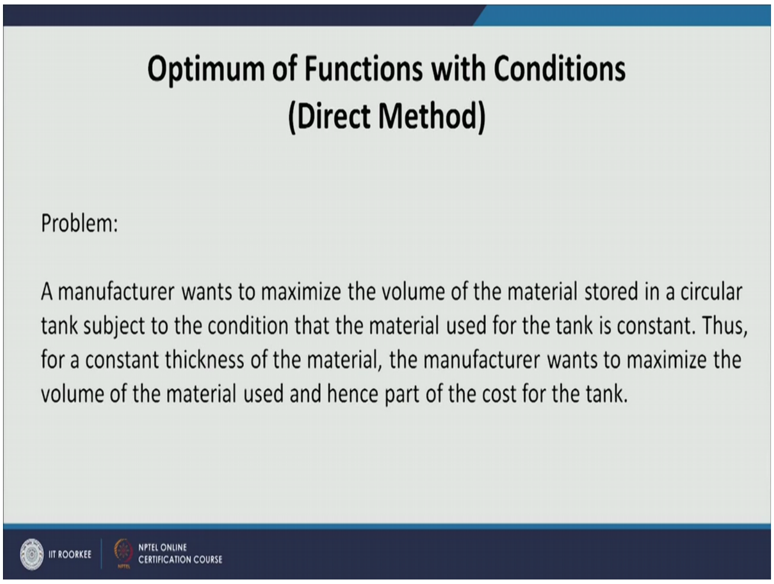

So, my problem here is given as: A manufacturer want to maximize the volume of the material is stored in a circular tank subjected to the condition that the material used in the tank is constant. Thus, for a constant thickness of the material, the manufacturer wants to maximize the volume of the material used and hence it has to reduce the cost.

So, in a given material if I will increase the volume. So, this means the storing cost I can reduce. So, what my problem is, I am given you can say a constant or a fixed material is given and from this material we want to make a circular tank, so that the maximum volume can be stored into this circular tank.

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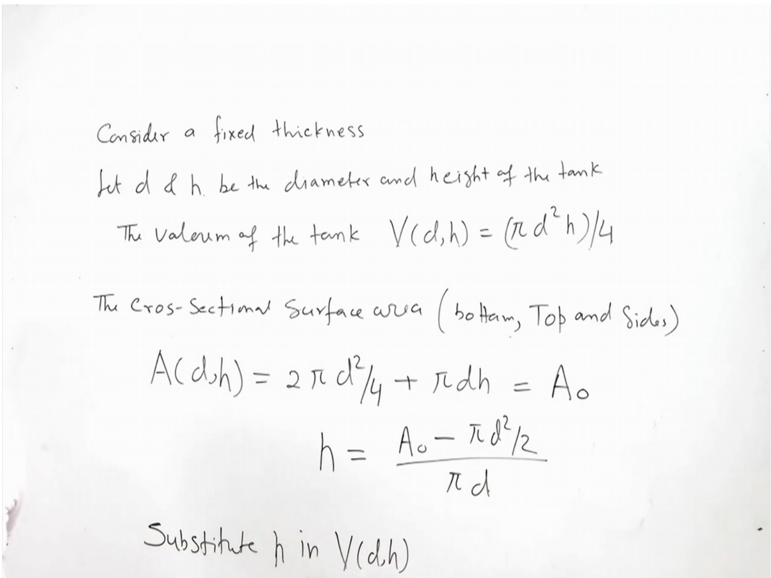


So, let us understand the problem in the following manner. Consider a fixed thickness, and let d and h be the diameter and height of the tank - where d is the diameter and h is the height of the tank. So, basically we have to find out what will be the diameter and what will be the height of the tank so that the maximum volume can be stored. While the surface area of the sheet by which we are forming the tank is constant. So, we can write.

So, if d is the diameter, h is the height then the volume of the tank is we can write here this is a function of diameter and the height and this is nothing but your pi d square h by 4. The cross sectional surface area of the tank will be, now what the cross sectional surface area? It will have the bottom, top and the sides. So, top and the bottom are my circular sheets while the surface is my cylindrical sheets. To this cross sectional area if I will write as a function of d and h, so pi r square is the area of my circular sheet are the top and bottom. So, I can write 2 times of pi d is the diameter d square by 4 this is my top and bottom plus pi d h is my sides and this is my surface area.

So, what is my problem? I have to find out the maximum volume, so this means I will maximize the volume by selecting the proper value of the d and h while my given surface area is constant, let say this value is A naught. So, this is my extremization problem with a given condition where my objective is to maximize the volume subjected to the condition given as A. V is function of the 2 variable d and h, because both d and h will be dependent variables. So, if I will treat one variable as a dependent other as the independent. So, I can place the value of the one variable I can find out the value of the one variable in terms of the other as my surface area is fixed from this equation I will write my height in terms of the diameter.

So, my h is A naught minus this is pi d square by 2, this is divided by pi d and I will substitute the value of h substitute h in volume expression in V d h. So, this means this h I am placing here and I will get.

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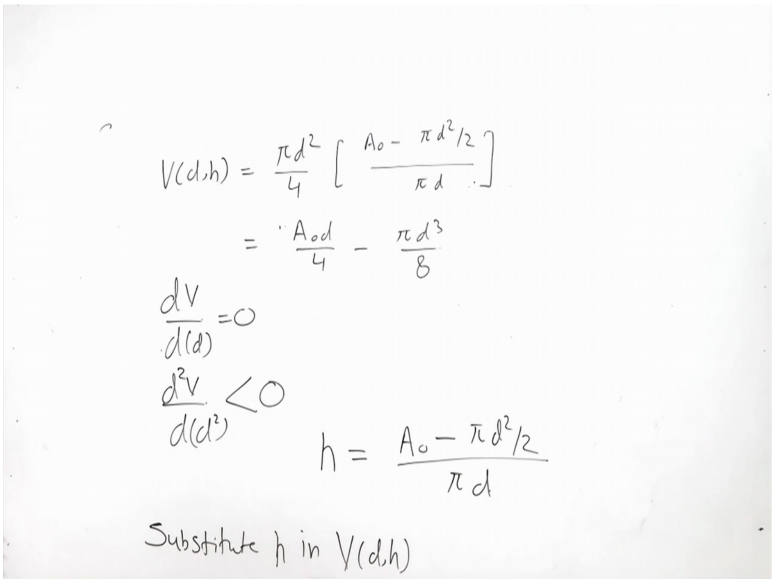


So, just by placing the value of h in the expression of V we get V d h as pi d square by 4, A naught minus pi d or simply I can write this as A naught d by 4 minus pi d cube by 8. So, this is, now I am writing my V only with one variable d, h and d are related by this expression, so what we have done? We have find out the one variable in terms of the other place the value of this variable into the volume expansion. So, my V is simply as n node d by 4 minus pi d cube by 8. So, we have eliminated the h from the volume expression.

There is a simple maximization problem. So, I have to maximize the V and find out the value of the d which will maximize my V. So, simply I can say my d V by d of d, this d for diameter that must be equal to 0 and if d V sorry; d 2 V by d of d square is less than 0. So, this will be the maximization problem.

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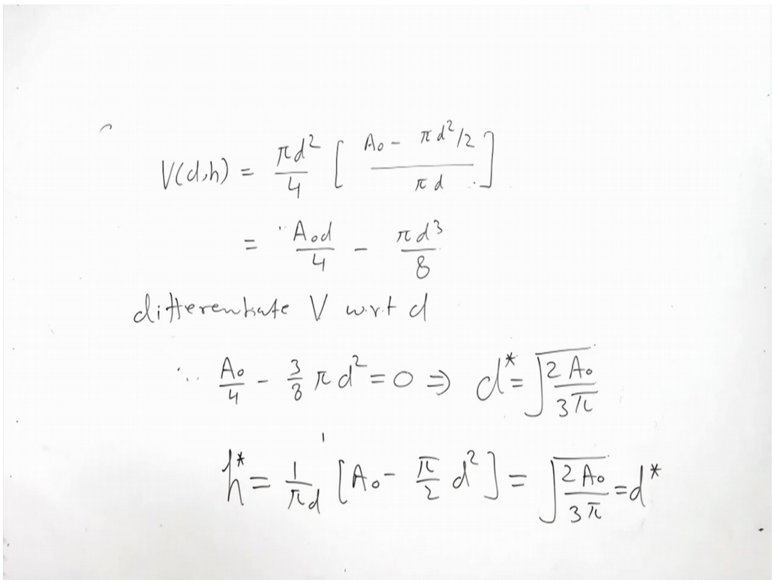


So, we will differentiate V with respect to diameter. So, differentiate V with respect to d what we get? We get A naught by 4 minus 3 by 8 pi d square equal to 0 this implies my d which is nothing, but my optimal value is square root of 2 A naught by 3 pi.

So, V to be maximum d 2 to be 2 A naught by 3 pi h we already have been determined in terms of the d. So, if d is known to me I can directly write the value of the h, this is 1 by pi d A naught minus pi by 2 d square and if you will simplify this again you will get this is 2 A naught by 3 pi which is nothing but your d star. So, this h we can also say my optimal value. So, for this particular case my h and d both have the value 2 A naught by 3 pi.

So, in this problem we can conclude we can maximize the volume of the tank, if we will make the diameter of the tank same as the height of my tank. So, this is one of the approach to find out the optimum value of the functions with conditions. My another approach is called the Lagrangian multiplier approach in which we will define a Lagrangian which is nothing but contain both my function and my condition which are related with a variable lambda which is called the Lagrangian multiplier. So, this approach we will discuss in the next class. So, in this class I stop it here.

Thank you very much.