Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture – 04
The Fixed End Point Problem

Welcome friends to this session of our discussion which we are extending to our basic variational problem, in the basic variational problem we are trying to find out the optimal value of a functional.

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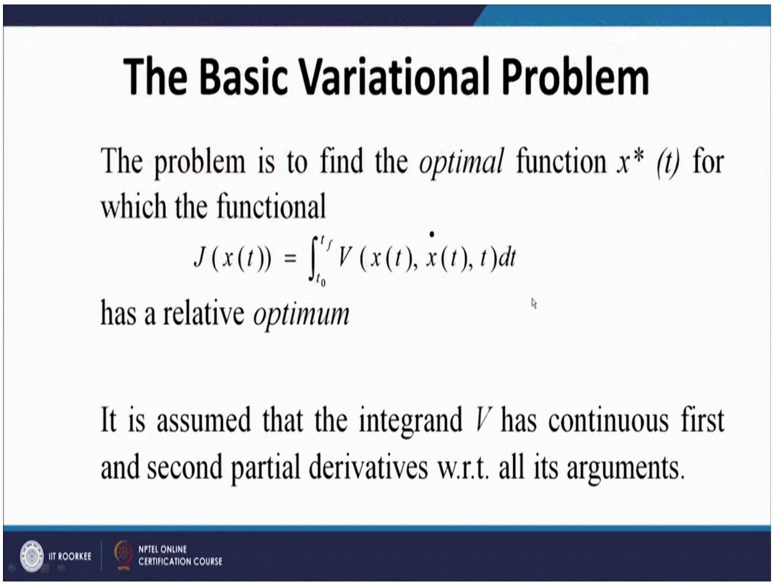

And we have consider the functional is J x t, V x x dot t.

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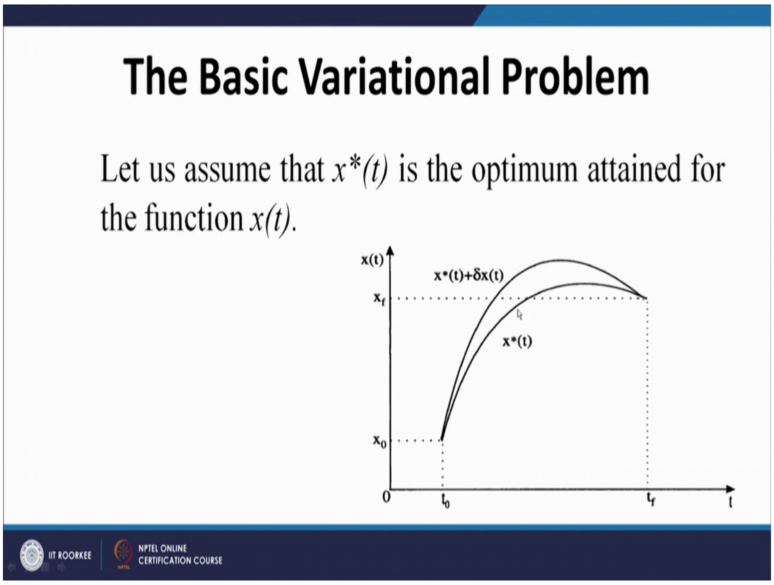

We start our solution with assuming that we will have the optimal value and the variation.

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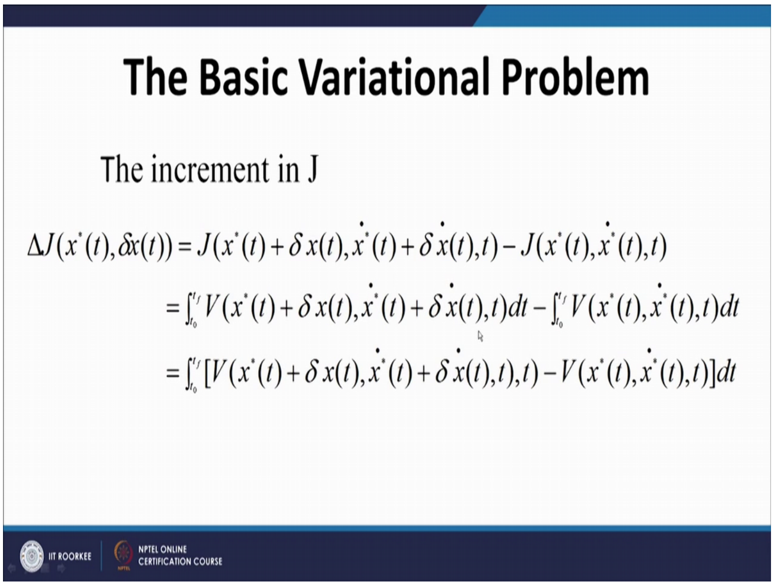

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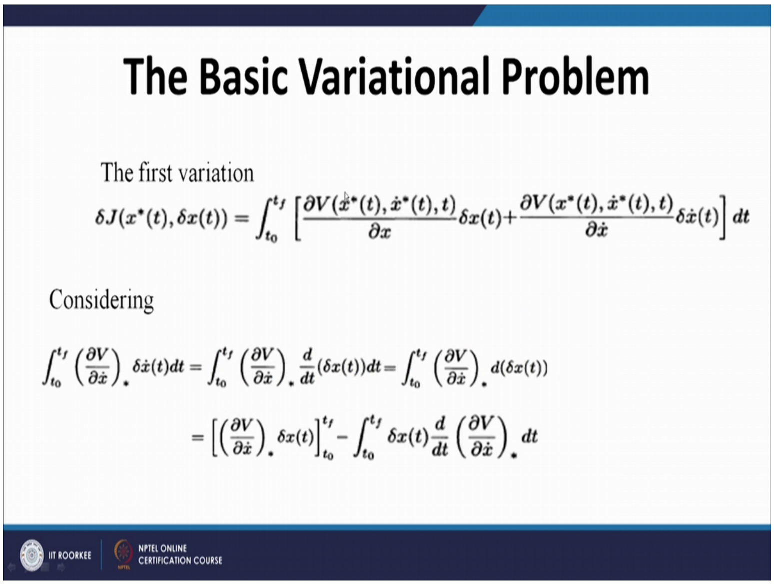

In this for this we will find out the increment J and write down the first variation as del V by del x, delta x t del V by del x dot delta x dot of t.

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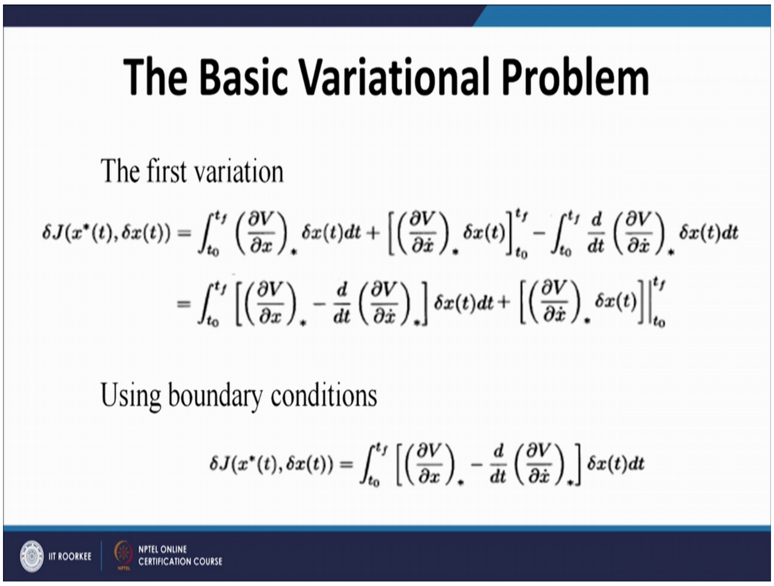

We expand the second term using the integration by parts to get the final value as del V by del x minus d by d t del V by del x dot delta x t first integral term and the boundary value terms. In this case my boundary values are fixed, we got t 0 and x t 0 is given t f and x t f is given. So, this means delta x of t 0 is 0 delta x of t f is 0. So, this term will not appear in the first variation, and that particularly again to remind for a two point boundary value problem, where initial and the final points are is specified and we write the first variation as del J equal to integral of this term.

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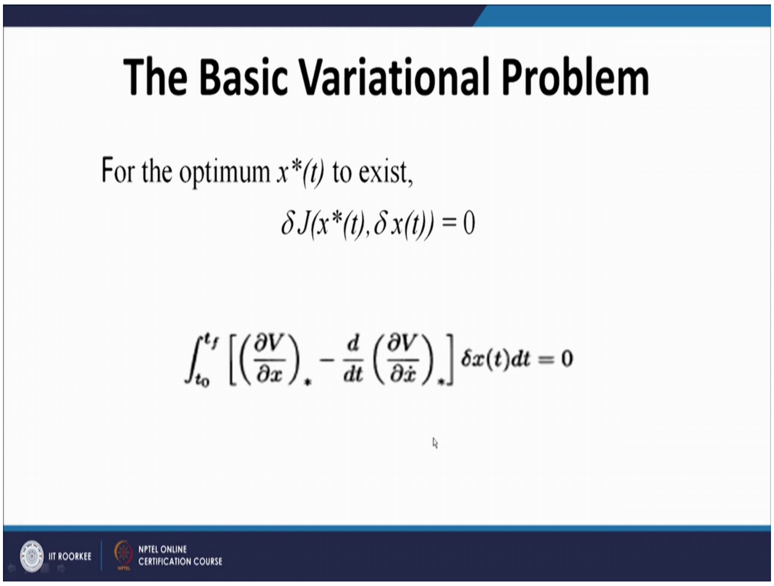

Then we use the fundamental theorem as del J to be equal to 0. So, my this term will be 0 we apply the fundamental lemma.

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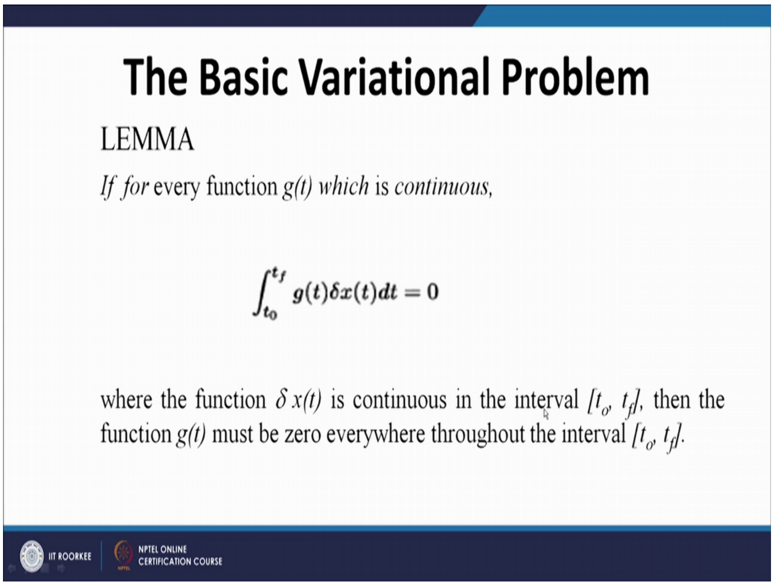

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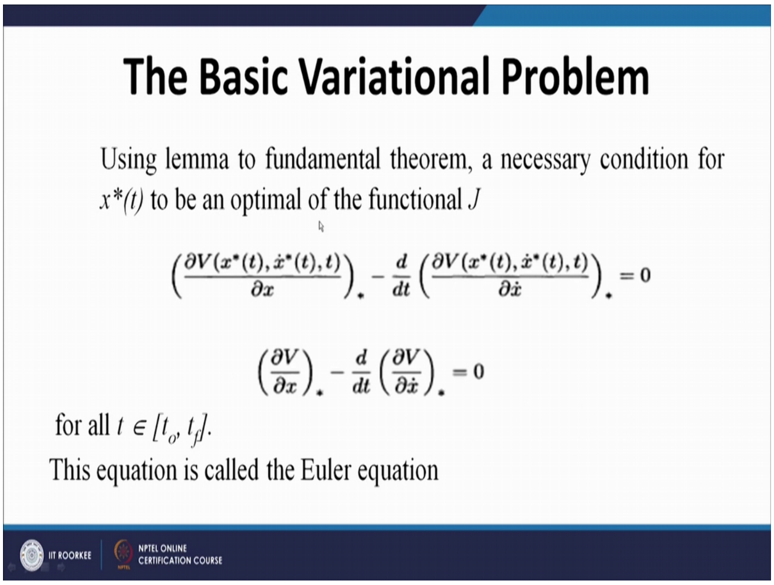

To get my Euler equation as del V by del x minus d by d t, del V by del x dot equal to 0. So, how to solve a problem based on this.

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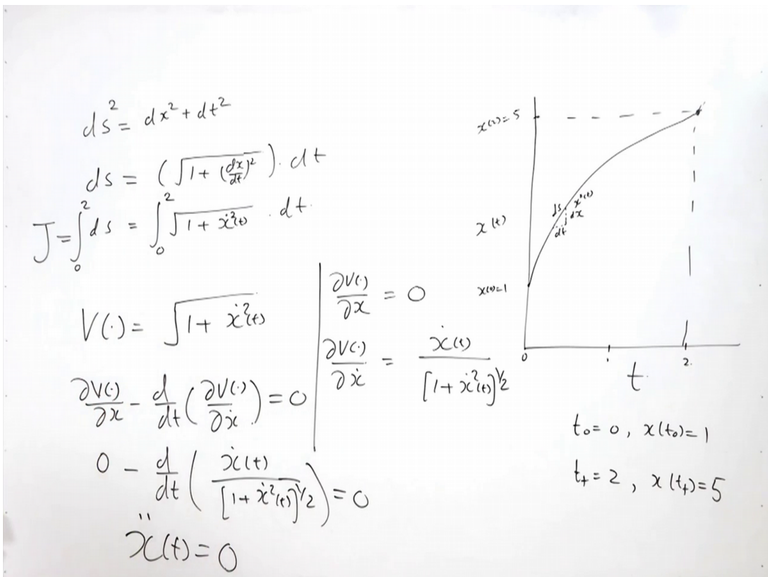

Let us take us an example and see how we can solve a two point boundary value problem. Our problem is very simple my objective is to determine the is smallest distance between two points, my points are as t 0 equal to 0, and x t 0 equal to 1 and t f equal to 2, and x t f equal to let say 5.

So, my first point lie at t equal to 0 is this point. So, my let say first point lie here, this is x 0 equal to 1, and my second point let us a lie here this is 0 let us a 1, this is 2 and this value say 5 and this is my second point which is I can say this is my x 2 which is equal to 5 So, I have to find the minimum distance between these two point, to get this distance let us consider a slope. So, what we say let this is my optimal value which I am saying this is my x star of t. So, at this let say this is my d s, this is d t, this is d x. So, this means my this d s from 0 to 2. So, I can write d s as d s square, d x square plus d t square or simply d s I can write as 1 plus d x by d t is square into d t; for I will simply writing this 1 plus x dot of t its square into d t, this is my d s. So, to find out the complete length I can integrate this from 0 to 2, 0 to 2, 0 is by t 0 point and 2 is by t f point.

So, in this way I can say this is my functional s or this is my objective function. So, my objective is to optimize the s for optimal value for a given x which will be my optimal value. So, this I can treat as, we have taken s I can take as say in place of s we can write this is my J. So, if I will equate it with my objective function. So, I can say my V is nothing but is square root of 1 plus x dot is square t, this is my V. So, if for this functional my e l equation must satisfy and what is my equation is this means del V by del x minus d by d t, del V by del x dot that must be equal to 0.

So, for this V if I will write what is my del V by del x. So, I can see this V is not a function of x. So, this will be nothing but 0 and del V by del x dot if I will find. So, I have to differentiate this with respect to x dot. So, what actually we will get. So, this is x dot of t, 1 plus x dot of t square whole to the power half. So, these two values I will place here. So, this is 0 minus d by d t, x dot of t, 1 plus x dot of its square t, 1 by 2 and this whole must be equal to 0. So, if I will solve this I will get nothing but due to this d by d t this is the x double dot of t equal 0.

And naturally if I am the solving this, this x is nothing but my optimal value of the x.

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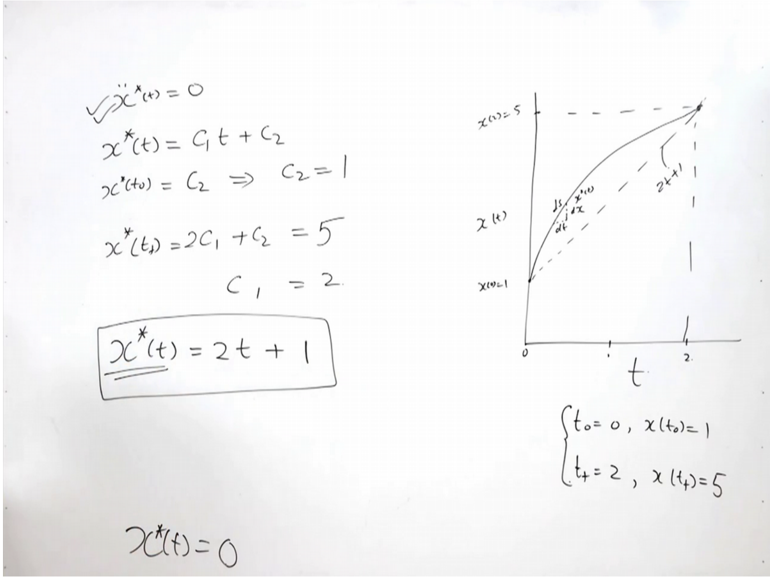

So, my differential equation is x double dot of t must be equal to 0, and to get the optimal value of the x I have to solve this differential equation subjected to the given boundary conditions. So, the solution of this is simple and I can directly write the solution as C 1 t plus C 2; where C 1 and C 2 are my constants constant of integration, and value of the C 1 and C 2 I can simplifying by using my boundary conditions. So, if boundary conditions will change naturally my C 1 and C 2 value will change.

So, for the given problem at t 0 I have x of t 0 as nothing but C 2. So, this implies and t 0 my x t 0 is 1. So, C 2 is nothing but 1 and at x star of t f, t f is 2. So, twice C 1 plus C 2 this is equal to by x t f is 5. So, this give me C 1 equal to 2. So, what is my optimal x? C 1 means 2 t plus 1. So, this is my optimal trajectory, and as we can see this is nothing but the equation of a straight line as I have assume. So, nothing but I will directly get this is 2 t plus 1line. So, in this way we can solve a 2 point boundary value problem, we are given with the boundary conditions we know this is my Euler equations I have to solve this equation.

So, first I will identify what is my I take del V by del x minus d by d t del V by del x dot and the solution of this e l equation nothing but gives me a differential equation in terms of my x, and I have to solve this x subject to the boundary condition to get the optimal value of the x. So, this is the general approach to determine the optimal value of a functional, but till now we got only the optimal value, our objective was to find out the minimum distance. So, whether this optimal value is my minimum value or not I have to check the sufficient condition.

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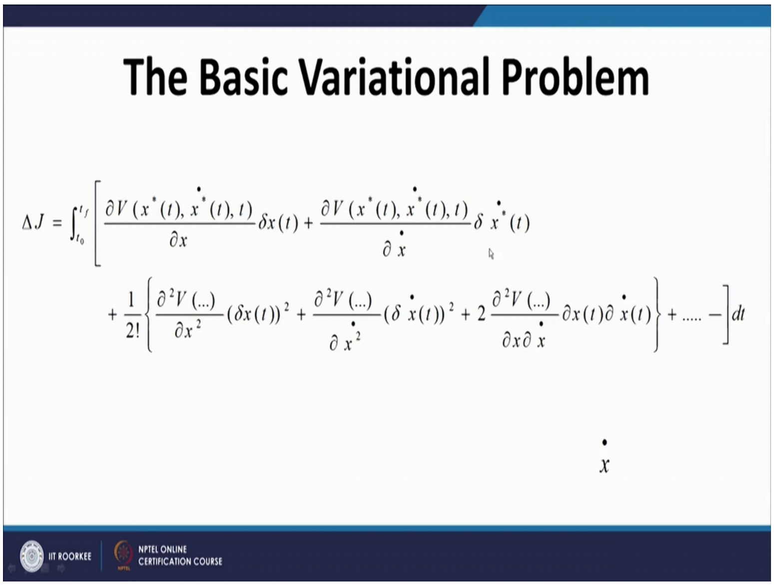

As you can recall the increment we have written as the first variation and the second variation; and what is the definition of the optimal of a functional says at optimum point my first variation will be 0 and the sufficient condition says the second variation should be positive and my second variation is nothing but my this complete term is defining nothing but the second variation.

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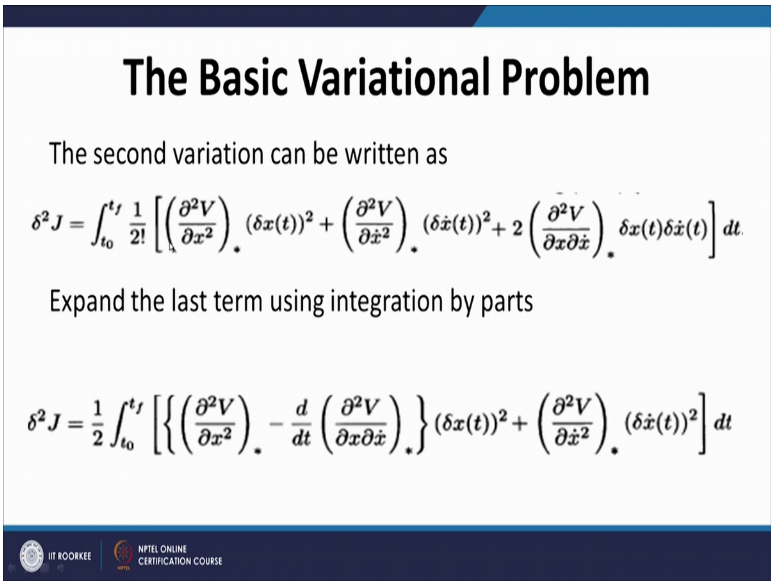

So, I can write this second variation as 1 by factorial 2, del 2 V by del x square delta x t square del 2 V by del x dot square, delta x dot t square plus twice del 2 V by del x del x dot del x t delta x dot of t. Now in this I can write the last term which is delta x delta x dot in terms of the sorry, I can explain this term using the integration by parts, considering del 2 V by del x del x dot delta x t this term as my u and delta x dot t as my V with as the two point boundary value condition we are problem we are taking. So, my initial and the final point will be given.

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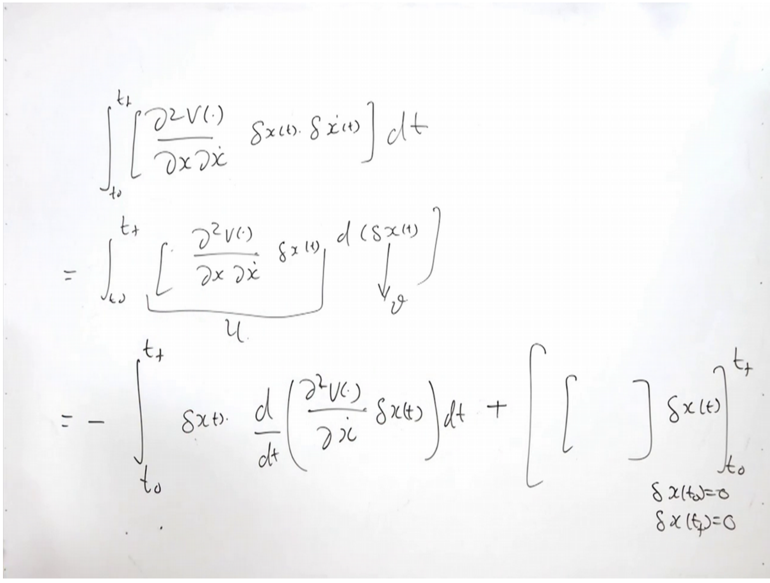

So, if I will write this value del 2 V by del x, del x dot, del 2 V by del x del x dot multiplied with delta x t, delta x dot of t. delta x t delta x dot of t d t what we are saying we are and this is nothing but t 0 to t f. So, t 0 to t f that we can write as del 2 V by del x del x dot delta x of t and this we will write d delta x t by d t into d t.

So this dt, dt will be canceled out. So, I can simply write this as delta x t and this whole I am taking as u and this a I am taking as v. So, this is u d v and u d v is integral t 0 to t f minus v d u. So, v is delta x of t into d u I will write del 2 V by del x dot delta x of t sorry there is no d t. So, if I will divide this by d t. So, this will be d t here this is cancelled out plus u v. So, u is I can directly write here, and V is delta x of t t 0 to t f. So, if I will place this value delta x of t 0 will be 0, delta x of t f will be 0. So, my this term will not appear only this term will appear and this term will be nothing but my d by d t del 2 V by del x del x dot delta x t square. So, because these two delta x t I can have as a square.

So, the whole term I can write as half of del 2 V by del x square, d by d t del 2 V by del x del x dot delta x t square, del 2 V by del x dot square delta x dot t is square. So, to be this whole term to be positive because this is a square term, this is a square term this will be positive. So, to make the whole term positive this means my first term should be greater than 0.

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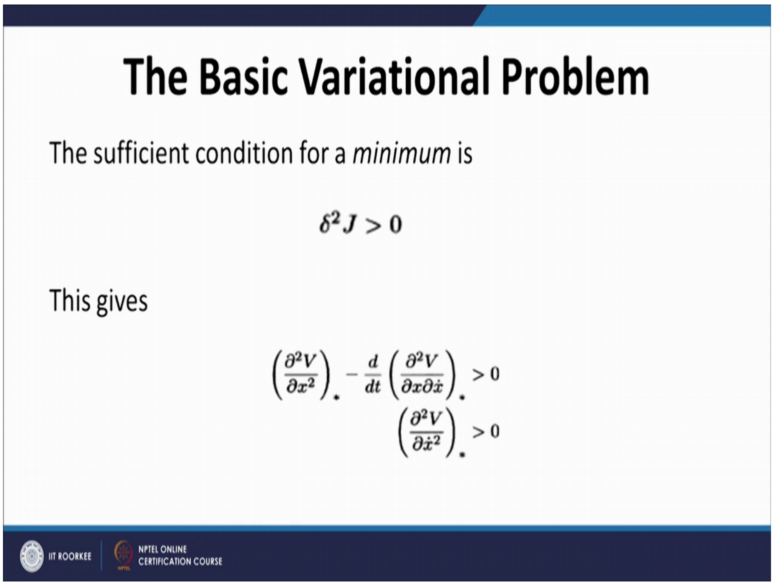

Simultaneously del 2 V by del x square must be greater than 0. So, this means if del square J is greater than 0, my first term del 2 V by del x square minus d by d t del 2 V by del x del x dot must be greater than 0, and del 2 V by del x dot is square this term also must be greater than 0. So, this two term must be greater than 0 then I can say this del square J will be greater than 0, and the similar is a applied if this is less than 0.

So, if del square J is greater than 0 my condition will be for minima, if this is less than 0 my condition will be for maxima. The same as we are getting here the this particular equation I can also write in the form of del x, del x dot, del 2 V, del x square del 2 V by del x, del x dot, del 2 V by del x, del x dot, del 2 V by del x dot square in terms of this say del x, del x dot, x x dot.

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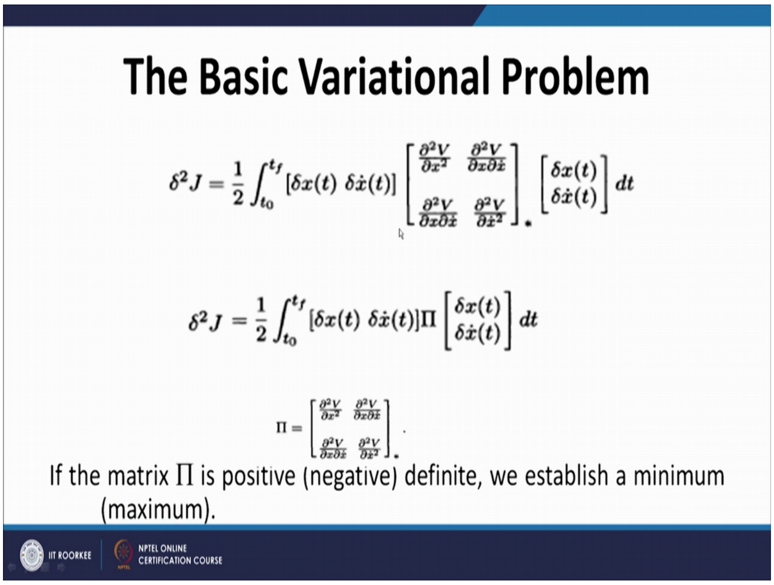

So, they always give me the square term. So, whether whatever be the variation in delta x and delta x dot, that always in del 2 J will be positive. So, the whole sin of del 2 J will depend upon what actually is the sin of my matrix given as pi. So, I can write this matrix pi. So, the definiteness of this matrix will decide whether my sufficient condition is for minima or for maxima. If this matrix is positive definite or greater than 0 then this is the condition for my minima and if this is sorry this is less than 0 or this is negative definite then this is the condition for my maxima.

So, in this way we can apply both necessary and the sufficient condition to get what actually will be my maxima or minima. So, what is the condition of my optimum point that we can decide by using the second variation. So, you can apply the second variational to the problem which we have discussed just now and you will find my second variation is nothing but positive. So, what the optimum x we have selected that is giving us the minimum value for the J or minimum so that particular length will be minimum.

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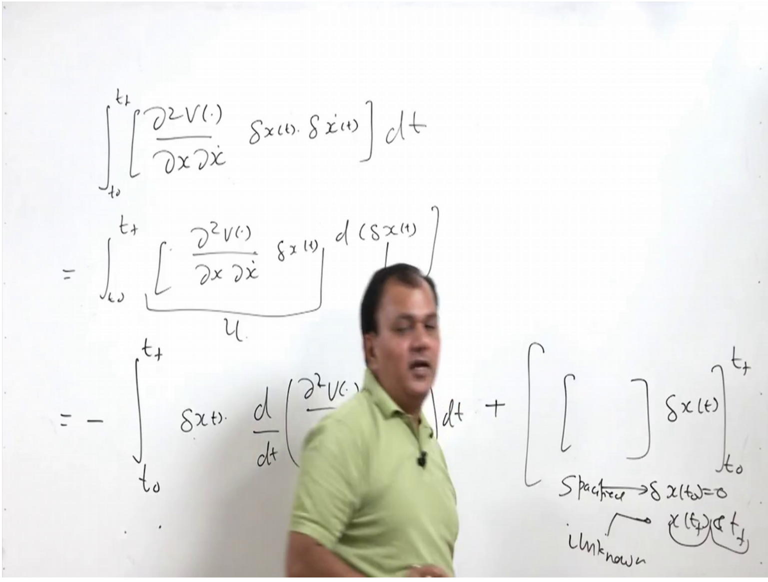

So, I stop my discussion here for this session and in the next session we will start our discussion on the free end point problem. In that problem we normally keep this is specified and delta x t f and t f are may be unknown. So, they are the different condition will appear on x of t f and t f. So, the different cases we will consider in the next class to discuss the free end point problem.

Thank you very much.