Optimal Control
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Lecture – 06
Free End Point Problem (Continued)

So, friends in lecture 6 we will continue from our discussion from the previous lecture in which we are discussing about the free end point problem. So, what was the free end point problem just to summarize it?

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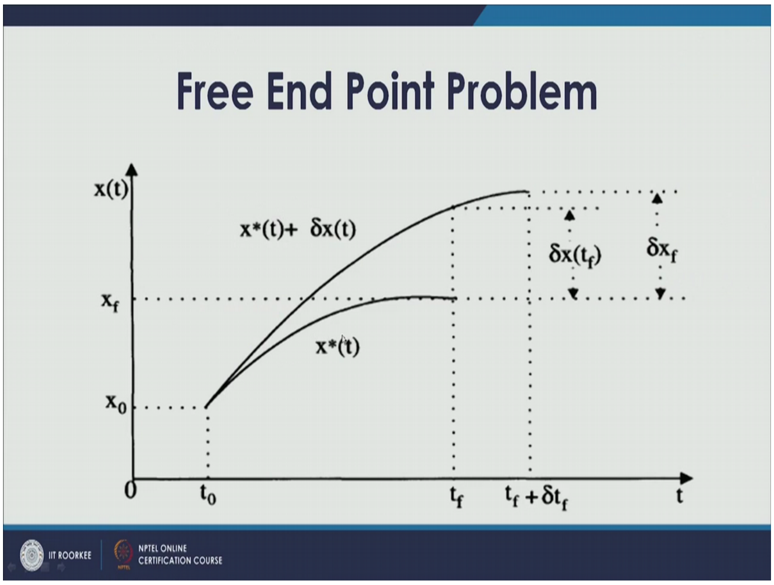

If we have the optimal trajectory x star t and x is varying from x to x plus delta x and t is varying from t to t plus delta t. So, my variation goes here up to the t f plus delta t f points and the basic variation problem we have to solve in which we are trying to minimize the functional V with the limits from t 0 to t f.

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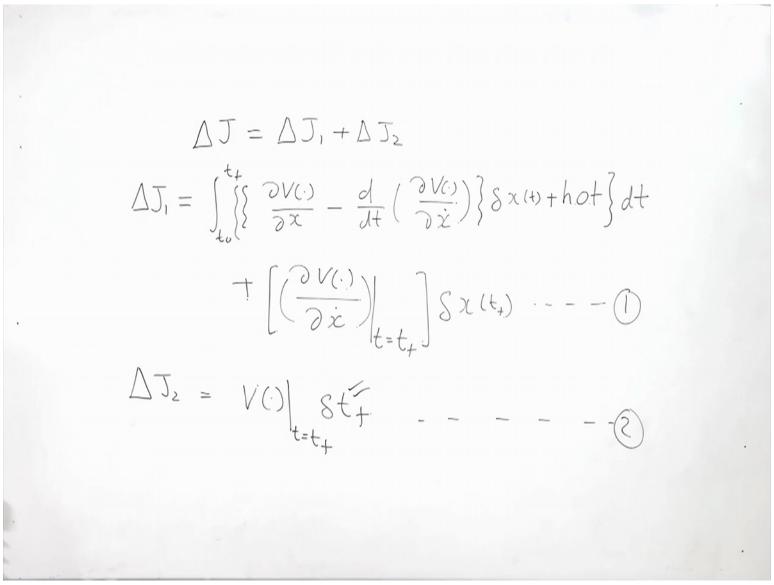

And if you recall in the previous class we have written that the variation can be the increment we are trying to find out the increment delta J which was equal to which we have written as delta J 1 plus delta J 2 where delta J 1 we have derived is integral t 0 to t f del V dot by del x minus d by dt del V by del x dot into delta x t. So, this was by first order variation plus the higher order terms into dt, and plus del V dot by del x dot evaluated at t equal to t f into delta x of t f.

So, this we have written as the delta J 1 which I am saying equation number one and delta J 2 we have derived is nothing but we can say the V dot at t equal to t f point delta t f. So, this was our expression for J 1 and J 2 this we are saying as equation number 2.

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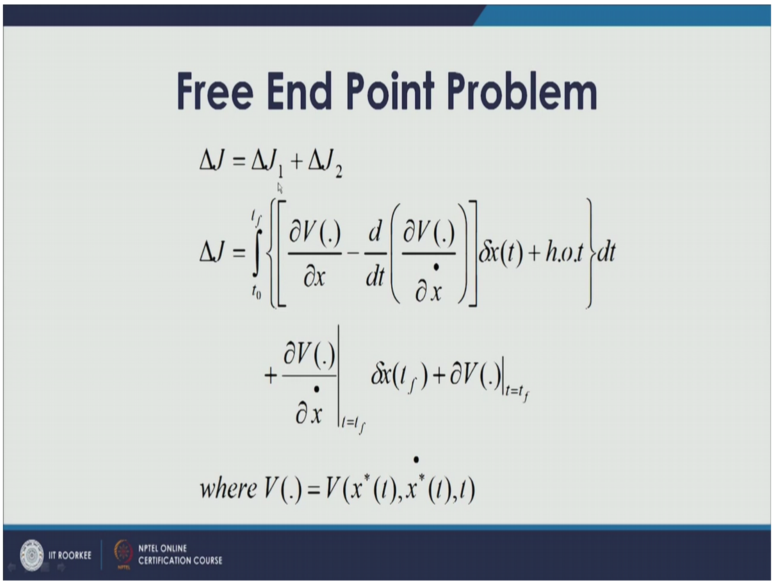

So, we can see that my delta J is nothing but delta J 1 plus delta J 2 where J 1 is t 0 to t f up to delta x of t f this is representing my J 1 and this is representing nothing but my J 2, where V dot here we can take as the x star t, x dot star t and t.

So, next we have to if we will see, this will have my boundary condition one is variation in delta x at t f and another is the variation at the delta t f say at this point delta t f is missing. So, like in this slide we are missing the delta t f here. So, this is multiplied with delta t f. So, this is my delta J 1 this is my delta J 2 summation of this will give me the delta J value. In this delta J value now we have to write delta x of t f, if we will see my final value is x f. So, I have to see whether delta x of t f can I represent in terms of the delta x f because by final termination is at t plus delta t f point where I will have this value as delta x f. So, next we will see can we relate this delta x of t f with delta x f.

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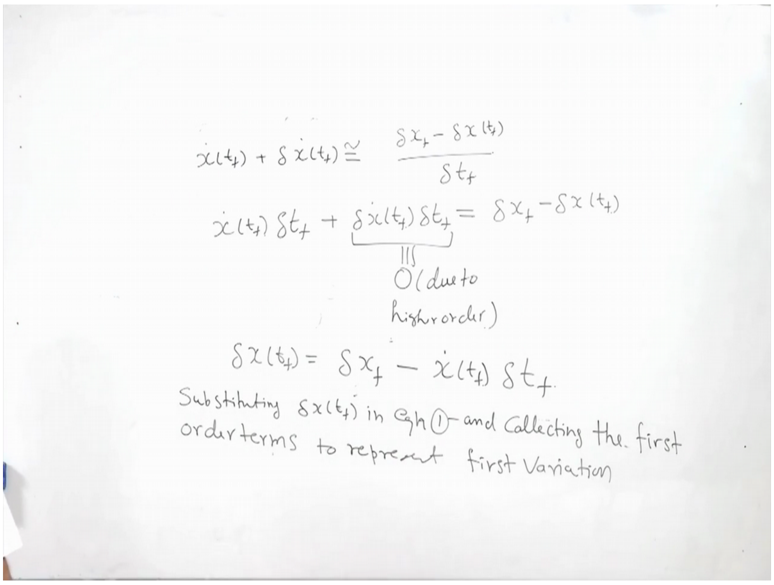

Say in this figure we can approximate x dot t f plus delta x dot t f as what. So, this is the variation at my terminal point at these points. Here we are trying to find out x dot plus delta x dot at the t f point. So, we can approximate this result as delta x f minus delta x of t f divided by delta t f. So, this is if we will see I am saying this is the delta x of f minus delta x of t f where delta x of t f is nothing but the value of the x plus delta x at the t f point and delta at t f plus delta t f point. My value is basically x f plus delta x f and this is x f plus delta x t f. So, this minus this divided by the time difference we are saying we are approximating this relation by this value. Cross multiplying I am having x dot of t f delta t f plus delta x dot of t f, delta t f equal to as my delta x f minus delta x of t f. So, I am trying to find out the value of the delta x of t f in terms of the delta x f.

Now if we will concentrate on this, this is the double variation we are finding at delta x dot and delta x t f. In our consideration we are considering only the first order terms. So, we can neglect this term or can approximate it to 0 value, this is due to higher order. So, we can say delta x of t f if I will try to find out this value this is nothing, but delta x f minus x dot of t f delta t f. So, this delta x of t f we have to place in this expression which we are saying this is my equation number one.

So, substituting delta x t f in equation one and collecting the first order term and collecting the first order terms. So, we will get nothing but the first variation, collecting the first order term to represent first variation. So, we are substituting the value of the delta x t f in equation number one, we are collecting the first order term so that we can represent the first variation.

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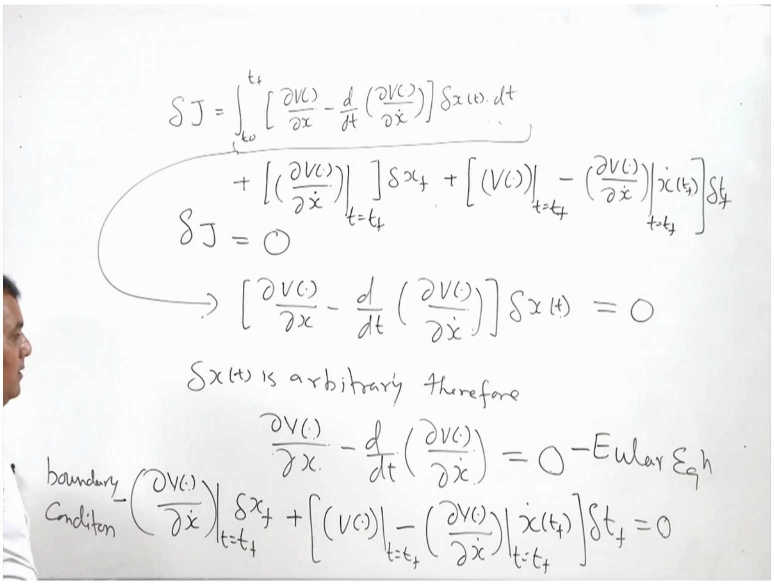

So, the first variation we can write it as, so delta j. So, what we are doing? We are substituting this delta x of t f as delta x f minus delta x dot t f in this equation. So, once we will place this value, it will have the one term with the x f other term with the delta t f. So, delta t f we will collect with the second term and the overall expression can be represented as integral t 0 to t f. So, the integral term will be del V by del x minus d by dt del V by del x dot this whole multiply with delta x t dt. So, this is the integral term plus once we will place it in to the equation number one, my next term will be del V dot by del x dot and this we have to evaluate at t f point multiplied with delta x f plus the delta t f term will contain the V dot evaluated at t equal to t f point minus del V dot by del x dot evaluated at t equal to t f point multiplied with x dot of t f and the whole expression multiplied with the delta t f.

So, this is my first variation which will contain the first one differential equation and the last terms will represent the boundary condition with the variation in x t f as well as the variation in the t f. Now to minimize our functional as per our fundamental theorem my first variation must be equal to 0. So, this delta J we can equate it to 0. So, equating this to 0 means what my integral term as well as the boundary condition, they all must be separately equal to 0 due to my fundamental remark as we discussed in the previous lectures. So, if delta J equal to 0, so I can say del V dot by del x minus d by dt del V by del x dot delta x t is equal to 0 and this we are writing from the first term. From this term we can write this. So, delta x t is arbitrary therefore, the coefficient of the delta x t that must be equal to 0 this means del V dot by del x minus d by dt del V by del x dot that must be equal to 0 and this is nothing, but my Euler equation.

Subjected to this I will have the boundary conditions this means del V dot by del x dot at t equal to t f point I am evaluating this with delta x f plus V dot at t equal to t f point minus del V dot by del x dot, again evaluating at t equal to t f point x dot of t f, delta t f that will be equal to 0. So, this is the boundary condition and this is my Euler equation to find out the optimum value of a given functional. So, if I will consider by a basic variational problem as a free end point problem I will lead to the Euler equation given by del V by del x minus d by dt del V by del x dot equal to 0, and the boundary condition is del V by del x dot delta x f plus del V sorry; V dot evaluated at t equal to t f point del V by del x dot evaluated at t equal to t f point x dot t f delta t f equal to 0.

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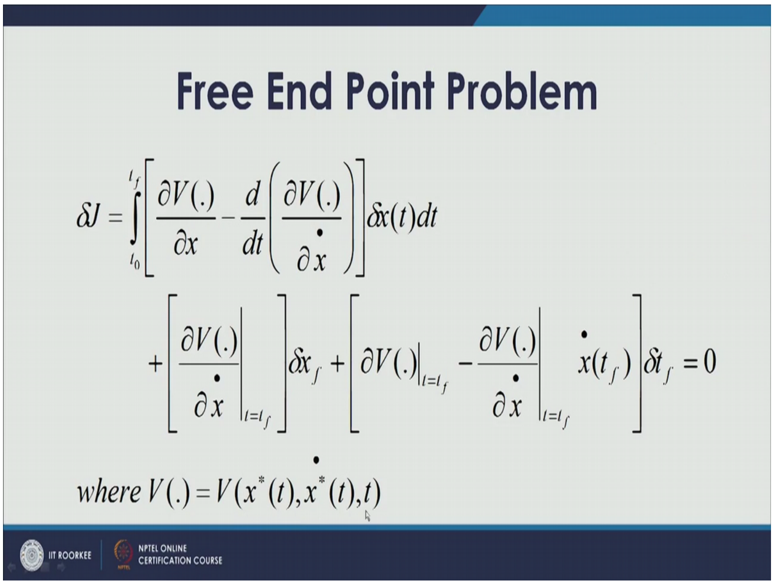

And we are V dot we are representing as nothing but the value of the V at optimal point x star and x dot star and t. So, this will we finally have these 2 equations.

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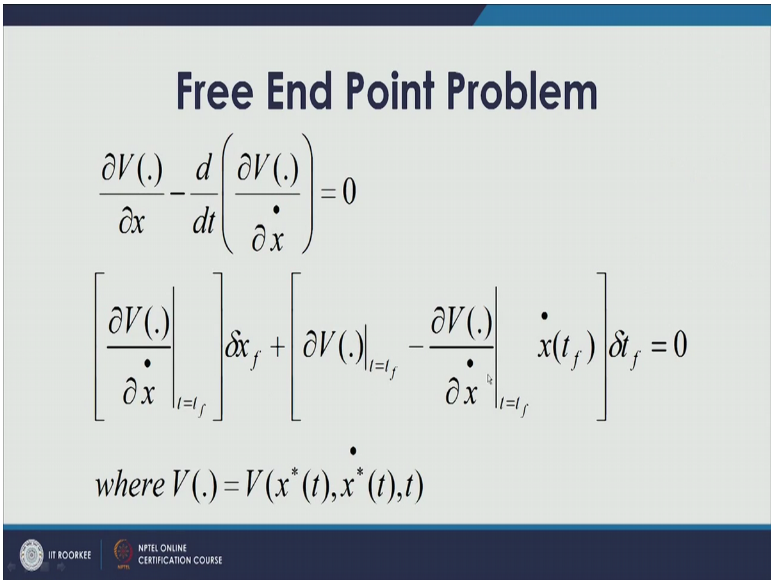

Now the free end point problem can appear in different manners. So, in the next we will see what may be the different cases as my final point is what we are saying, in this case we are saying my x t f is free. So, it can terminate to some other point simultaneously the t f is free. If you can recall in the first case we have taken that the fixed end point problem, in the fixed end point problem my initial and the terminal point was fixed. So, we will try to apply we will apply our different cases to my final result and see what may be the different cases may be for a free end point problem.

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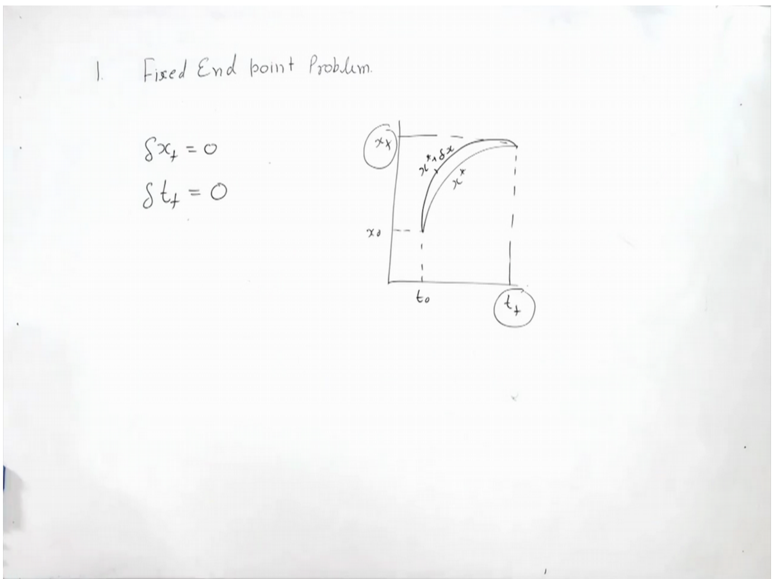

So, we take the first case which was the fixed end point problem. So, what is my fixed end point problem? I am given with the t 0 this is x 0 and let this is my x star, so my t f is also fixed as well as the final, x f is also fixed. So, in this case if I will have the variation as x star plus delta x, so if my terminal points are fixed I can say my first equation will be my Euler equation and the boundary condition as my terminal point is fixed. So, delta x f, delta x f will be 0 and with the same logic my delta t f will also be 0 because there is no variation in t f at this point and there is no variation in the x at this point. So, this particular case my these 2 will be 0 and to find out the minimal we have to solve only my given Euler equation.

The one example we have seen in the previous class once we are trying to find out the minimal length of the between the 2 points. So, these 2 points we already have been fixed. So, this problem already we have been taken previously. So, this general problem can be converted into the fixed end point problem if we will consider the delta x of f and delta t of f to 0. So, my second equation will not appear in that particular case.

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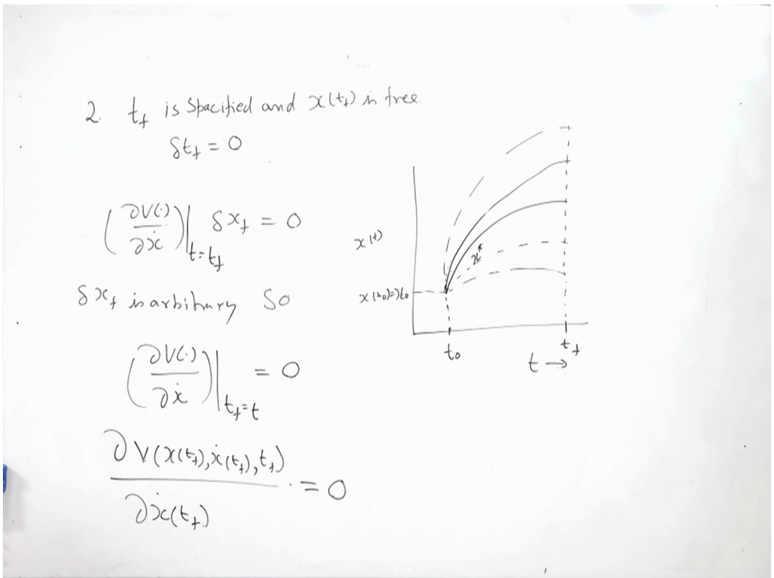

Next we will take the another case. So, in second case I have t f is specified and x t f is free. So, what is this case? If this is t, this is x t, so what we are saying; my t f is specified. So, this means let say this is t 0 this t f, so this t f is specified. In addition to this we are also given with the t 0, x t 0. So, my trajectory let say this is my optimal trajectory x star.

So, by the variation of this, this can terminate to any point because my x t f is free. So, if there is a variation on the x star. So, my final trajectory can terminate to any point on the t f line. So, in this case either t f is fixed, naturally my delta t f this delta t f will be 0. So, this means delta t f will be 0 in this case and my problem is, so end point is left with only the del V by del x dot evaluated at t equal to t f point delta x f equal to 0 and here delta x f is arbitrary. So, this means del V by del x dot t f t, we are taking as t f this will be 0 or this, we can also write as this means delta V x of t f, x dot of t f, t f by delta x dot of t f this must be equal to 0.

So, to solve such a problem first we will solve the EL equation given as del V by del x minus d by dt del V by del x dot equal to 0 this will lead to a differential equation which we have to solve subjected to the initial condition given as the t 0, x t 0 and the final condition if t f is fixed, x t f is free. So, this will represent my final point condition.

So, how to implement this? That we will see using an example, just to consider the case 2 we take an example.

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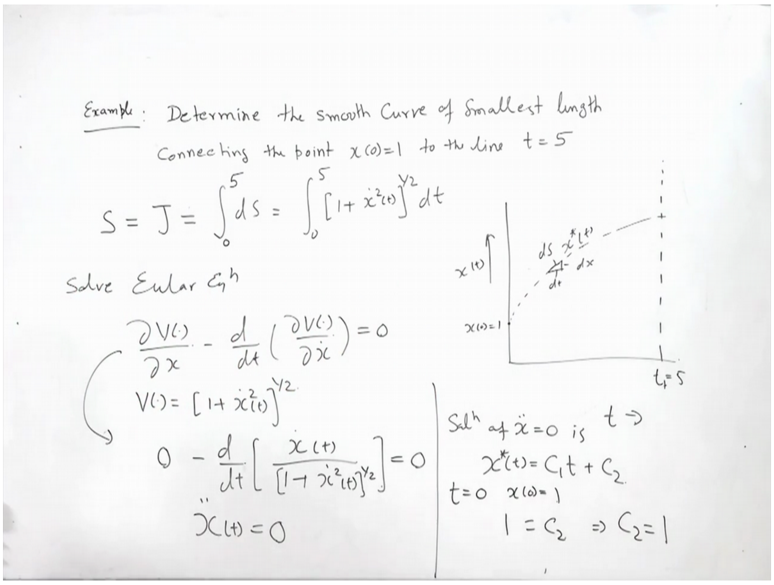

Our objective is to determine the connecting the point x 0 equal to 1 to the line t equal to 5. So, my problem here is I have to determine the curve of the smallest distance which is given as x 0 equal to 1 from this point to a line given as t f equal to 5 and this is my x t. So, this point can terminate to this let this is my optimal x star t.

So, if this is terminating to this my objective is to find the shortest distance, so that the x 0 equal to 1 will have the line t f equal to 5. So, in the first problem we have taken that between the 2 point we are trying to find out the shortest distance. So, this is my x t, I can write my distance which is nothing, but my objective function, in the similar manner as we have done it before from 0 to 5, with ds and what was the ds? My ds was 1 plus x dot square t square root into dt. So, in the similar manner we are taking a small ds if this is my dt and this is my dx and this is my ds. So, this ds integrating from 0 to 5 I am finding out what is the length of my curve. My objective here is I have to determine the minimal length of this curve means, what is my optimal x star t that is my objective to find out.

So, in this case my t f is fixed, but x t f is free for the minimal it can go to anywhere. So, in such case my first step is, first I will solve Euler equation which is del V by del x minus d by dt del V by del x dot that must be equal to 0. In this case I am given with the V is nothing but 1 plus x dot square t. So, if I will find use V here in EL equation. So, what this will give me del V by del x it is independent of this 0 minus d by dt of x dot t 1 plus x dot square t this must be equal to 0 and this nothing, but can be written as x double dot t equal to 0. So, the solution of this Euler equation giving me x double dot equal to 0.

Now, the solution of this, solution of x double dot equal to 0 is this we can directly write as x t equal to C 1 t plus C 2 and nothing but this is my optimal x star. So, this optimal x star is in the form of the C 1 and C 2. So, my objective here is next to find the value of the C 1 and C 2 using the end point conditions. My initial condition is given with to be t equal to 0, value of the x equal to 1 at t equal to 0 at x 0 equal to 1. So, this means 1 equal to t 0 this is 0. So, C 2 this means my C 2 is nothing but equal to 1. So, the one constant C 2 I can directly find out using my initial condition.

My next aim is to find out another constant C 1. C 1 we can find out if my terminal condition is given. So, in this case what is my terminal condition? My delta t f is 0, so del V by del x dot t f that must be equal to 0.

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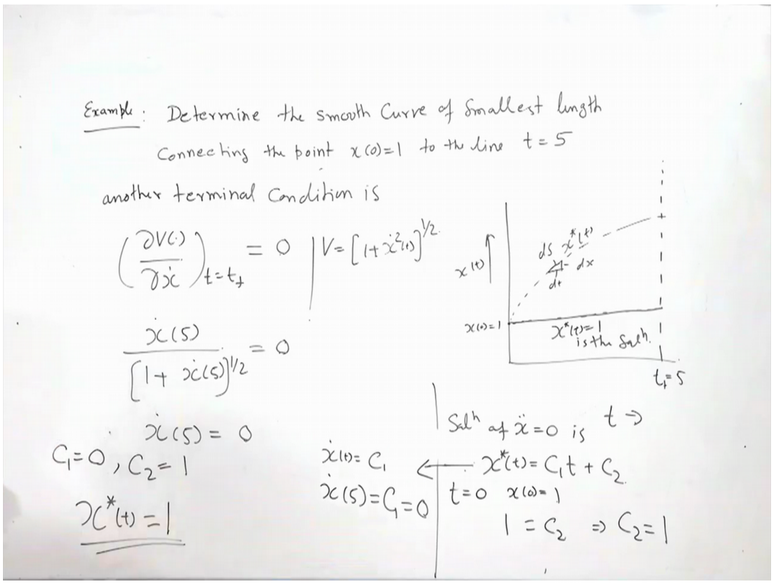

So, another terminal condition is del V dot by del x dot which we are evaluating at t equal to t f point that is equal to 0. I was given with the V as 1 plus x dot square t 1 by 2. This is V, so del V by del x dot if I will take at t equal to t f point this is nothing but because t f is 5, 1 plus half equal to 0 this means this is giving me x dot 5 equal to 0, x dot 5 equal to 0, but this is my x t if I will differentiate this, so this give me x dot t as nothing, but C 1.

So, this means x dot 5 is nothing but my C 1 and which is the value 0. So, I have my 2 constant as C 1 equal to 0 and C 2 equal to 1. So, my optimal trajectory will be x star t is nothing but as C 1 is 0 C 2 is 1. So, x star t equal to 1 is my solution of this and this we can see it here the minimal distance between this point is nothing but a line parallel to t axis and this is my x star t equal to 1 is the solution. So, by this we find the optimal solution for my given problem is x star t equal to 1 and this x if we will use in this, so I can find out the minimal length of this my value of the J with x star t equal to 1, J can be determined which give you the what is the minimal length between these 2 points.

So, we stop this lecture here and we will continue the free end point problem in the next class.

Thank you very much.