Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture - 40
Time Optimal Control System (Continued)

Welcome friends to our last session of this course. In the previous session we have started our discussion on the Time Optimal Control System, and we have seen that for a linear time invariant system x dot equal to A x t plus B U t in which my U is bounded as mod of U t is less than or equal to 1.

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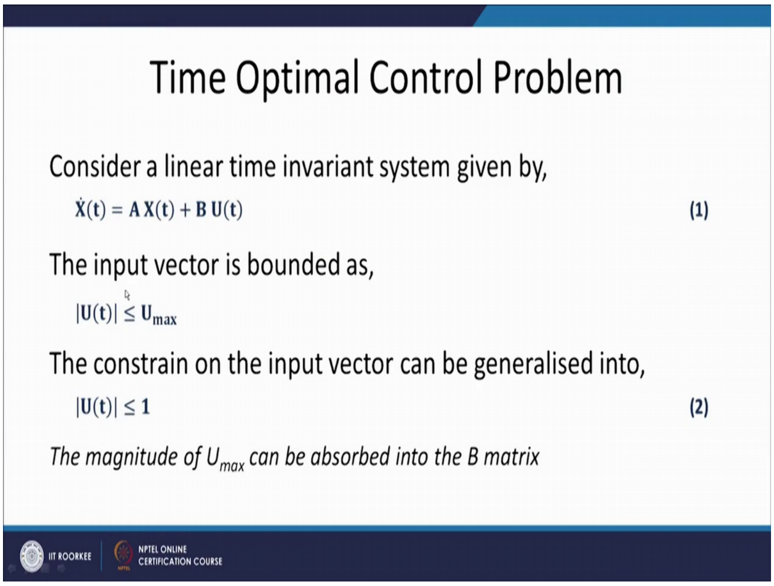

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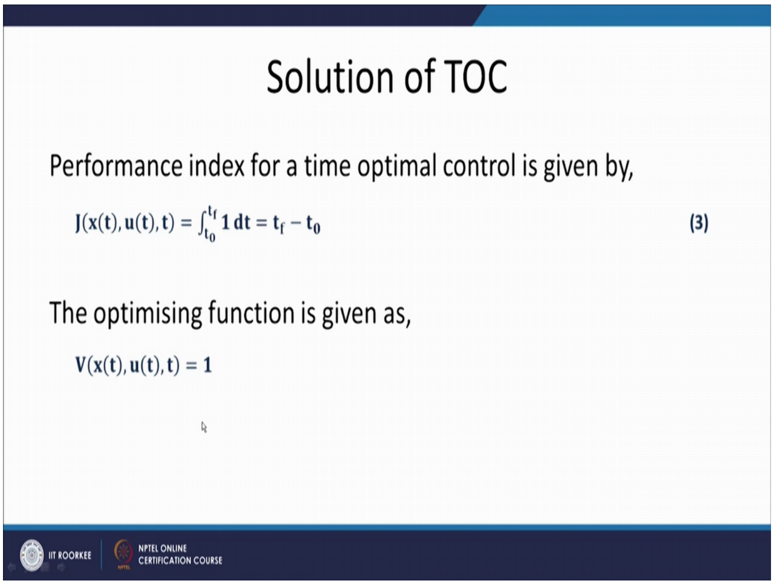

We can define our Hamiltonian as H 1 plus lambda prime t A x t plus B U t.

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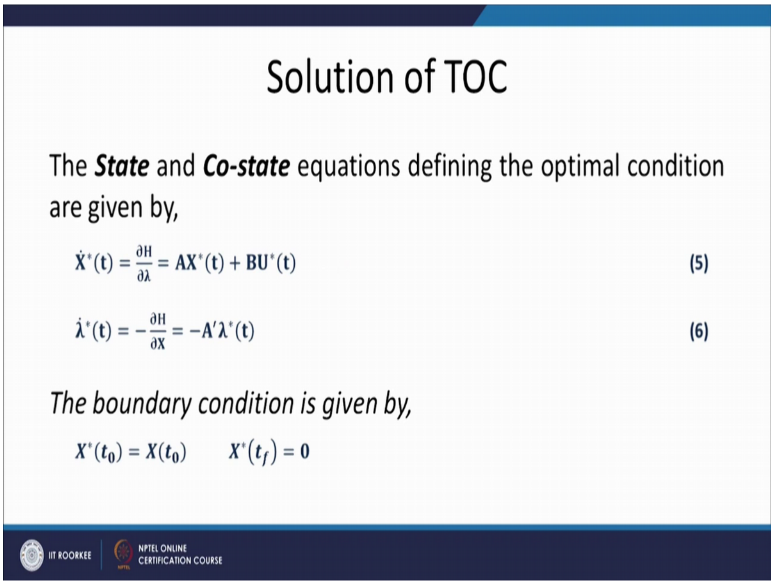

We got our state and the co-state equation co-state equation is lambda dot t as minus a prime t lambda star t. So, this means this equation can directly be solved in terms of the lambda t and the Pontryagin minimum principle we have applied as my Hamiltonian 1 plus A x t lambda dot t plus U prime t must be less than or equal to a journal value of the U if U is as U star and U is the journal value of the u.

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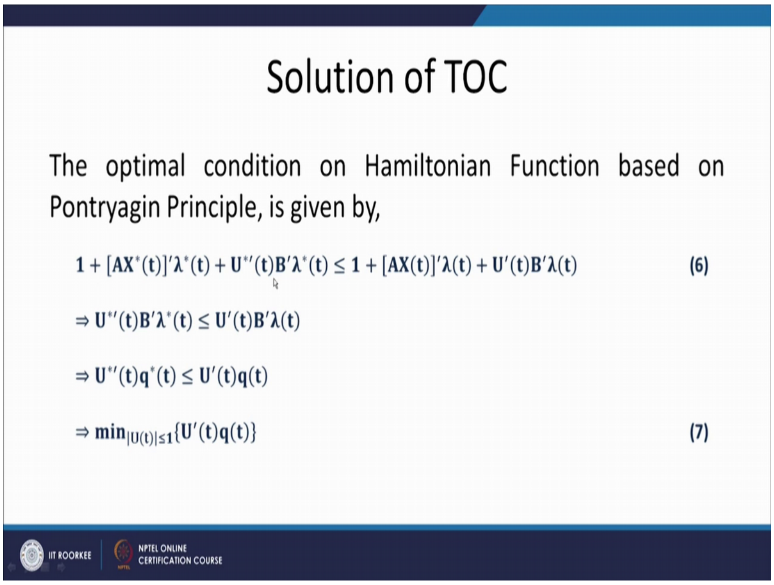

So, this will be less than this and we have shown that this can be minimized by selecting the U as plus 1 or minus 1 depending upon the value of the q t and q t is nothing but our B transpose lambda t. So, this B transpose lambda t will decide my switching value q from 0 to 1 and we have shown that my system should be completely controllable.

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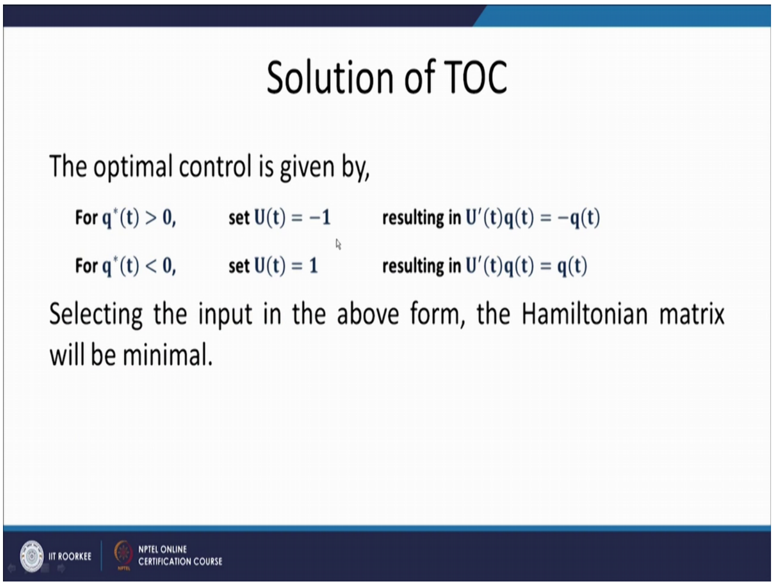

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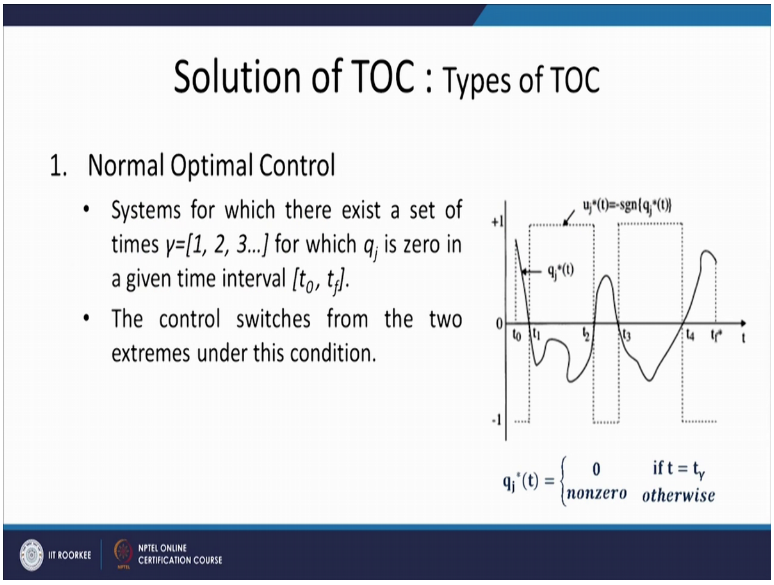

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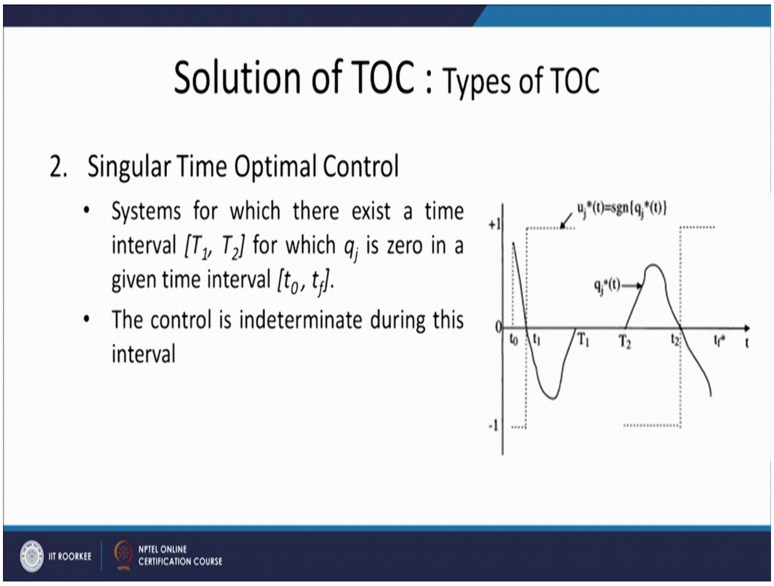

This means no singular time optimal control system can be controlled for minimum time because here the U is undetermined between the period T 1 and T 2.

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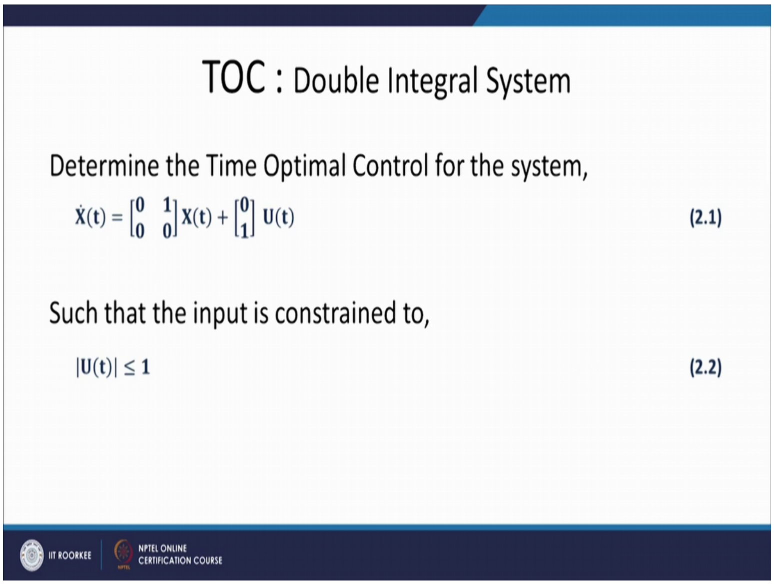

In the next we will see the application of a time optimal control system to a double integral system and if you will model the double integral system in state space it can be represented as x dot t equal to 0 0 0 1 x t 0 1 U t. So, by this we can define a double integral system U is bounded as less than or equal to 1.

So, if we will first check the controllability of the system you can try yourself then it shows my system is completely controllable because to check the controllability.

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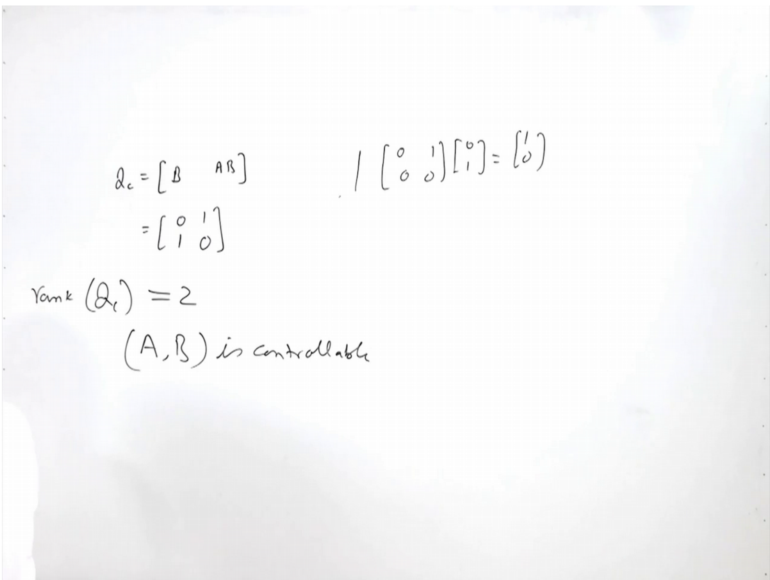

We have the controllability matrix B A B for this system my B is 0 one and A B will be A is 0 1 0 0 and B is 0 1. So, what we will get 1 and 0. So, this is 1 0 and the rank of q c is that is clear from this, this is equal to 2. So, a B pair is controllable. So, it is simple to show my A B is controllable this means I can apply the time optimal control solution to this problem.

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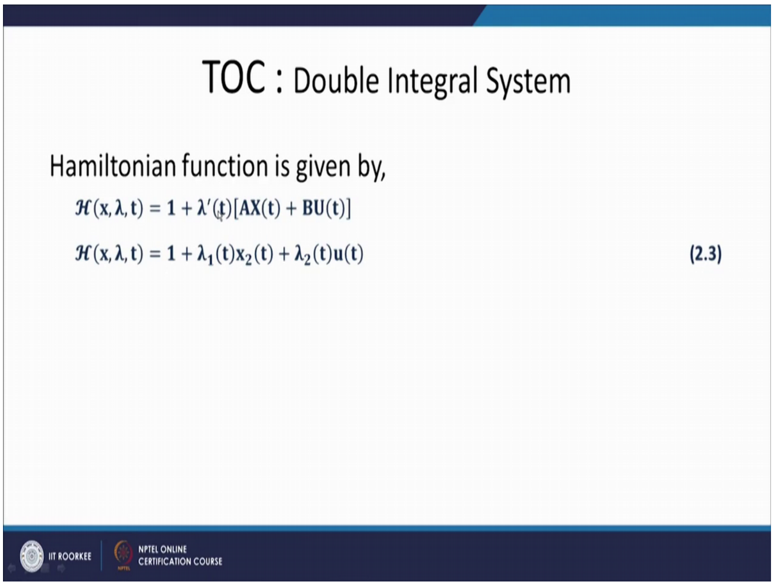

So, as a first step I will write my Hamiltonian as 1 plus lambda prime A x t plus B U t and if I will write the value of A and B. So, my H will comes out to be 1 plus lambda 1 t x 2 t plus lambda 2 t U t objective is to minimize this Hamiltonian this means I have to check the value of lambda 2 t it should be if lambda 2 t is negative U will be positive if lambda 2 t is positive U will be negative.

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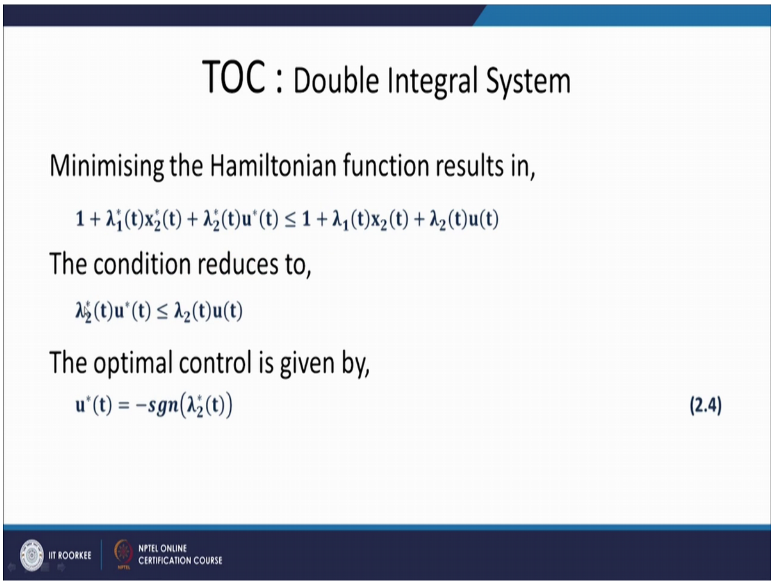

So, this means I can decide my U t as the signal function of lambda 2 t because the sign of lambda 2 will decide the sign of my U if this is positive U will be negative and if this is negative U will be positive.

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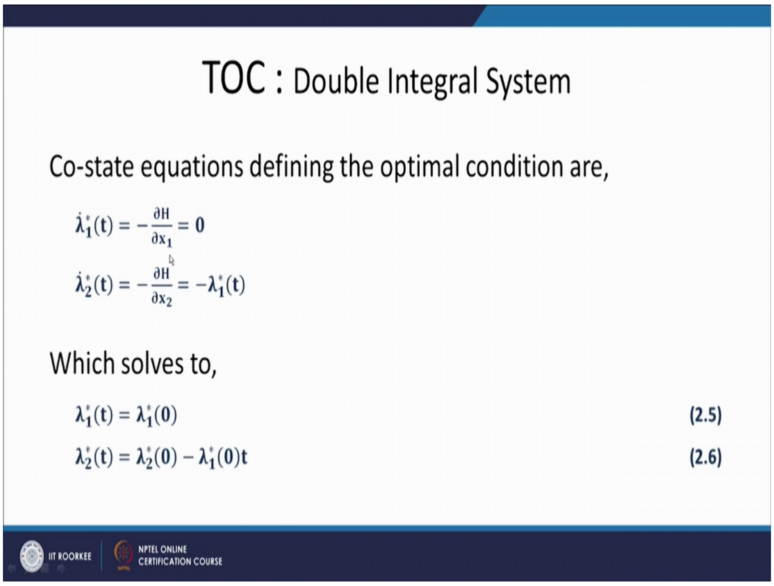

So, these are I have lambda 1 t as del H by del x 1 means this is my co-state equation.

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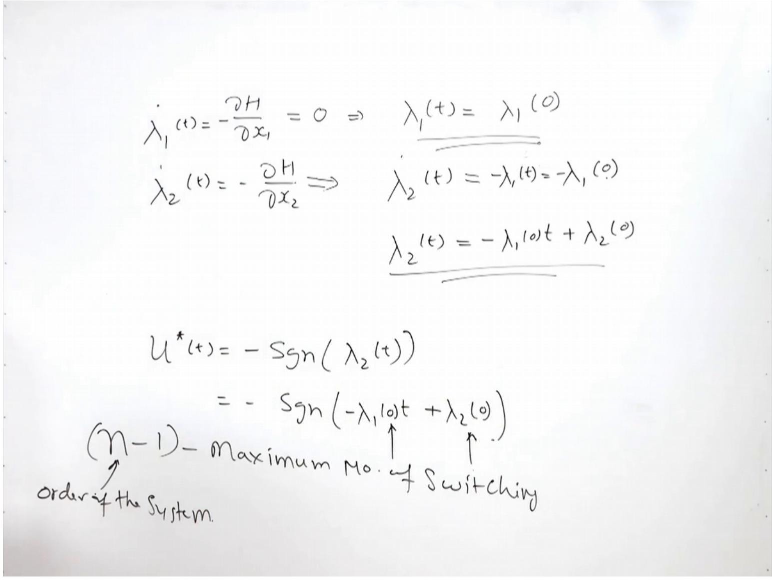

This is equal to 0 this means I am writing my lambda 1 dot t as del H by del x 1 with negative sign and what is my H if you will see my H is one plus lambda 1 t x 1 plus lambda 2 t U t. So, this is H is not a function of x 1. So, this means my lambda 1 t will be 0 this implies the solution of this is lambda t is sorry lambda 1 t is nothing but lambda 1 0 where lambda 1 0 is my initial value of the lambda 1.

Similarly, my lambda 2 dot t is minus del H by del x 2 this implies my lambda 2 dot t is what means del H by del x 2. So, this is nothing but lambda 1 t. So, this is nothing but my lambda 1 t which is nothing but my lambda 1 0. So, if I will write the solution of this; this is lambda 2 dot t which will be sorry this is negative sign. So, this will be negative this will be negative. So, minus lambda 1 0 t plus lambda 2 0 will be the solution of. So, this is the solution of lambda 2 and this is the solution of my lambda 1.

So, this we are representing here my lambda 1 t solution is lambda 1 0 and lambda 2 solution is lambda 2 0 minus lambda 1 0 t.

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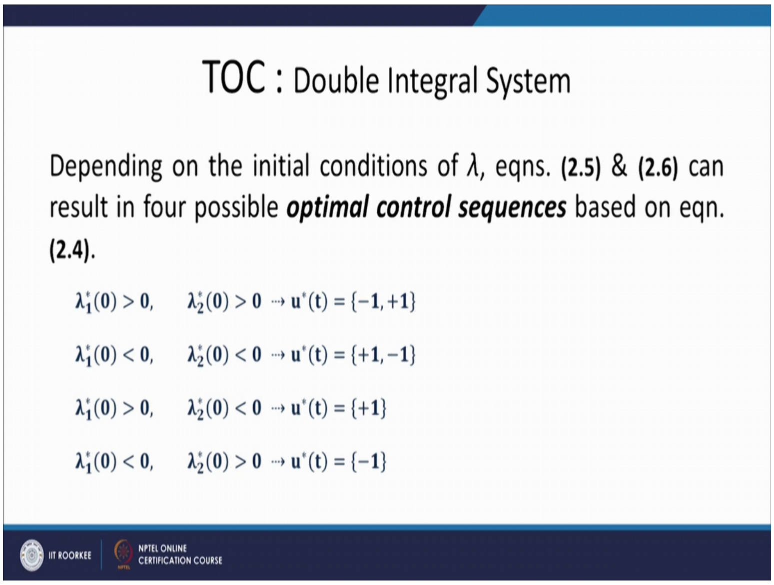

So, means now we have to decide what should be the value of switching my what is my control law control law is U t equal to minus signum of lambda 2 t this means we have decided my optimal control as signum function of lambda 2 t this means lambda 1 0 t with negative sign plus lambda 2 0. So, what the switching point will depend what is the value of lambda 1 0 and lambda 1 lambda 2 0.

So, now depending upon the initial condition of lambda my these 2 equations means lambda 1 t equal to lambda 1 0 or lambda 2 t equal to lambda 2 0 minus lambda 1 0 t. So, my U t is nothing but the signum function of lambda 2 t which can be written as minus lambda 1 0 t plus lambda 2 0 t. So, based on the initial value of lambda 1 and lambda 2 we can define the switching.

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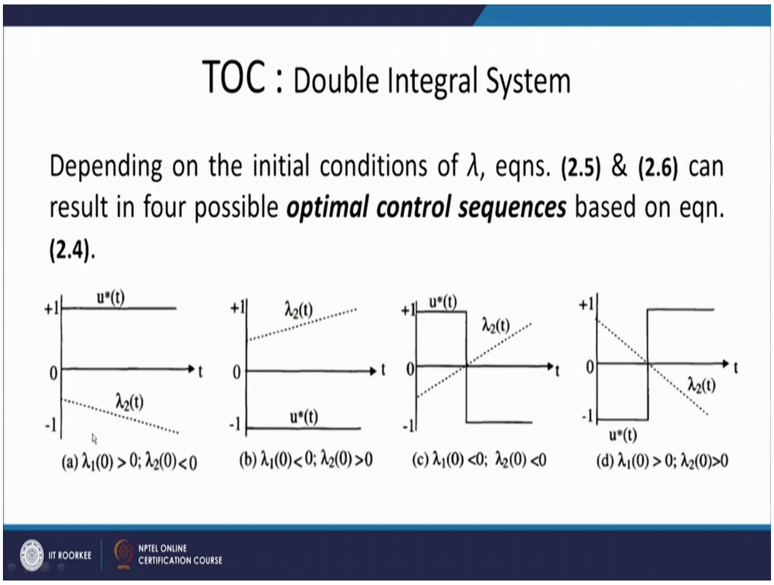

See lambda 1 is greater than 0 lambda 2 is less than 0 means lambda 2 is negative if lambda 2 is negative lambda 1 is positive. So, this means the whole term will be negative. So, my U always will be positive in this case.

So, if my initial condition exists in this way my U will always will be positive the second case if you will see lambda 1 0 is less than 0 and lambda 2 0 is greater than 0 means lambda 2 is positive in this case lambda 1 is negative if lambda 1 0 will be negative. So, my; this term will be positive and the whole term will be positive. So, lambda 2 t always remains positive in this case. So, U will be always negative another is if lambda 1 0 is less than 0 and lambda 2 0 is also negative. So, what we are saying lambda 2 is sorry lambda 1 is negative, lambda 2 is also negative. So, my; this term will be negative, this term will be because this is negative. So, this will be positive.

So, this is a line with positive slope. So, my lambda 2 t will be a positive slope. So, initially at t equal to 0 if I will see my lambda 1 is lambda 2 is negative lambda 1 is lambda (Refer Time: 10:38) will be 0. So, will start with the negative and with respect to time this slope will go in the; with a positive value and sometime it will cross only once the t axis. So, once my lambda 2 is negative my U is positive and this will be 0 it will switch to negative. So, my switching in the U will be positive to negative when both are negative.

So, U is plus 1 then will become minus 1 if lambda 1 is greater than 0 lambda 2 is greater than 0 this means my lambda 1 and lambda 2 both are positive value with respect to t at t equal to 0 I have the positive value only. So, I have a negative slope of the line as shown in the last figure here. So, once this is positive your U will be negative and then this will switch to the positive value. So, this we can see here if both are positive then U is minus 1 and then plus 1 this means if both are positive U is initially minus 1 then switch to plus 1 both negative then U will switch from plus 1 to minus 1 as we have seen in this; this figure representing this case lambda 1 positive lambda 2 negative I have plus 1 this is representing this case and lambda 1 negative lambda 2 positive U will be minus 1 as represented in this case.

So, this basically this will decide the switching and in a time optimal control system maximum number of the switching will be n minus 1 is the maximum number of switching where n is the order of the system. So, we have considered a second order system. So, maximum switching is one. So, in the one switching this means I can transfer my n initial state to origin this we have to show. So, this is the switching we have seen.

Now, we will see how this switching can take place. So, for a given system we have to consider the phase plane trajectory.

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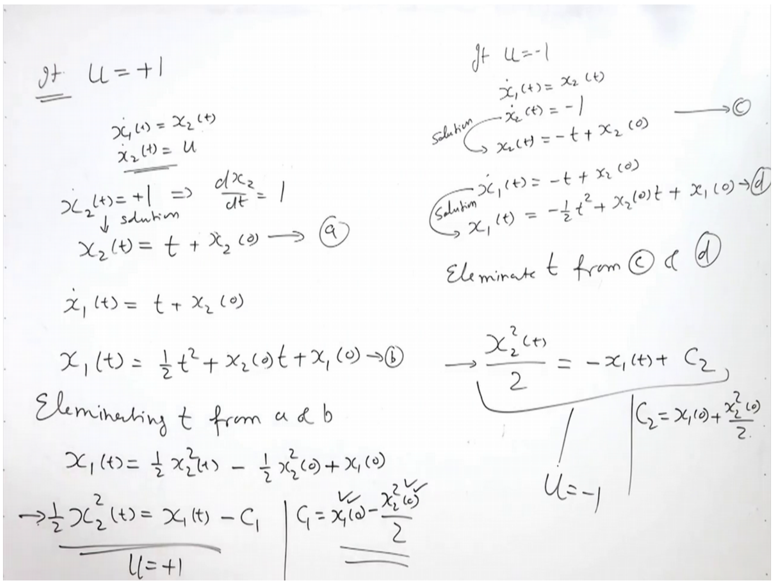

So, this means my U can switch either to plus 1 or to minus 1 now we take if U is one then what will be my system my system is x 1 dot t is x 2 and x 2 dot t is nothing but u.

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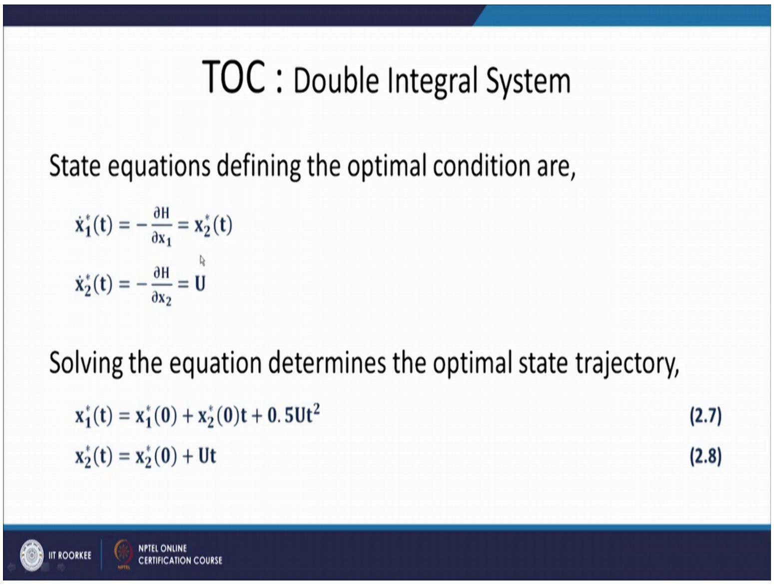

So, x 1 dot t is x 2 t and x 2 sot t is nothing but U this is by system a double integral system which we have considered. So, in this case if you will see I have x 2 dot t U we have considered as plus 1. So, the solution of this is. So, I can simply write my x 2 t is nothing but x 2 0.

So, this is the solution of my second equation say this is one this means my d x 2 by d t equal to 1. So, the solution of this will be t plus x 2 0 where x 2 0 is nothing but my constant which we are determining. So, the solution this will be t plus x 2 0 if I will write my x 1 dot t this is my x 2 t. So, I can write this as the t plus x 2 0. So, the solution of this equation will be 1 by 2 t square plus x 2 0 t plus x 1 0 where x 1 0 and x 2 0 is nothing but my initial conditions. So, this is by evaluating the integration time constant I am writing these values.

So, from this equation, so, I can say this is my equation A this is my equation B. So, I can eliminate t from these 2 equation eliminating t from A and B I can write x 1 t as 1 by 2 x 2 t square minus 1 by 2 x 2 square 0 plus x 1 0 or by rearranging this I can write my equation in terms of x 2 square 1 by 2 x 2 square t as x 1 t minus c 1 where c 1 we consider as x 1 0 minus x 2 square 0 by 2.

So, I will solve these equations with U equal to plus 1 I will get equation in x 1 x 2 as half of z 2 square as x 1 t plus sorry minus c 1 where c 1 is my constant which will depend upon my initial condition. Similarly, we can solve if U equal to minus 1. So, my equations will become x 1 t sorry x 1 dot t this remain x 2 t, but x 2 dot t is U which will become now minus 1.

So, x 2 t the solution of second equation, so, the solution of this will give me x 2 t as minus t plus x 2 0 substituting x 2 in x 1 dot t. So, x 1 dot t is nothing but my t plus x 2 0. So, the solution of this will become x 1 dot t equal to minus 1 by 2 t square plus x 2 0 t plus x 1 0. So, this is sorry this is the solution of this equation. So, the solution of x 1 sorry x 2 dot t equal to minus 1 x 2 t is minus t plus x 2 0 and the solution of x 1 t as minus t plus x 2 0 is x 1 t equal to minus half of t square x 2 0 t plus x 1 0 t.

Let us say this is my c equation this is d equation similarly eliminating t from c and d eliminate t from c and d equation we get a equation in x 1 and x 2. So, I am directly writing this I can write this as x 2 square t by 2 this is minus x 1 t plus c 2 and c 2 will be x 1 0 plus x 2 square 0 by 2. So, this will be my another trajectory equation for with U equal to minus 1 now if we will make a plot of x 1 x 2 with U equal to plus 1.

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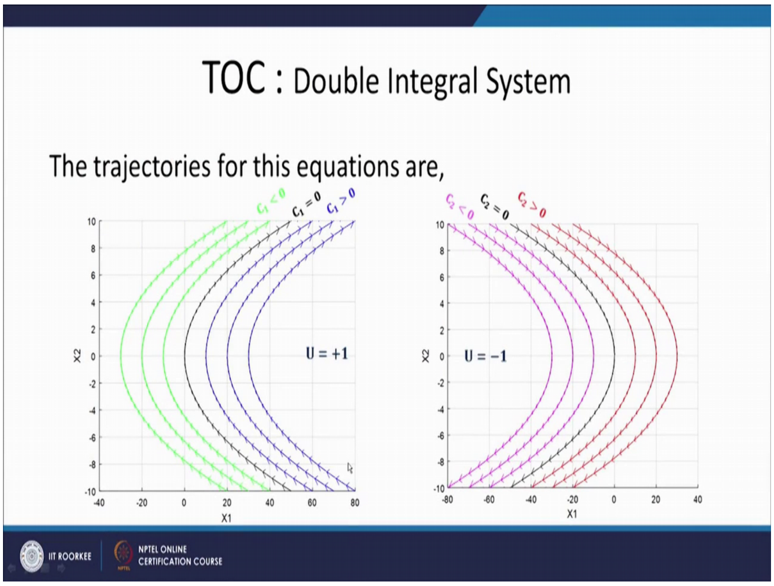

So, now, if U equal to plus 1 and my x axis is the x 1 y axis is the x 2 this means I am trying to plot this equation for U equal to plus 1 and I will plot this equation with U equal to minus 1.

So, this is nothing but the trajectories with if U equal to plus 1. So, this is nothing but my x 1 x 2 trajectory which will move in as a parabola if c equal to 0 this line represent the c equal to 0 if c is greater than 0 my parabola will be on this side and if c 1 is less than 0 my parabola will be on this side and where the c 1 depends c 1 depends upon the value of x 1 0 and x 2 0.

Similarly, for U equal to minus 1 my this equation for U equal to minus 1 and here again if you will plot the trajectory this is the reverse parabola I am getting at c 2 equal to 0 less than 0 greater than 0. So, my; these 2 equation give me the x 1 x 2 trajectories in terms of the parabola.

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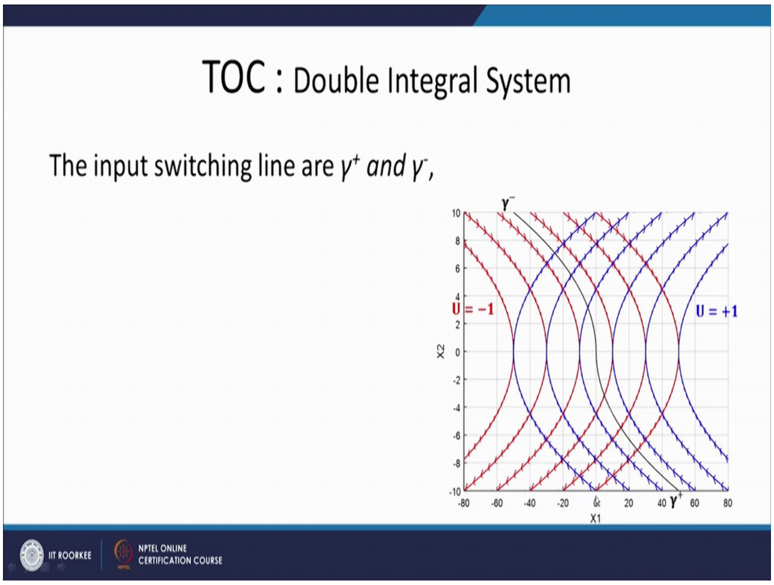

Now if I will draw this 2 on a same graph. So, my whole of this graph will basically cover my initial condition initial condition is x 1 0 and x 2 0 they can adjust anywhere in this space and if you will simple concentrate here this is the value of my this is the 0 axis 0 axis are at this particular point. So, with this; this means this means my; this whole space my initial value can adjust.

Now, I have to see if I have my initial value anywhere in the space what type of the control I can have.

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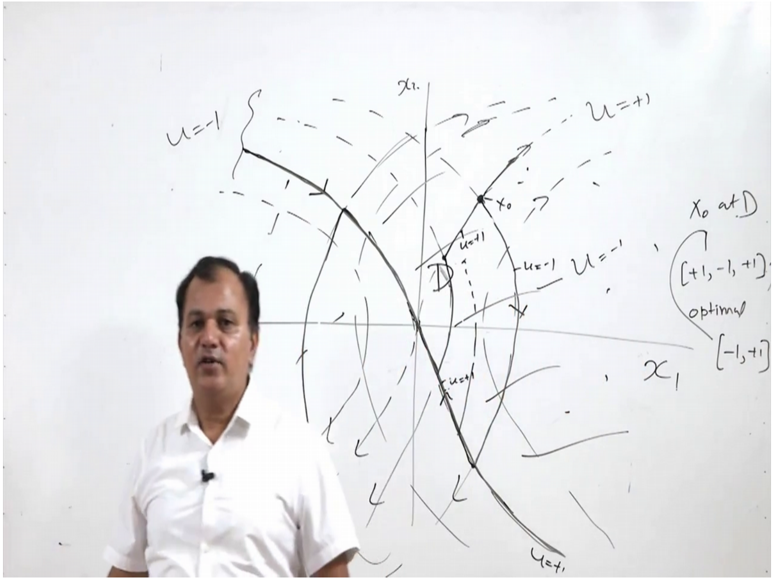

So, let us draw this say if this is my space for U equal to plus 1 I have parabolas like this I will take only the 2 case if U equal to. So, these trajectories are spreaded over. So, something like this; this will go in this way; in this way. So, this is U equal to plus 1 and this is U equal to minus 1 suppose my trajectory sorry initial point adjust at this point this is my x 0 it will have the value of the x 1 and the value of the x 2.

So, what should be my switching at this particular point this means if it is U equal to plus 1. So, this means my trajectory will go away from this. So, my control cannot be positive and if I will apply the U equal to minus 1 my trajectory will move in this way reach to the switching line here and then move to this way. So, this will be the switching point.

So, what we are saying? See, now my all these trajectories are spreaded over the throughout the curve with U equal to minus 1 as shown here in the blue line and U equal to minus 1 as shown by the red line.

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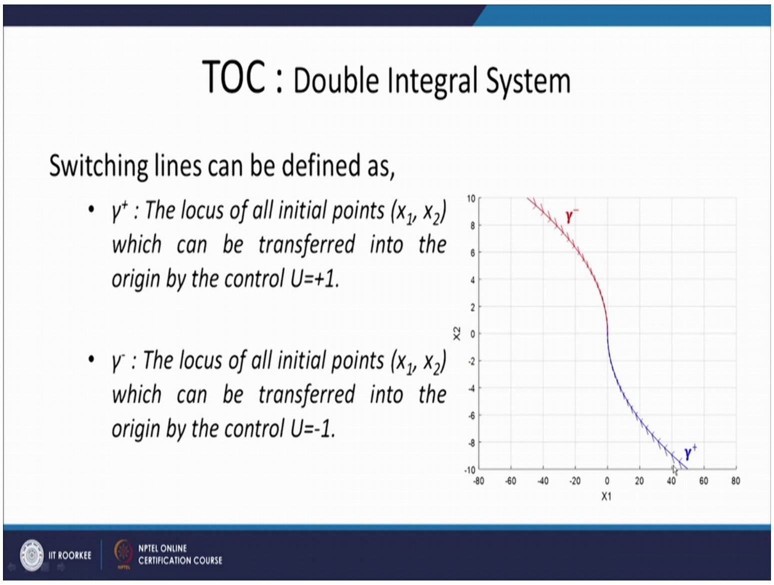

So, at the blood line which is here its shows my switching curves. So, my switching curve is which is driving the state to the origin and it will drive the state to origin only when. So, this lower portion is this is for U equal to plus 1 and upper portion is for U equal to minus 1. So, if somewhere I am in this region. So, means I have to force this initial state to move towards this line.

So, this means this will be my positive region. So, sorry this region will be when I will apply my U equal to minus 1 if I will apply the U equal to minus 1 beyond this line. So, this means I am forcing my all the trajectory to move towards the switching line. So, means my all trajectory will move towards the switching line. So, I will apply the; you equal to minus 1. So, they will come here and then I will switch to U equal to plus 1. So, they will drive this to the origin this means as we have seen here any trajectory. So, I can say this bold line which is going through the origin and this is for my U equal to minus 1 and this curve lie on the U equal to plus 1.

So, beyond this line this whole region I have to select that for I will apply U equal to minus 1 if anywhere in this space my initial condition exists because they will drive this to the switching line and then I am switching my control to bring it to the origin similarly those are in this region let us say this is my initial condition. So, this means I have to apply the U equal to plus 1 which will drive it to the switching line and this line here I change my control to minus 1. So, this will drive it to the origin. So, this gamma line which is shown here as the gamma I can say this is my switching line region right to the gamma line is 4 U equal to minus 1 and region left to the gamma line is 4 by U equal to plus 1.

So, here we can see the switching line this portion is for U equal to minus 1 this portion is for U equal to plus 1. So, once when I am in this region. So, I am placing U equal to minus 1. So, this means they are driving using this trajectory to my switching line for U equal at the switching line my U will be plus 1 and I will drive it to the origin. So, gamma plus the locus of the initial point which can be transferred into the origin by the control line plus 1 and similarly we define the gamma minus.

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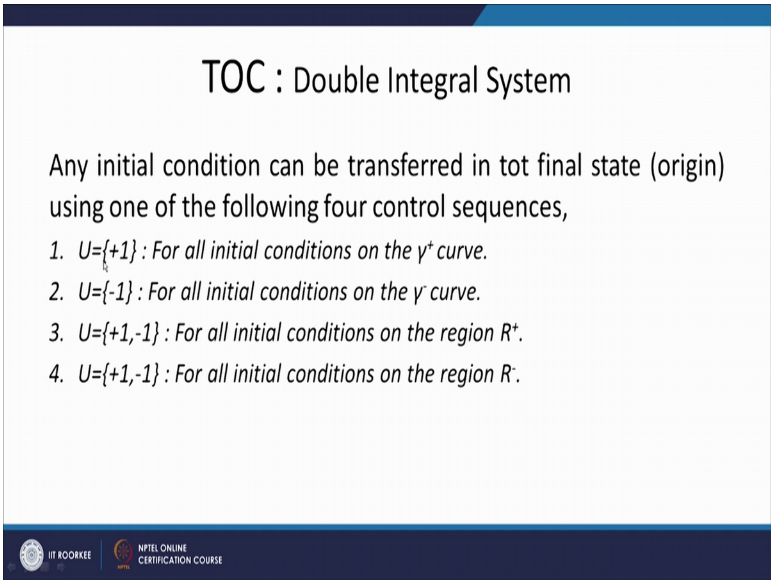

So, U equal to plus 1 for all initial condition on the gamma plus curve this means this is my gamma plus curve. So, my initial condition will lie anywhere on this line. So, I have only 1 control that is U is equal to plus 1 similarly if my initial condition lie on this my only 1 control U equal to minus 1 if at gamma minus curve U equal to plus 1 and then minus 1 for initial condition on the region R plus.

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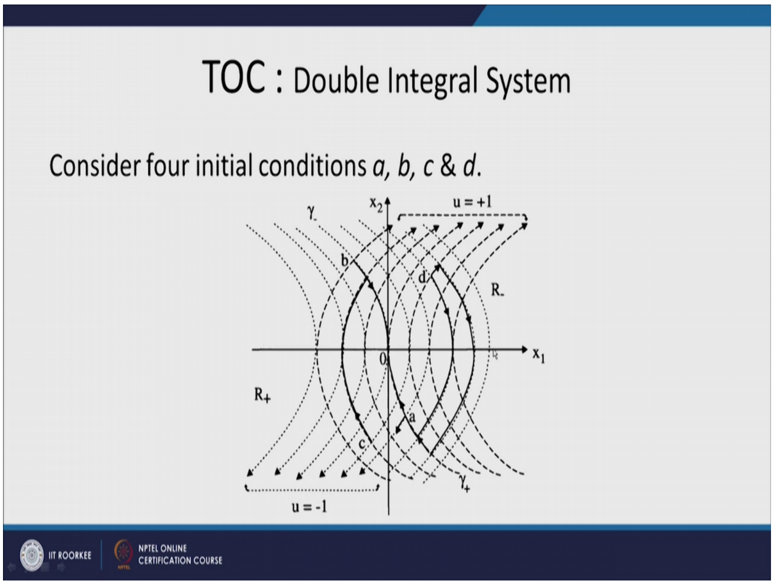

So, which region we are saying R plus the region this region R plus is here or this whole region which is left to the switching line is my R plus region. So, this is in R plus region my control is U equal to plus 1 to drive it say U equal to plus 1. So, it will drive this to the gamma line and then minus 1 to transfer origin.

Another case this is U equal to for region R minus. So, this should be negative it is wrongly written here. So, it is minus 1 to plus 1; this just be reverse. So, this means this is my R minus region which I am driving U equal to plus 1 first and then sorry U equal to minus 1 first at the switching line it is U equal to plus 1. And I will reach to origin if I an somewhere follow the other path means suppose my initial condition is this, so, my path also maybe U equal to plus 1 then. So, this is for U equal to plus 1 this is U equal to minus 1 and again U equal to plus 1.

So, for let us say for 4 point d if x 0 at d and my switching is plus 1 minus 1 plus 1. So, this is plus 1 minus 1 plus 1 again I am reaching to the origin, but not in the minimum time. So, my optimal switching optimal switching is I am in the region R plus. So, it should be simply minus 1 plus 1. So, if I will select minus 1 plus 1 this means from d point I am coming here and reaching to this. So, this will be my optimal switching at this we have to reject. So, in this way we can implement the Pontryagin minimum principle to the time optimal control system. So, this was a very simple example more bigger problem can be solved utilizing the Pontryagin minimum principle.

So, at this point we will end our course hopefully you will be benefitted by this course. So, your feedback will be required.

Thank you very much.