Optimal Control
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Lecture - 39
Time Optimal Control System (Constrained Input)

Welcome friends to this session of our discussion. In the previous class we have started our discussion on the Pontryagin minimum principle. What the Pontryagin minimum principle says? If we will find out the Hamiltonian at a given control which is u star plus u t and my control is bounded.

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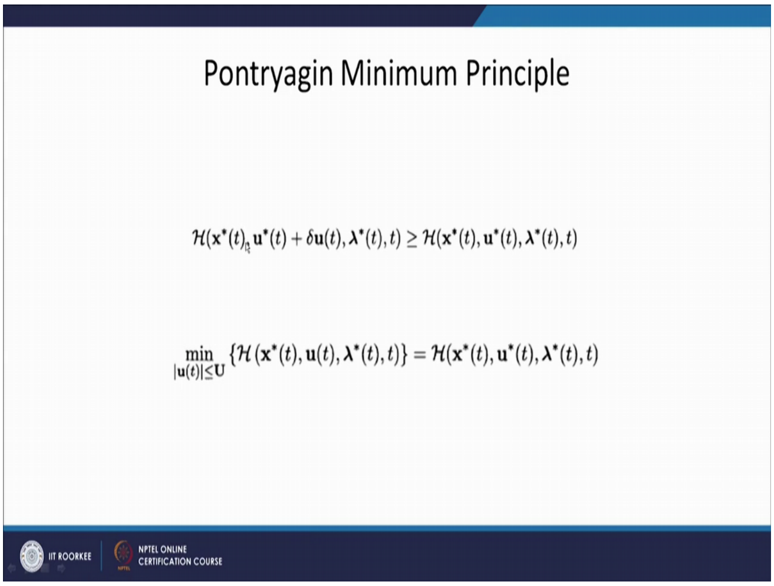

So, in that case Hamiltonian at any control point is always greater than or equal to the Hamiltonian at the optimal point so this means we can minimize the Hamiltonian determined at a journal u to get the minimum of my H. So, minimization of H with respect to u means del H by del u equal to 0 if we are placing. So, to find out the optimal value of the u as we have done in the previous case, but if u is constrained then its minimum value is nothing but the minimum of my Hamiltonian.

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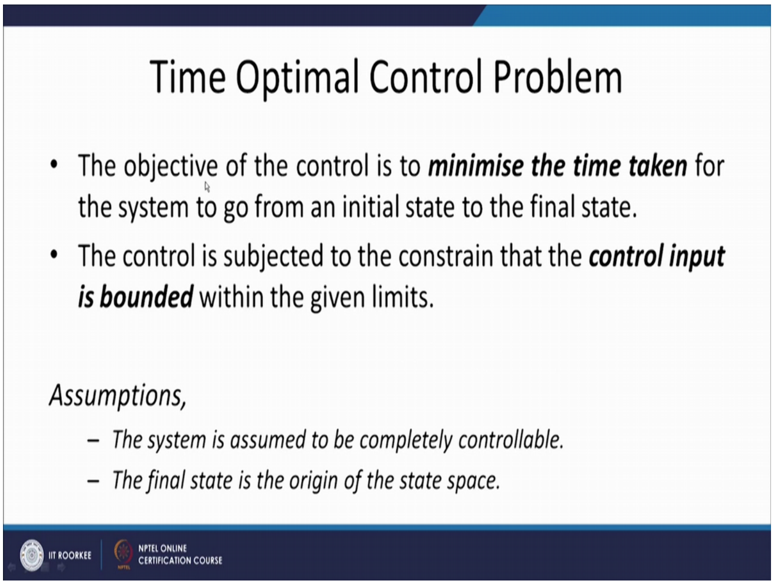

Now this we will apply to a time optimal control system the objective of minimum time control problem or the time optimal control system is we have to transfer the state of a system from its initial value to final value in minimum time.

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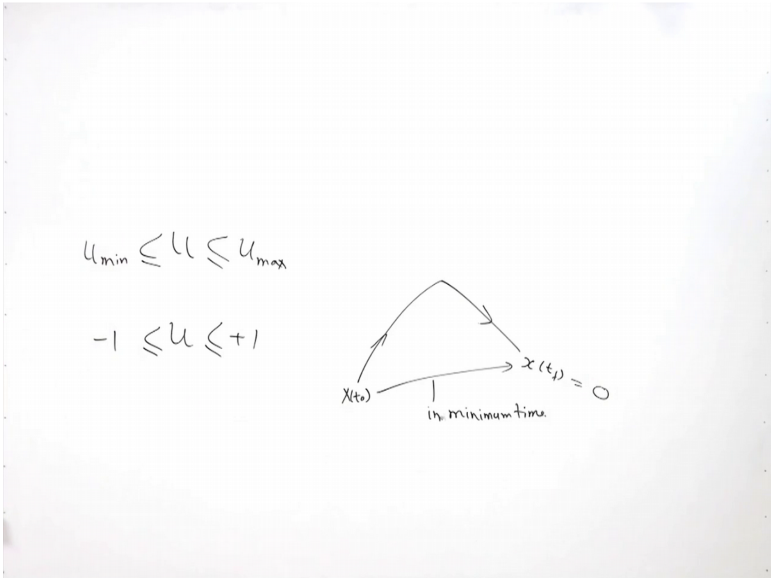

So, we have to minimize the time taken for the system to go from an initial state to the final state and in this case my input is bounded by its minimum and the maximum limit. So, with bounded input my objective is to transfer a initial state which maybe x t 0 to some final state which maybe x t f. So, I have to follow the path which will give the. So, I have to transfer this in minimum time. So, this is my time optimal control problem for time optimal control problem we initially assume that my system is completely controllable. And the final state is origin of the state space this means x t f we can consider to be at origin or means my system has to return to its operating point from any disturbance in minimum time.

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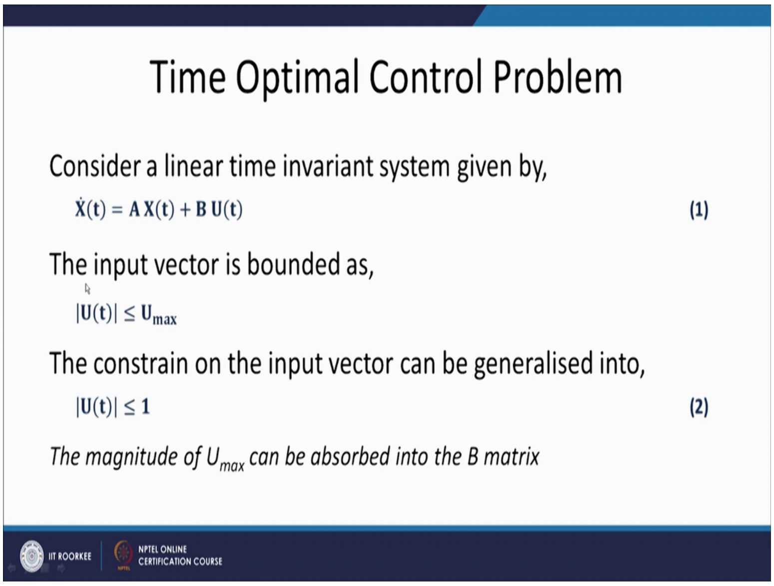

So, for this we will consider a linear time invariant system defined as x dot t is A x t is B U t, my input is bounded as mod of U t is less than U maximum or otherwise we can say my U is less than or equal to some U minimum and U maximum, so, if I can normalize this bound just to have us to be in between minus 1 to plus 1. So, whatever be the magnitude that I can absorb in my matrix b. So, this means mod of U t less than U max I can write as mod of U t less than or equal to 1 by absorbing the magnitude of U max in matrix B.

So, this is a problem statement I have a linear time invariant system in which my U is bounded I have to determine the optimal value of the U which will transfer my state from initial state to the final state in minimum time.

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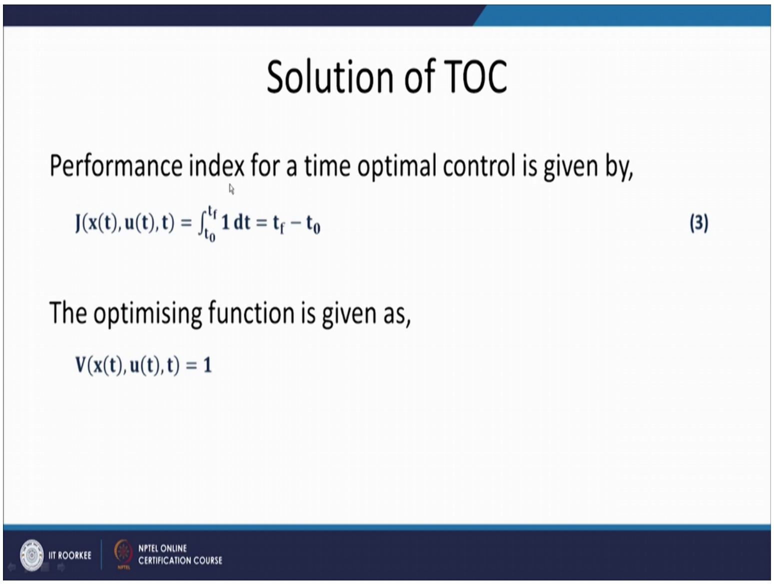

So, I can define my performance index as we have discussed for a minimum time problem my objective is to minimize the time so I will take this as t 0 to t f 1 d t which is nothing but t f minus t 0 because this time I have to minimize. So, if this is my J and I will take consider what will be my V. So, my v is nothing but a unity which we have considered like we have seen our; we have apply the Pontryagin minimum principle. So, this means we have to minimize the H.

So, first of all we have to define what will be my Hamiltonian as we can see my Hamiltonian is H 1 plus lambda prime t A x t plus B U t because this is my V lambda prime f f is A x plus B u.

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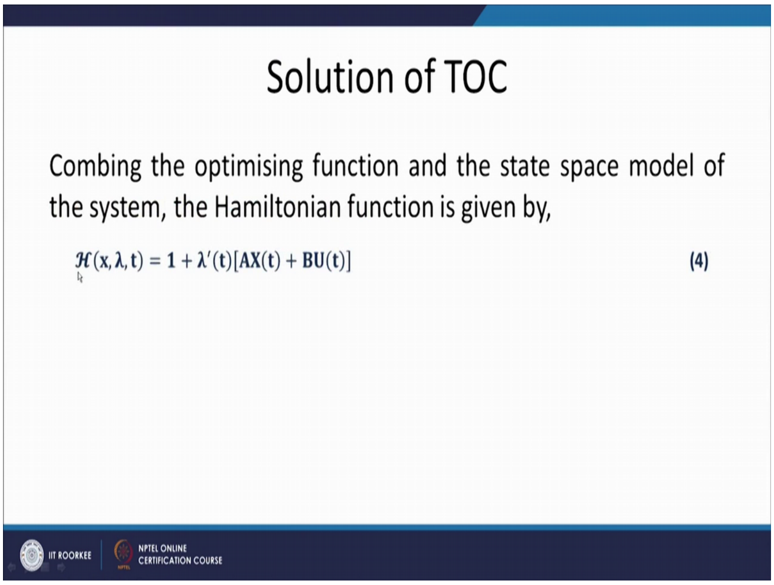

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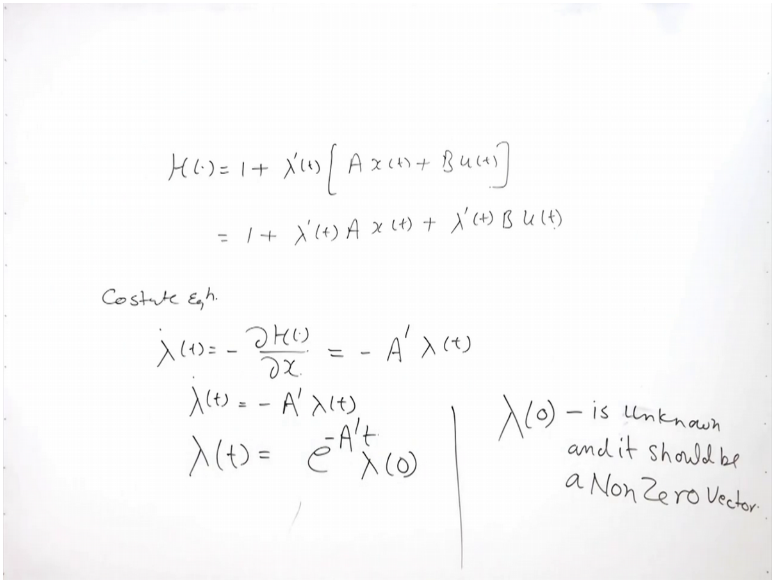

So, H we are defining as 1 plus lambda prime t A x t plus B U t or in other way I can simply say this is lambda prime t A of x t plus lambda prime t B of u t.

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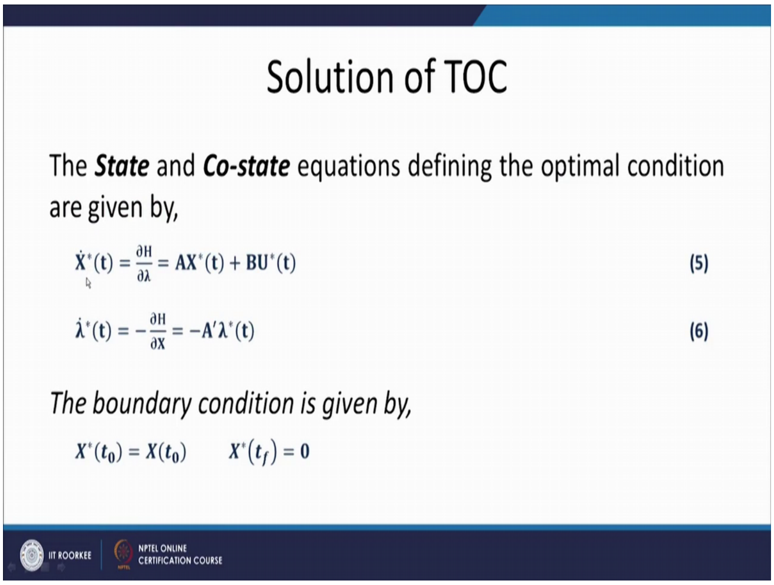

So, I define my Hamiltonian with this value of the Hamiltonian what is my state equation state equation is x dot t equal to del H by del lambda which is nothing but my A x t plus B U t and lambda dot t is minus del H by del x a prime lambda t. So, my co-state equation if I will write this is my lambda dot t is nothing but minus del H by del x and if I will take this minus del H by del x this is a prime lambda t.

So, this is nothing but minus a transpose lambda t. So, it is a simple first order differential equation if I will see this is lambda dot t is nothing but minus a transpose lambda t. So, I can write what will be my lambda t e to the power sorry I can write e to the power minus a transpose t lambda 0. So, the solution of this equation is nothing but e to the power minus a lambda prime t a 0. So, we can write this in terms of initial value of the lambda. So, lambda 0 which is the initial value of my co-state vector is unknown and it should be a nonzero vector.

So, this is our; another condition we can say lambda 0 we have to consider to be the no 0 vector why it is nonzero that we will see little bit later on. So, we can consider it we cannot consider it to be a 0 vector. So, the solution of my co-state equation is e to the power minus a transpose t lambda 0 which we will utilize little bit later on.

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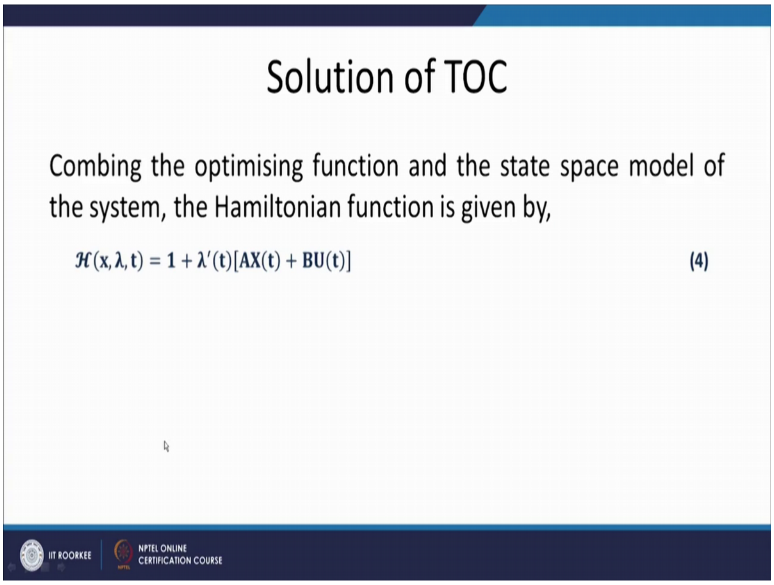

So, in this case if my if I say my Hamiltonian is 1 plus lambda prime t A x t plus B t u t. So, what is my del H by del u my del H by del U, in this case will be lambda prime B because this is the term which is related to U. So, lambda prime B this is lambda transpose B and this is equal to 0. So, this means what this equation shows this equation is independent of the U.

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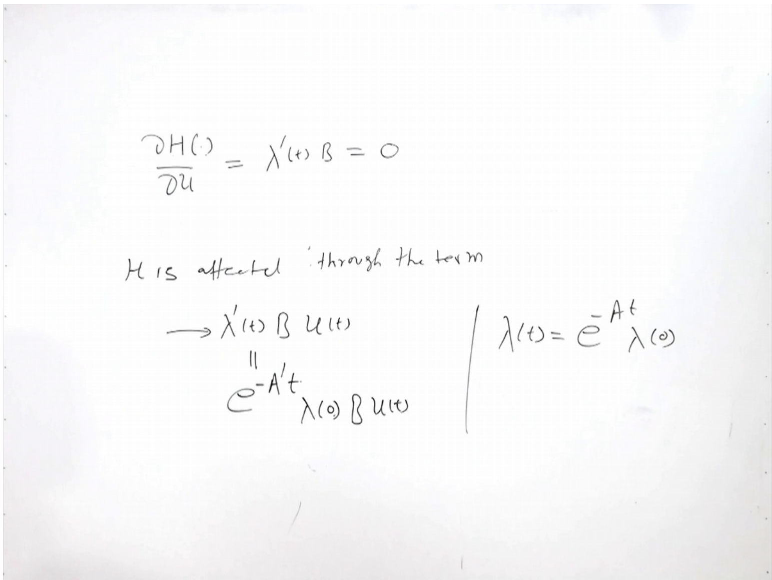

So, we cannot have any particular value of the u which will get optimized because if you will see my H is nothing but a linear function of u if I will take the first derivative of this.

So, that will be independent of the u. So, how to minimize the H and if you will see this H is affected by the u through the term. So, H is affected through the term which is my lambda prime B U t.

So, this is my term which is affecting the H and my objective is to minimize the H or in other words I can say this is equal to my lambda t we have taken as. So, because my solution of lambda t was e to the power minus A r lambda 0 if I will take the lambda prime t. So, I can write this as e to the power minus a prime t lambda 0 B U t.

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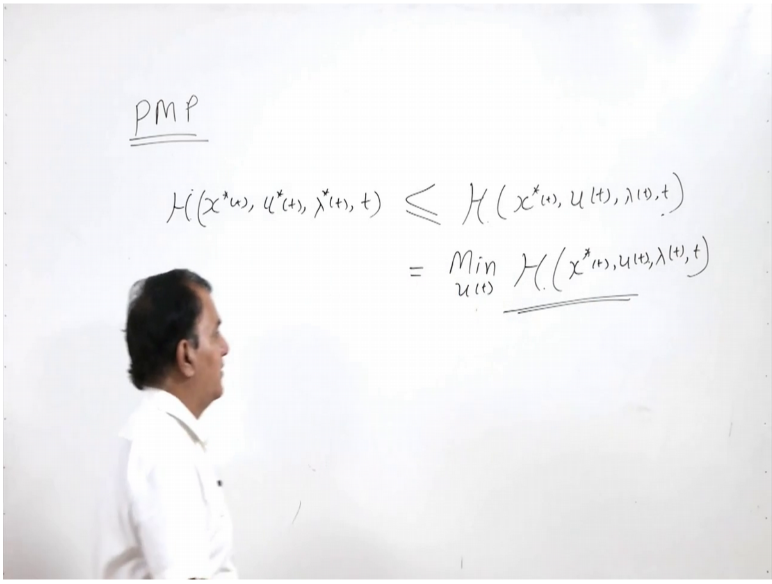

Now, if you I will apply the Pontryagin minimum principle. So, what my Pontryagin minimum principle states they shows that my H it should be less than or equal to the H which is at a journal control point u t lambda t and t and my objective is or this is equal to minimum of this H at x star t u t lambda t t. This means to get the optimal value of H I have to minimize this, and H if you will see what I am getting my H is 1 plus lambda transpose t A x t plus B U t and what the lambda t was my lambda t was e to the power minus a transpose t lambda 0.

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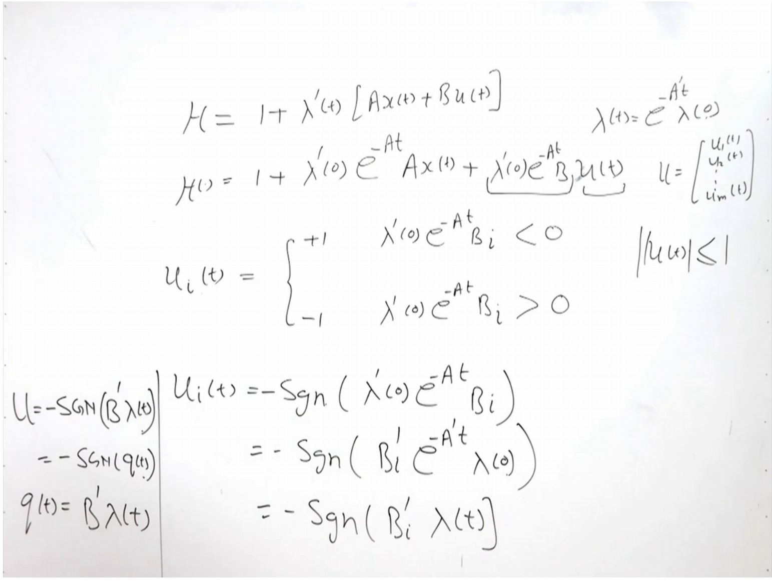

So, if I will write this; this is 1 plus I have take the transpose of this lambda transpose 0 e to the power minus a t is my lambda prime t and this I will write as A x t plus lambda prime 0 e to the power minus a t B U t, so, for H to be minimum if my H is minimum. So, I have to subtract a maximum value of my last term from the first 2 term u is bounded as less than or equal to 1. So, I have to see what should be the value of u t value of u t should be can be plus 1 or minus 1. So, this means to get the H m minimum I have to subtract the maximum value of this; this means if u is plus 1 then this value should be negative.

If u is plus 1, it should be negative if u is minus 1 it should be positive. So, if I will follow this sequence I can subtract the maximum value of this from these 2 term to get the minimum value of the u. So, I can write the i-th column of the u as plus 1 if lambda prime 0 e to the power minus a t B is less than 0 and I can write this as minus 1 if lambda prime 0 e to the power minus a t B is greater than 0.

So, I can select that particular u to be either positive or negative because u is a vector of u 1 u 2 and say let us say the m terms are there u m t. So, this means in this vector I will select the positive or the negative vector depending upon the value of sorry it should be B i. So, what is that particular u t should be u I must be positive or negative in this case or in general I can write this u i t as the minus signum function of lambda prime 0 e to the power minus A t B i. So, if this sign is negative u will be positive if this sign is positive my u will be negative.

The same we can write as if I will take the transpose of this B I transpose e to the power minus a transpose t lambda 0 or simply I can write this is nothing but the signum function of B I transpose this is my lambda t or in general if I will write. So, I can write my u to be the signum function of B transpose lambda t.

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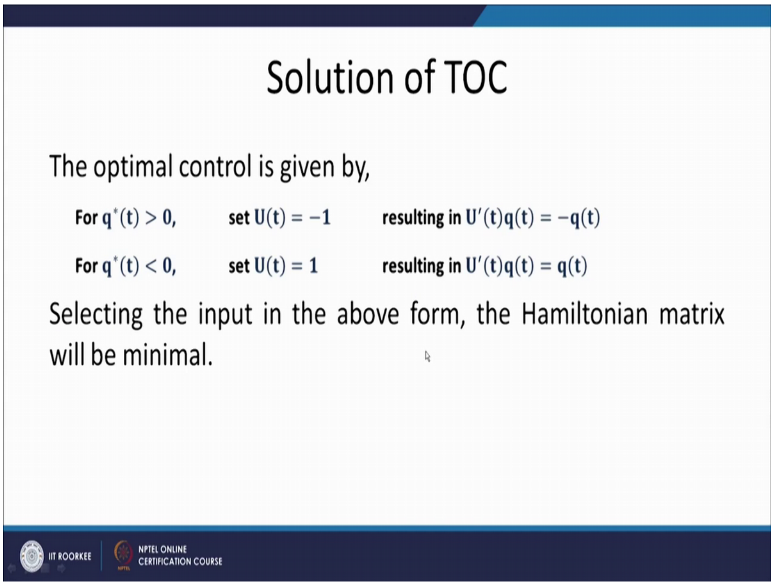

So, I can define oh this B transpose lambda t as signum function of q t where q t I am nothing but taking as my B prime lambda t.

So, if q t greater than 0 we said u to be minus 1 if q t is less than 0 I said my u to be plus 1. So, selecting the input in the above form we can minimize the Hamiltonian matrix H. So, this H can be minimized if I will select this q t which is nothing but B prime lambda t to be greater than 0 or less than 0.

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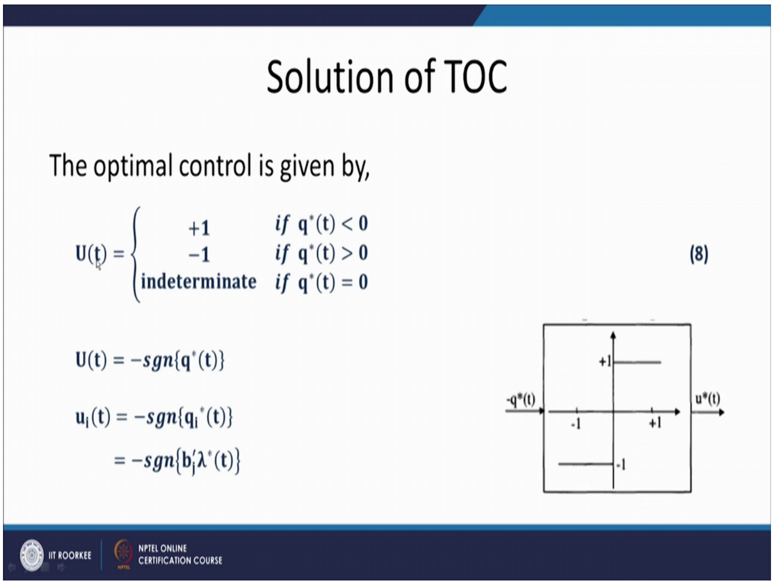

So, if u t is plus 1 my q is less than 0 u t is minus 1 q t is greater than 0, but what is if q t will become 0 then my u is undetermined and this logic we can represent by this on of logic as given here with minus q t is the output and u t is the input.

So, this switching function will decide the value of the u and u will switch between plus 1 and minus 1 due to this switching my time optimal control also known as the bang bang control say if u t is 0 when q t maybe 0 this is B prime see if I will concentrate this term B is my input vector. So, there is nonzero element into this e to the power minus a transpose t this is nothing but my transition matrix; it cannot be a null matrix. So, if lambda 0 I will select a 0 vector for lambda then my; this term maybe 0. So, that is why we have considered initially that lambda 0 should be a nonzero vector.

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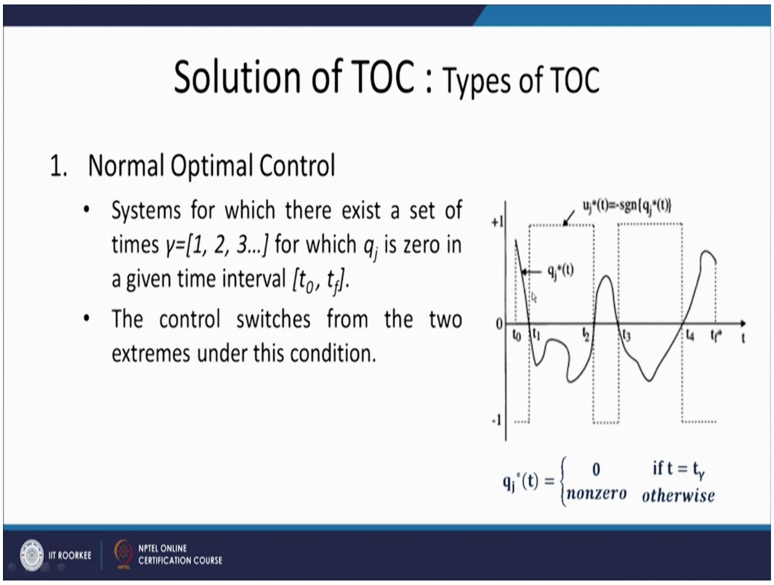

So, my; this term is normally nonzero and if q t now if you will see here if this is the variation in the q t and what is our logic if q t is positive my control will be negative. So, I will get a negative control up to the point where it is crossing the 0 as this will cross the 0 only at this moment the q t will cross the 0 from its minimum value to maximum value which will be switching point and from when this is negative my q t will be positive sorry my u will be positive. So, by the variation of the q t my control will switch to the minimum to the maximum value.

Either plus minus 1 or plus 1. So, between these 2 my switching will take place such type of the system we call the normal time optimal control system. So, this is the normal time optimal control system, but if anytime my q t is going to be 0.

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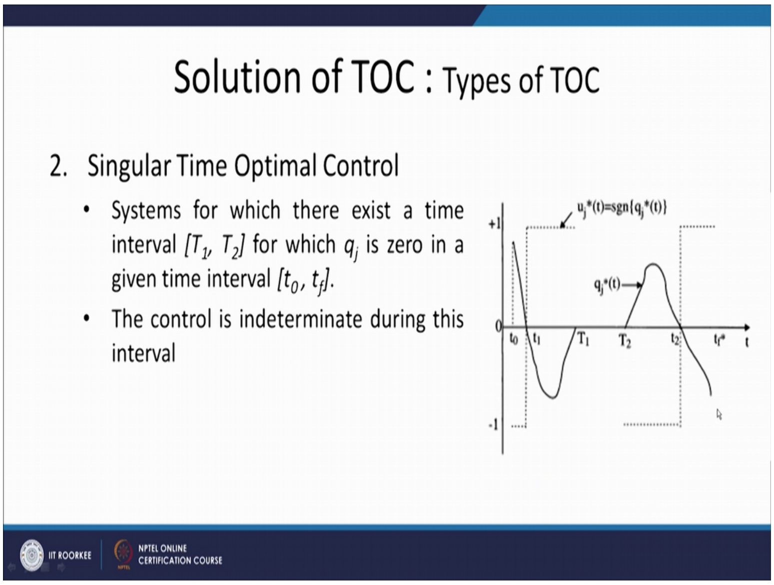

So, let us consider another case; we will have this curve as a q T which is 0 between the time period T 1 and T 2. So, between T 1 to T 2 this is 0 and further; this is again changing. So, what will be my control it will be negative then goes to positive, but between T 1 and T 2 it is undefined.

So, during this period we cannot define any control input. So, this is the indeterminate case as we have said here. So, when q T is 0 my u is undetermined. So, in this case we can prove this if this is the; at this condition will adjust my system will be uncontrollable. So, you can follow the literature just to get the proof of this. So, a and such type of the system we call the singular time optimal control system.

So, that is why initially we assume that my system should be completely controllable if the system is not controllable then there a time period can exists in which the u will be undetermined. So, this is the basics of a time optimal control system and the application of this we will see in the next class. So, I stop this class here.

Thank you very much.