Optimal Control
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Lecture – 05
Free End Point Problem

Welcome friends in this class. In the previous class we have discussed about the fixed end point problem. We are trying to find out the optimal value of a given functional subjected to the end points are fixed. Today we will see the another problem which we can say the free terminal point problem in which t 0 is fixed or the initial point is fixed, but the final point is free, free with respect to the time and with respect to the x. So, both x and t we will consider to be the free then we will see the different cases which can arise in such a problem.

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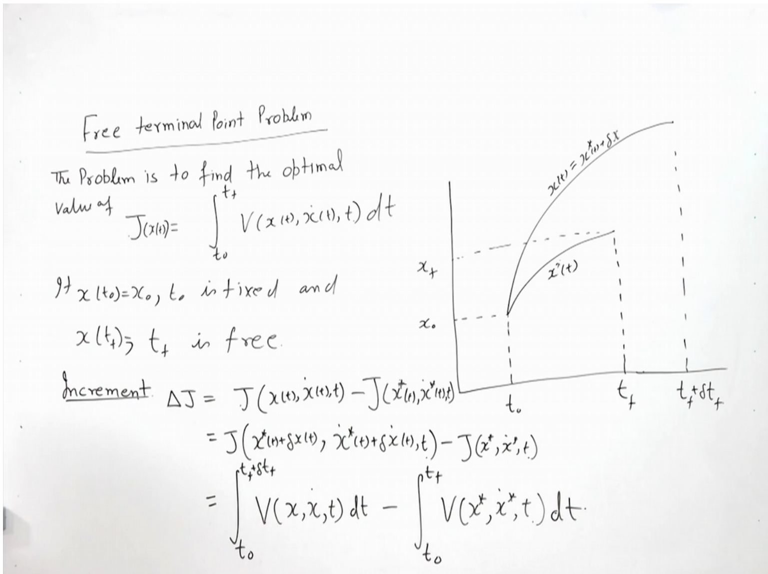

So, we will have a free terminal point problem. So, we can define t 0 and x t 0 value of x at the t 0 is fixed, I define this by optimal trajectory x star and this is the variation x t can be simply written as x star t plus delta x of t. My optimal trajectory is terminating it t f point and see the trajectory will go beyond the t f point t f plus delta t f at t f point let by optimal trajectory will have value x f. So, the problem is to find the optimal value of J my performance index which is simply defined as t 0 to t f V x or in more general form we can write it as like J of x t, V of x t, x dot of t t dt, if x t 0 x 0 and t 0 is fixed and x t f t f is free. So, this is our problem, we have to find the optimal value of the J if by boundary conditions x t 0 x 0 and t 0 is fixed x t f t f is free.

So, basic approach to solve this problem is same as we have discussed for the fixed end point problem means first a of all we have to find the increment, from this increment we will write the first variation as for the fundamental theorem first variation must be equal to 0, from there we will get the conditions for this V to be optimal for the given boundary conditions.

So, our first step is to find out the increment, how we define the increment? Delta J is nothing, but the value of the J at x t, x dot of t t minus value of the J at x star of t x dot star of t t. What is x t in this case? This is x star plus or in other way I can also write this as x star of t plus delta x of t x dot is star of t plus delta x dot of t, at delta x is there. So, my t is nothing but t minus J I am dropping the t x star x dot star t. So, this J star is I can directly write because in this case my t is varying from t 0 to t f, but once am writing J for x star plus delta x t my time is varying from t 0 to t f plus delta t f. So, this can be written as integral t 0 to t f plus delta t f V x x dot t here x is x star plus delta x. So, I am simply writing the V this into dt minus for J x star this is t 0 to t f V x star, x dot star and t dt. So, this equation gives me the increment in J which we can find. So, my increment is delta J is given by this equation.

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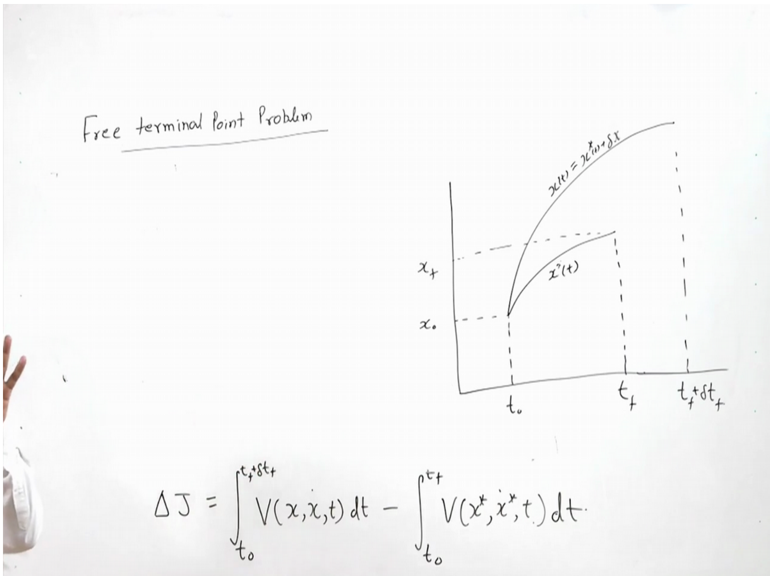

So, now in the next what we are doing. Here we have the integral limit from t 0 to t f plus delta t f; this will break in t 0 to t f and then t f to t f plus delta t f.

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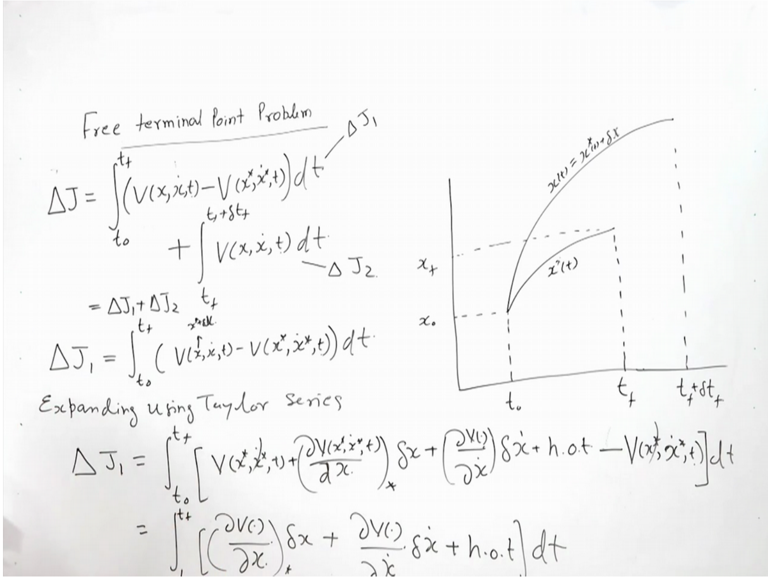

So, this integral we are breaking into the 2 integral - the first integral we can plug with the second one because both will be from t 0 to t f or I can write my delta J as t 0 to t f, V x, x dot, t minus I am writing for this V x star x dot star t dt plus the breakup of this from t f to t f plus delta t f V x x dot t dt. Now if we will see this is the first integral is similar to what we will have in our the fixed end point problem. So, this we will say let say by first integral give me the delta J 1 and the second integral give me the delta J 2.

So, first we will take. So, my delta J I am simply writing is delta J 1 plus delta J 2. So, I will we will each term independently let us say delta J 1 is given as my t 0 to t f. So, this V x x dot t minus V x star, x star dot, t dt because this x is nothing, but my x star plus delta x. So, expanding this term using Taylor Series, Taylor Series what we will get. So, delta J 1 t 0 to t f we are simply going to get V x star x star dot t plus V is the function of x star x dot star t by del V by del x obtained at optimal point into delta x plus del V by del x dot I am writing the dot here the simply because I have the same value into delta x dot plus higher order term minus V x star x dot star t dt.

So, these 2 term will get cancelled out and we are left with del V by del x delta x plus del V by del x dot delta x dot plus some higher order terms into dt. So, if we will see for J 1 I will get the same expression as we are getting in the fixed end point problem. So, delta J

1 I can represent it by this. So, now, in this case I will expand this term using integration by part.

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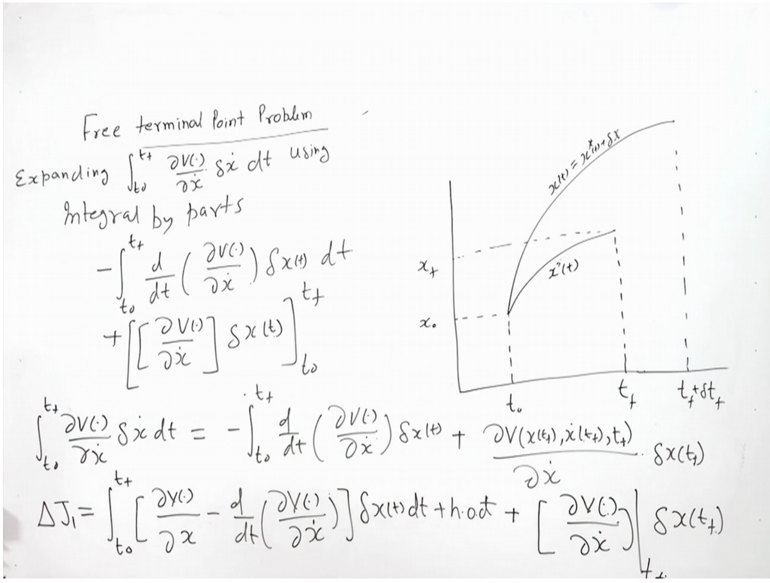

Expanding t 0 to t f del V by del x dot delta x dot dt using integral by parts. So, the second term gives us. So, we will get plus del V by del x dot into delta x t t 0 to t f, as my initial point t 0 is fixed. So, delta x t 0 will be 0 and this term means I have to evaluate only at the point t f. So, my this second term t 0 to t f del V by del x dot delta x dot dt can be written as minus t 0 to t f d by dt del V by del x dot delta x t plus del V at x of t f x dot of t f, t f by delta x dot into delta x of t f.

So, this will be the second term which we are expanding. So, in J 1 expression if I will club this, so the total J 1 expression I can write is. So, delta J 1 can be written as t 0 to t f del V by del x minus d by dt del V by del x dot delta x t dt plus higher order terms plus del V. So, this, the value of the V by del x dot and this whole is evaluated at t f point multiply delta x of t f. So, the first delta J we have divided into the 2 terms J 1 and J 2. So, delta J 1 can be written as given by this last expression. So, the J 1 we have written by this.

Now next we will see how we can write the second integral this we will keep with us the last part.

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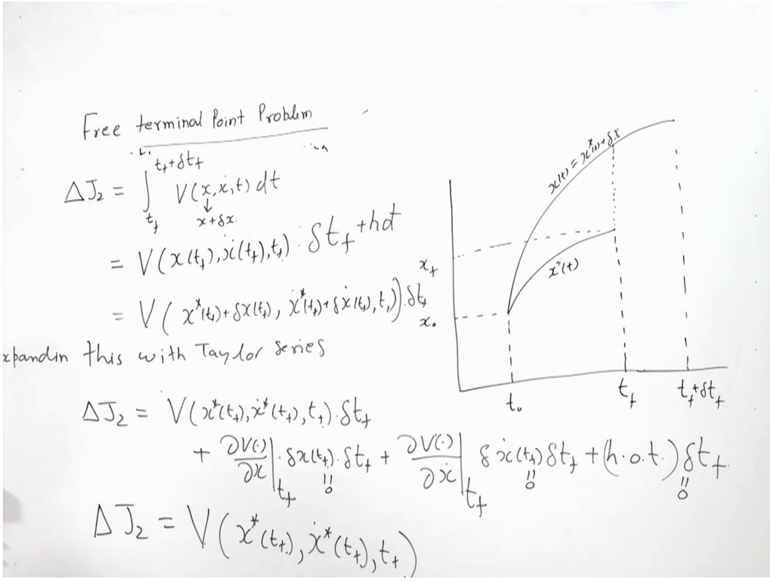

So, what we have taken as the J 2? So, delta J 2 is considered as this is my integral from t f to t f plus delta t f V x, x dot, t dt. What actual this integral represent here? Say to find this value let us see this, we are integrating from the time t f to t f plus delta t f and V is x is my x plus delta x. So, this means x plus delta x at t f point is this t f plus delta t f point is this. So, I can say this is nothing, but area under this curve.

So, we can recall that the delta J 2 we are representing as t f to t f plus delta t f V of x, x dot, t dt. So, what this integral represent here? My limit is from t f to t f plus delta t f and the V value which is dependent on the x which is x t, x star delta t to this. So, at this point the value of the x will be the value x star of t f. So, this integral is nothing but the area under this curve and area under this curve can be simply represented as V x star of at the point t f x dot of t f, t f into delta t f plus some higher order terms, sorry. So, this is x of t f, x dot of t f, delta x of t f, at t f point we will have the variation into this. So, this can be approximated as V x star of t f plus delta x of t f x dot star of t f plus delta x dot of t f, t f. So, this can be written by as given here.

Now expanding this with Taylor Series what we can write for the J 2 this is V x star of t f, x dot star of t f, t f multiplied with delta t f which I have missed here delta t f. This is the first term plus del V by del x evaluated at the t f point into delta x of t f, delta x of t f multiplied with delta, t f plus del V by del x dot evaluated at the t f point, delta x dot of t f delta t f plus what is my higher order terms and multiplied with delta t f.

So, if we will see in this delta V multiplied with the delta t f is the only first order term here, all other terms are the second order and the higher. So, this can be approximated to 0 this can be approximated 0 or we can neglect this, this can be approximated to 0. So, delta J 2 simply can be represented as V x star of t f, x dot star of t f and t f. So, we are evaluated the delta J 1 and delta J 2.

Today, in this lecture we stop here. In the next lecture we will drive the complete expression for the J and complete our free end point problem in the next class.