Optimal Control
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Lecture - 08
Free End Point Problem (Continued)

Welcome friends to this class which is the continuation of the previous class in which we are discussing about the free end point problem.

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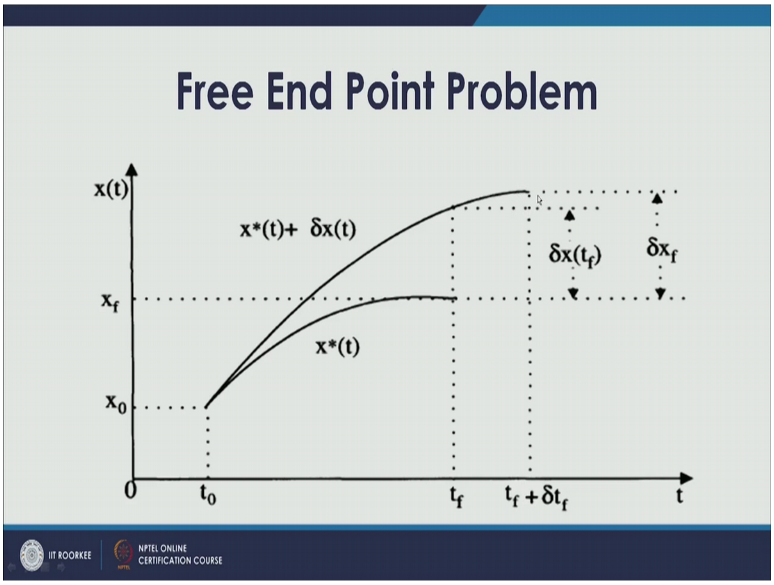

In the free endpoint problem our final point can terminate to any of the point if there is a variation in the optimal value of the x.

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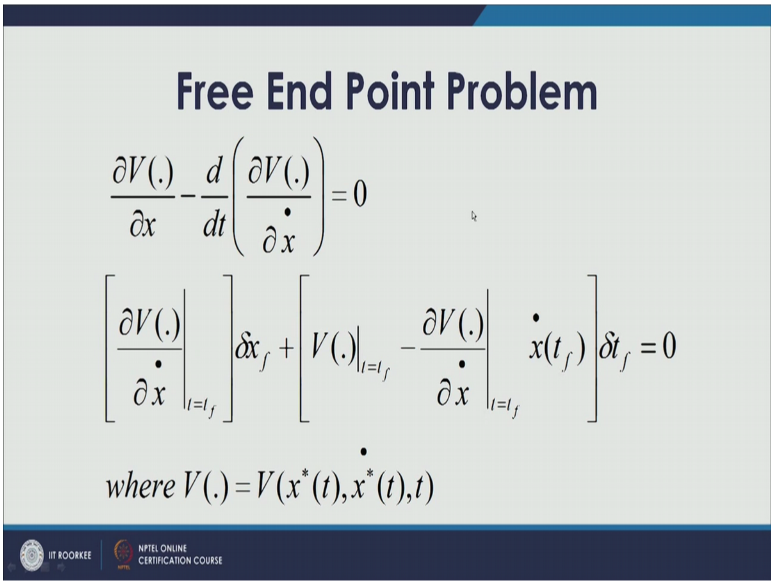

So, our general condition appear as the first equation I can see it here which is del V by del x d by dt del V by del x dot equals to 0 this is nothing, but my Euler equation and the general boundary condition is given as V dot by del dot x evaluated at the t equal to t f point delta x, then V dot evaluated at t f point minus del V by del x dot again evaluated at the t equal to t f point x dot t f delta t f equal to 0. So, this is the general condition.

For this general condition we have the different cases, like we can have the fixed endpoint problem in the fixed endpoint problem delta x f and delta t f both are 0. So, this means the second equation will not appear in that case and we have to solve the problem using the EL equation which will lead to a differential equation with given boundary conditions. So, in this case my boundary condition is the initial condition as well as my terminal condition. The second case we have considered that in which t f was fixed. So, delta t f was 0 and the x t f was free. So, in that case if this, delta x f variation is arbitrary my del V by del x dot will be 0.

Third case we have taken that in which my t f is free but x t f is fixed, if x t f is fixed this means my delta x f will be 0, if delta x f is 0 delta t f is arbitrary the coefficient of the delta t f that will also be 0. So, these three cases we already have been discussed in the previous classes. So, today we will take the case four in which my t f and the x t f are free.

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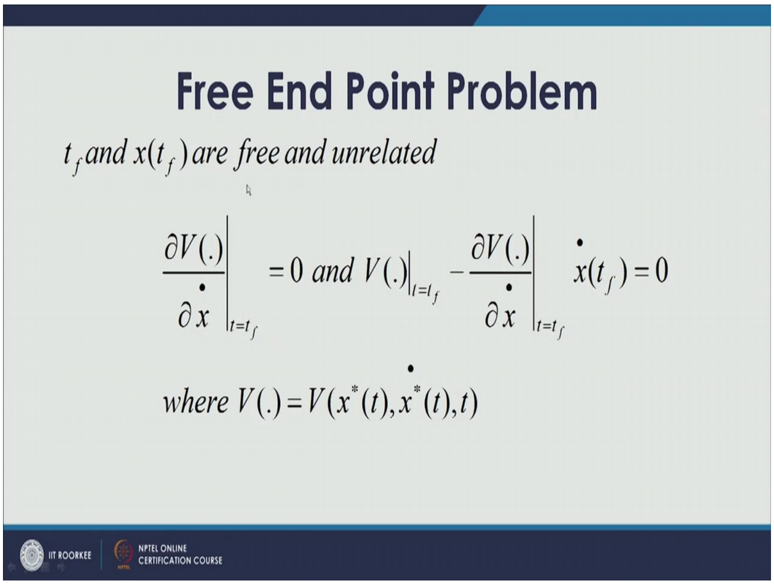

If t f and the x t f are free, so we have my general condition which is given by these 2 equations. So, my first equation will appear in the similar manner which will my Euler equation and terminal condition will be given by second equation.

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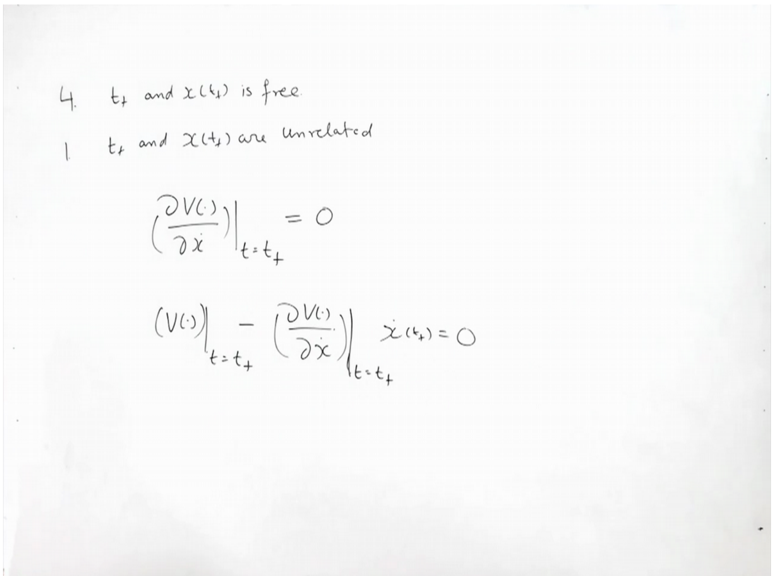

So, we can say my fourth cases t f and x of t f is free this will further lead to the 2 sub cases the first one is if t f and x t f are unrelated.

This means my t f and the x t f both are the independent variable if both are independent. So, naturally in this case whatever be the coefficient of the delta x f that will be 0 and what is the coefficient of the delta t f that will be 0 because both delta x f and the delta t f these will be arbitrary. So, in case of my first sub case in which t f and the x t f are unrelated, this means I can write delta V dot by delta x dot evaluated at t equal to t f point that will be 0 and V dot evaluated at t equal to t f point minus del V by del x dot again evaluated at t equal to t f point x dot t f that will be equal to 0. So, as we are seeing in this case my first term del V by del x dot this is nothing, but the coefficient of the delta x f and the second term is the coefficient of the delta t f, both delta x f and delta t f are arbitrary. So, their coefficient will be 0.

So, my first case in which the t f and the x t f are unrelated, my boundary condition will be governed by the del V by del x dot equal to 0 and the second equation is del V by sorry; V evaluated at equal to t f point minus del V by del x dot x dot t f that must be equal to 0. So, in this case my V dot is nothing but V evaluated at x star x star dot and t points. So, this is my first case.

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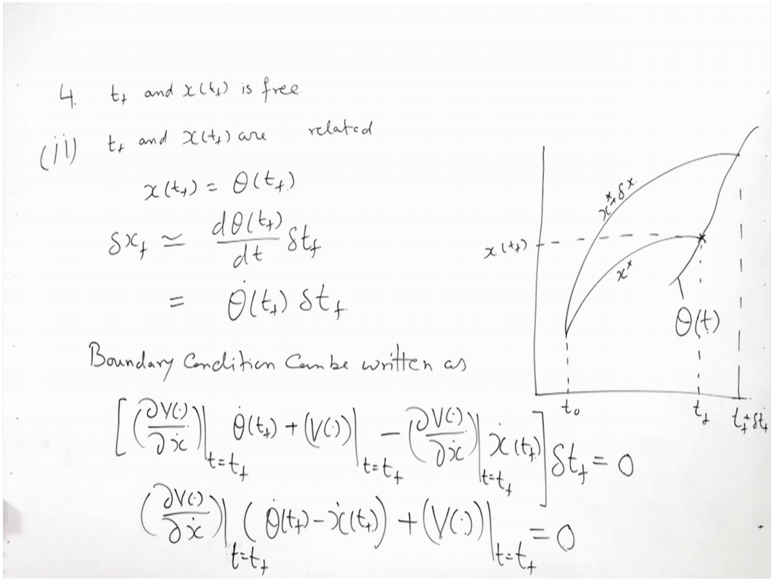

In the second case if this 2 will be related. So, the second case we are saying if t f and the x t f are related means there may be the case that anyone of may be the dependent variable of the other. So, one if I am treating as a independent the other may be the dependent on the first variable.

Let us take an example in this case let say this is my x star, this is my general boundary condition and let say this is my another curve which is given as theta t. So, this is t 0 this is nothing, but my t f. So, this is the same as we have discussed in the previous class t f plus delta t f, this is my x star plus delta x the variation into this and this is my x of t f. So, as we have seen t f and the x t f are related to each other let say my optimal trajectory terminate to a curve given by theta t. So, in this case my x of t f nothing, but will be equal to theta of t f. So, my x t f will be equal to theta t f this means at the endpoint my trajectory is terminating at this given point.

So, if this is terminating then how we can relate this. So, delta x f I can approximate as d theta at t f point by dt into delta t f. So, by this I can approximate or simply I can write this as theta dot at t f point delta t f. So, x f can be approximated in terms of the theta f if there is a relation between x and theta at any given condition. So, here we have the condition which we have considered as my trajectory is terminating to the theta t curve. So, x t f is equal to theta t f. So, again if I will consider my general condition like given in this case, so this delta x f I am trying to relate with the delta t f.

So, if I will place the value of the delta x f calculated on the board into this expression, so I will get is simple equation in delta t f which will be. So, now, boundary condition can be written as. So, I am just simply placing the value of the delta x f in my last equation, if I will place this value this is my equation will become del V dot by del x dot this value evaluated at t equal to t f point theta dot of t f plus V evaluated at t equal to t f point minus del V dot by del x dot at t equal to t f point x dot of t f and the whole multiplied with delta t f and this must be equal to 0.

So, by substituting the value of delta x f in my last equation I get this value, so del V by del x. So, I can combine this term and can write my boundary condition treating delta t f to be the arbitrary, so the coefficient of the delta t f will be 0. So, my final boundary condition will be del V by del x dot which I am taking out from this expression sorry; this is at t equal to t f point this whole multiplied with the theta dot of t f minus x dot of t f plus V dot at t equal to t f point equals to 0. So, this will be my final boundary condition.

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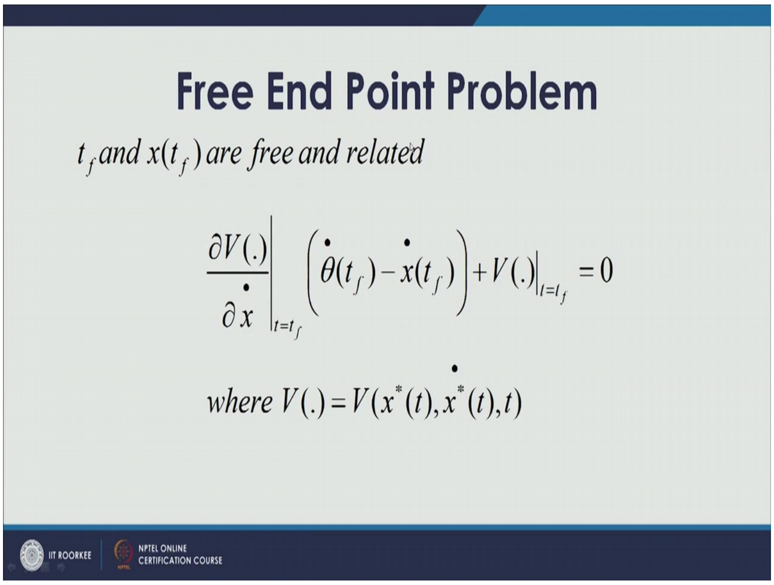

So, this we can say that if t f and the x t f are free and related to each other by the curve theta t and my trajectory is terminating to the curve theta t. So, my endpoint x t f will be equal to theta of t f then my final boundary condition will be del V by del x dot theta dot minus x dot t f plus del V equals to 0. So, this is my fourth case, let us take an example to understand this more clearly.

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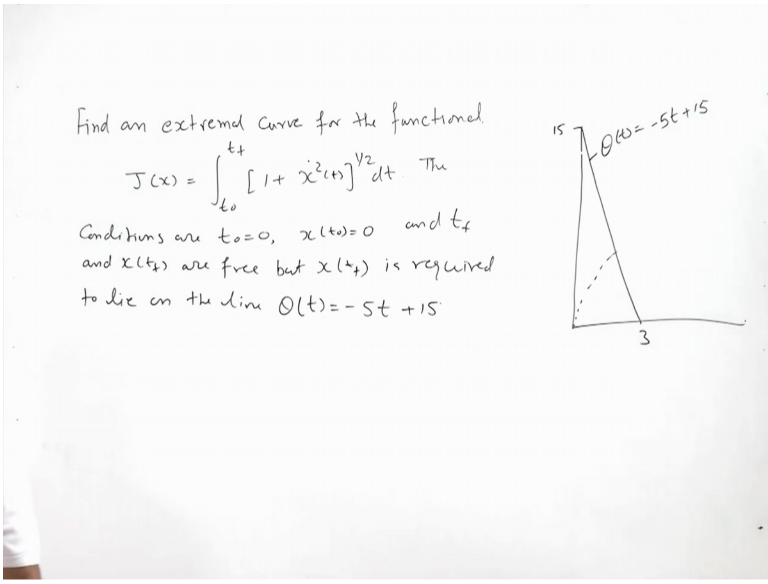

So, for this second sub case we will taken problem the statement of the problem is find an extremal curve for the functional J x equal to the power half dt the boundary conditions are t 0 equal to 0 and x t 0 equal to 0 and t f and x t f is x t f are free, but x t f is required to lie on the line theta t equal to minus 5 t plus 15. So, my problem is we have to find an extremal curve for the given functional with the boundary condition given as t 0 equal to 0, x t 0 equal to 0 and t f and the x t f are free, but here we are putting a condition - x t f is required to lie on the line theta t equal to minus 5 t plus 15.

So, now, if we will see what actually my problem is say I have to start with the origin t equal to 0 x t 0 this is my theta t line which is minus 5 t plus 15. So, t equal to 0. So, this value is 15 this is if theta t is 0. So, this will be nothing, but 3. So, this is my theta t line. So, we have to find; what will be my optimal curve for? So, this distance should be the minimal distance. So, this problem we have to solve. So, as you can see what is my functional this is nothing, but if you can recall our previous problem we are trying to find out the minimal curve which is joining the 2 points.

So, from there we have taken this functional. So, this we are directly treating at this point to solve this problem we will take first solve the EL equation and then we will apply our final condition to get out the sorry, to get the value of my constants.

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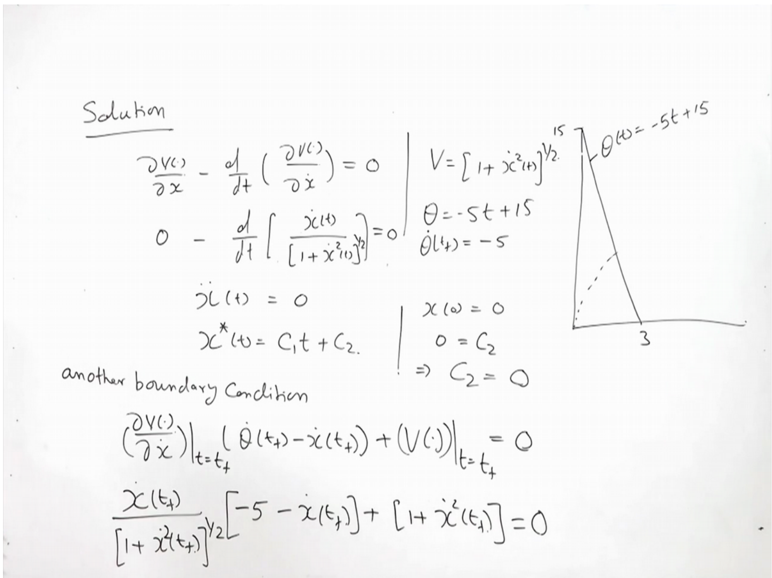

So, the solution of this problem, naturally first we have to write del V by del x minus d by d t.

So EL equation we will write first del V by del x dot that must be equal to 0. In this case my V is one plus x dot square t to the power half. So, del V by del x is 0, we already have been solve this problem d by dt x dot t by 1 plus x dot x square t the power half that is equal to 0. So, this nothing but gives us x double dot t equals to 0 and the solution of this which is nothing but my optimal x star t is C 1 t plus C 2, and using the boundary condition my objective is to find out the constant C 1 and C 2. So, my first condition is directly is x 0 equal to 0. So, if t is 0, so C 1 will be 0 nothing but C 2 this implies my C 2 value coming out to be 0. So, C 2 is 0 that we can find out from my initial conditions which is given x 0 equal to 0 and C 1 can be determined utilizing the end point condition which we have derived just now - del V by del x dot at t equal to t f point theta dot t f minus x dot t f plus V at the point t equal to t f and equating it to 0.

So, let us see what we will get in this. My another boundary condition, so I have del V by del V dot by del x dot will let just write here del V by del x dot at t equal to t f point theta dot t f minus x dot t f plus V dot t equal to t f point that must be equal to. So, this is my another boundary condition. So, with the given V I will find out del V by del x which is nothing, but given by this equation at t f point. So, I can write this is as x dot of t f 1 plus x square dot of t f, a square root of this; theta dot of t f what was my theta? My theta was minus 5 t plus 15. So, theta dot t f is nothing, but minus 5 so this is minus 5 minus x dot of t f and what is V at t f point? 1 plus x dot square of t f and that must be equal to 0. So, we are given with my last condition.

So, if I will solve this. So, I am getting this minus 5 x dot of t f plus 1 that must be equal to 0. What was my x t? X t is C 1 t plus C 2. So, x dot of t f is nothing, but my C 1.

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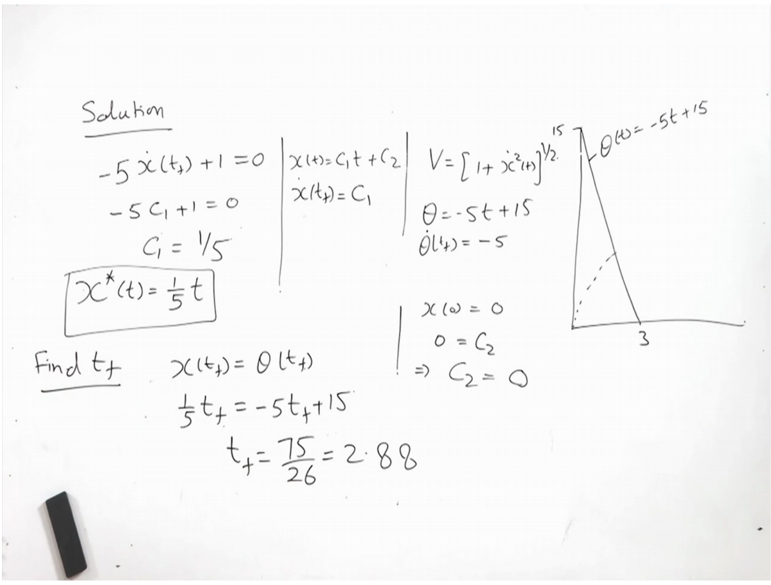

So, x dot of t f you I will find that will be nothing, but my C 1. So, this is minus 5 C 1 plus 1 equal to 0. So, C 1 nothing but is my 1 by 5. So, this means my optimal trajectory x star t which is C 1 t nothing, but 1 by 5 t, C 2 is 0 so this will be my optimal value of the x. Similarly they can extend this now find t f, and t f we can find because we know my terminal condition x t f equal to theta t f, x t f is 1 by 5 t f because x t is x star t is this 1 by 5 t f and theta is minus 5 t f plus 15 and this give you t f equal to 75 by 26 which is equal to we can write as 2..88. So, using this, this will be my final solution. So, my final solution I can find out in the following manner.

So, in a free end point problem we will try to find out the different cases this means normally we are given with the initial point and where my terminal point is will be. So, the four cases we have discussed - the first one is the fixed end point problem, second is my x t f is free t f is fixed, the third x t f is fixed t f is free and when both are the free.

So, these four cases along with the example we have seen here. So, this is the free endpoint problem when we are simply trying to extremize a given functional without any condition. So, in this case we have not considered the considered any condition. In our next part we will consider the variational problem with the given condition. So, this lecture I stop it here and in the next lecture we will start our discussion on the extremization of the functional with condition. We will start with the function extremization and then we will lead to the functional extremization.

Thank you very much.