Optimal Control
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Lecture - 38
LQR System Using HJB Equation

Welcome friends to this session of our discussion. In the previous session we have discussed about the HJB equation; Hamilton Jacobi Bellman equation; this equation we have derived using the principle of optimality and particularly for a continuous time system.

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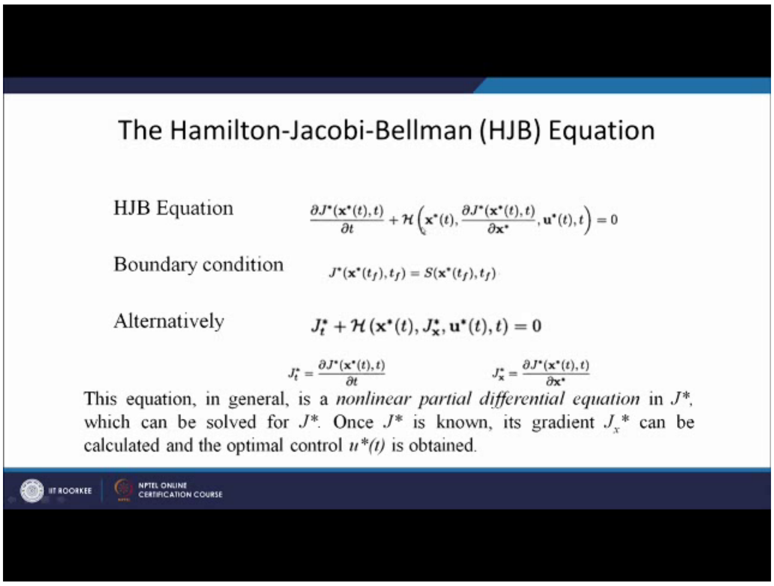

We have derived the equation del J star by del t plus H, H is this is my optimal value of the H because the optimal u I am utilizing here. So, with optimal value of the u my; this is my Hamilton Jacobi equation which we have to solve with the terminal condition which is J star x star of t f equals to s x star t f t f and this boundary condition is known to me.

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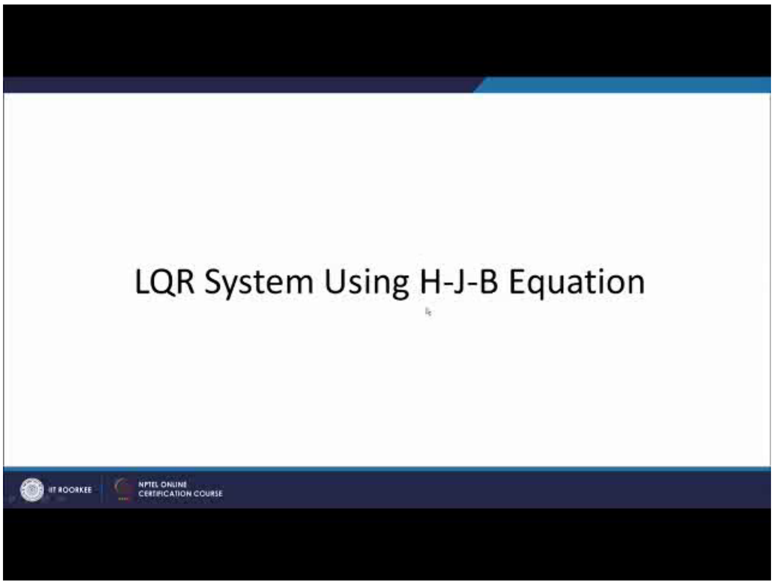

So, in this discussion we will apply the HJB equation to my LQR system means we will consider a linear time in variant system or it may be time in variant or to time varying. So, we are applying HJB equation to my LQR system we are considering the system as x dot t equal to A t x t plus B t u t.

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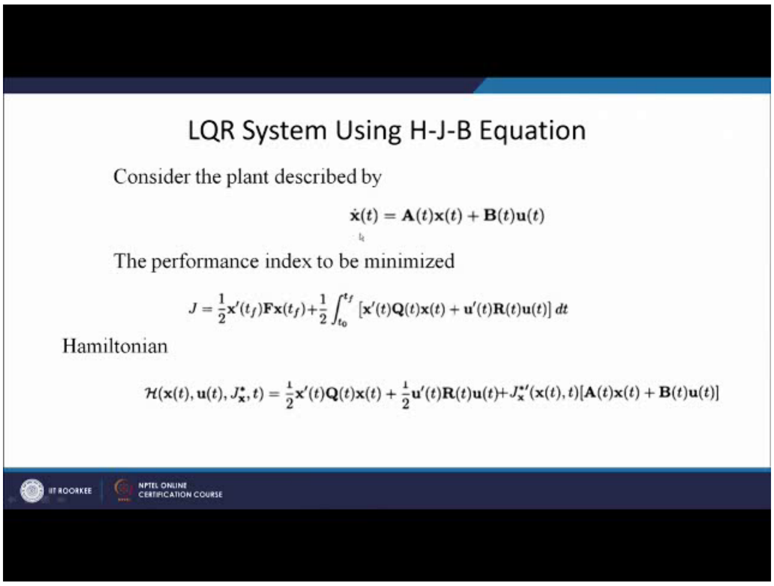

So, we are considering a time varying system and the performance index we have considered as a quadratic performance index defined as half of x prime t f f of t f x of t f. So, this f should also be the function of time which is missed here. So, this is the f of t f plus half of t 0 to t f x transpose t Q t x t plus u transpose t R t u t d t this is our standard as we have considered in the previous case the same system and the performance index we are considering here what is my HJB equation HJB equation is J plus H.

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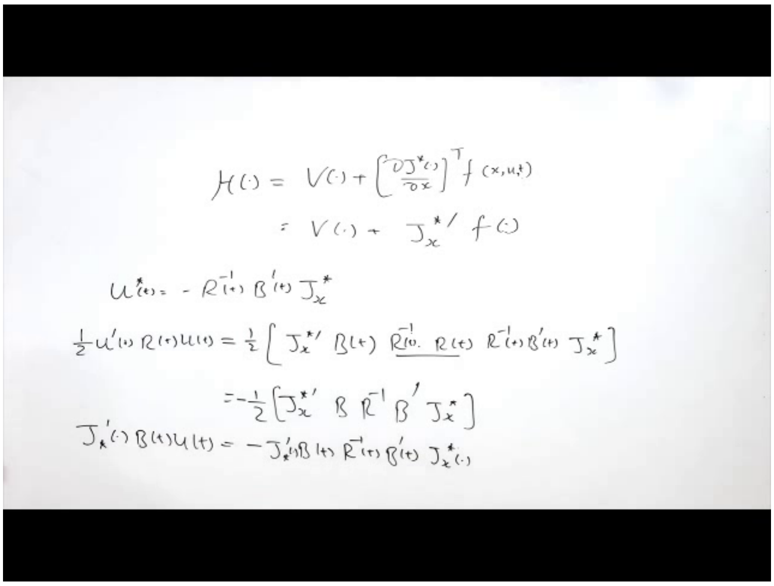

So, first of all I have to define what will be my Hamiltonian and what is Hamiltonian we have defined we have defined the Hamiltonian as if you recall H we have defined as my V plus del J by del x transpose f x u t or f naught we can write.

So, if the V plus del J by del x transpose or simply we can also write as V plus J x star transpose f now in our system which we have considered here it is a linear time varying system. So, this is my f is A t x t plus B t u t is nothing but my f and V is x prime Q x plus u prime R u. So, we are defining this H at the journal of control point u t we have considered as half x prime Q x plus half u prime R u this is nothing but my V plus J x prime A t x t plus B t u t which is nothing but my f. So, we can define Hamiltonian in this way my next step is I have to find the optimal value of this H this means del H by del u must be equal to 0.

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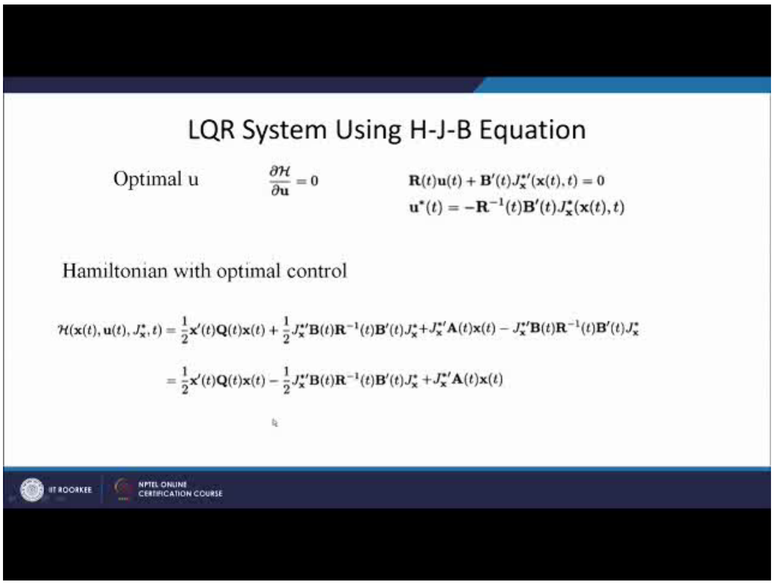

So, for optimal u my del H by del u is 0 which gives me R t u t plus B prime J x prime x t t this equals to 0. So, I got this equation if I am differentiating this H with respect to u. So, my; this first term is independent of u. So, this will give me 0 this is a quadratic terming u. So, this give me R u and J prime A t x t is independent of u. So, this will be 0 and J prime B u. So, this will be this will give me B prime j. So, this we are getting as B prime J should be J x t.

So, by this equation I can find out my u star as minus R inverse B prime J x t now if I will substitute this u in this H. So, I will get my optimal value of the h. So, in the next step we are finding the Hamiltonian by substituting the value of the u star in H my first term remain the same half x prime Q x this is u prime. So, what we have taken my optimal u is minus R inverse d prime J x star and the term which we are placing that is the half u star u prime R u which is my second term of the Hamiltonian.

So, if I will take the half of u prime u prime is J x star prime B R inverse is giving me the first term u prime then I will have the R then u u is R inverse V prime J x star . So, u prime R u I can write in this way this will be my unity. So, this is nothing but with minus half J x star prime B I am dropping the t B R inverse B prime J x star and this we are writing as my second term half J star prime B R inverse B prime J prime.

Now, what is the next J star a x t by multiplying this J star with the first term. So, I got this term as J star a x t my next term will be J star B t and multiplied with u t. So, this is J star prime B t u t and u t we replace with. So, this is J star prime B t u R inverse d prime J x star. So, my last term is J x prime B R inverse B prime J x star. So, these 2 term if you will see this is the half of this and this is one times of this. So, by subtracting by these 2 terms I get minus half of J x star B t R inverse V prime J x star plus J x star a t x t.

So, this is my optimal value of the H if this is my optimal value of the H.

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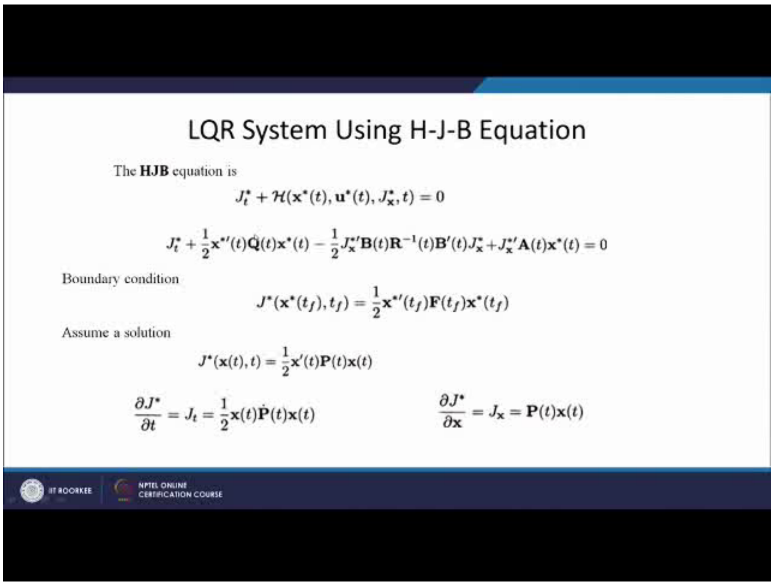

So, I can write my HJB equation as J t plus optimal value of the H equal to 0 this is the J t what is the optimal value of H is my half of x prime Q x minus half of J prime V R inverse V prime J x J x star A t x t equals to 0.

So, I got this as my HJB equation. So, my HJB equation I have to find out the value of the J star I have to solve this equation with my terminal condition as J star x of t f t f is half of x star t f f of t f x of t f. So, for a linear system for non-linear system no direct solution you can get for this HJB equation then we have to solve these equations using the numerical techniques, but for the linear system as we have seen my terminal cost at the terminal my cost function is J star x star t f t f which is equal to a quadratic function of x.

So, I can assume a solution which is similar to my terminal solution and this is nothing but J I am assuming as half of x prime P x where this optimal cost is a function of x and t if J is a function of x and t.

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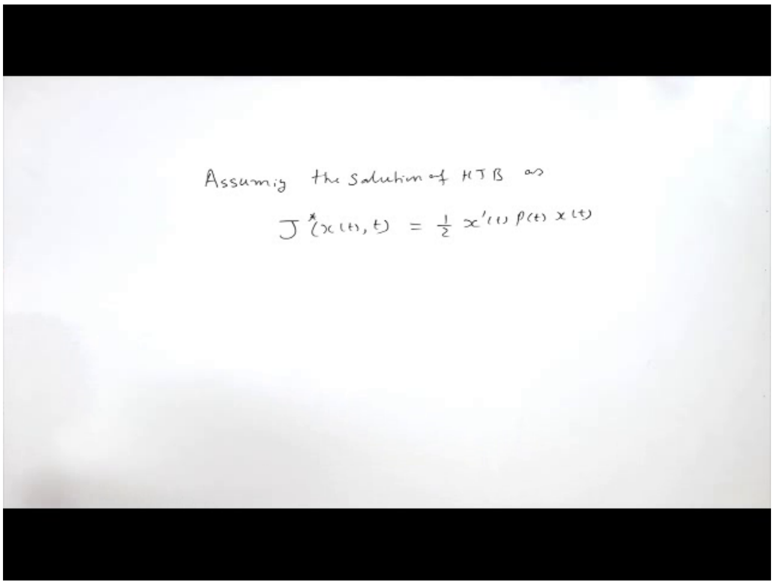

So, I can find out del J by del t and del J by del x what we are saying we are assuming the solution of HJB as let the optimal cost is the half of x prime t x. So, I am assuming my cost as this which I have take considering analogous to my terminal cost and with this value of the J I am finding out del J by del t and del J by del x. So, with this my del J by del x is half of because x and t are my 2 variable which we are considering t is a function of time.

So, this will be half x P dot x and del J by del x is simply my P t x t. So, this J t and J x I am replacing by half of x t P dot t x I am replacing the J t by this. So, my HJB equation will become. So, my first term is J t; J t I am taking as half of x prime P dot of t and x t.

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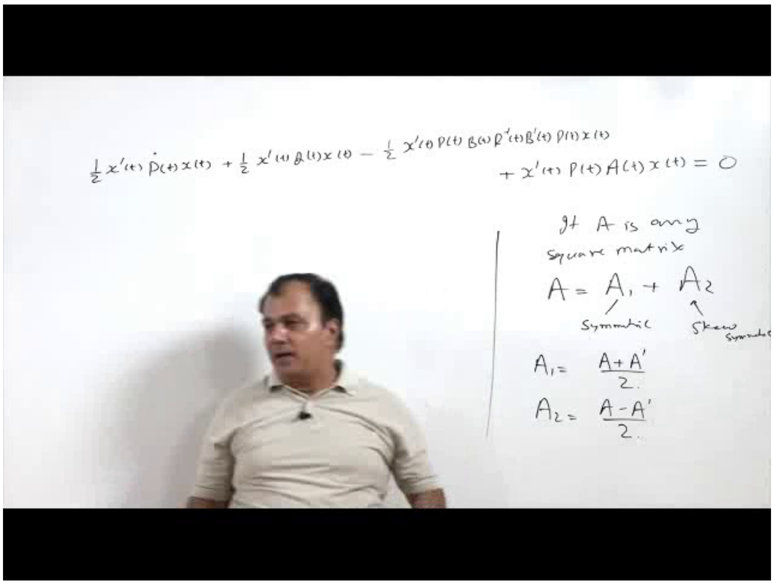

This is my J t my text term is half x prime Q x plus half x prime Q x next term is minus half of J x prime B R inverse B prime J x and what is the J x my P t x t. So, half of J x prime J x prime is P t x t. So, I am writing this as x prime t P t because in this case again the P t we have considered as a symmetric positive definite matrix then J cost, because why positive definite because J must have some positive value of the cost if my minimum cost cannot be negative. So, cost will be positive. So, P will be x prime x is square term P should be a positive definite.

So, if P is a positive definite and symmetric and as we have considered in the previous case the same analogy we can take it here. So, transpose of this will be x prime t x prime t B R inverse B prime J inverse P t x t this is my third term of the HJB equation and the last one is J x prime A t x t J x prime is x prime t. So, I can write this as last term plus x prime P t A t x t equal to 0.

So, the x J V equation given in this form with my assume solution as J as half of x prime P x I find out J del J del t del J by del x substituting these value in this equation.

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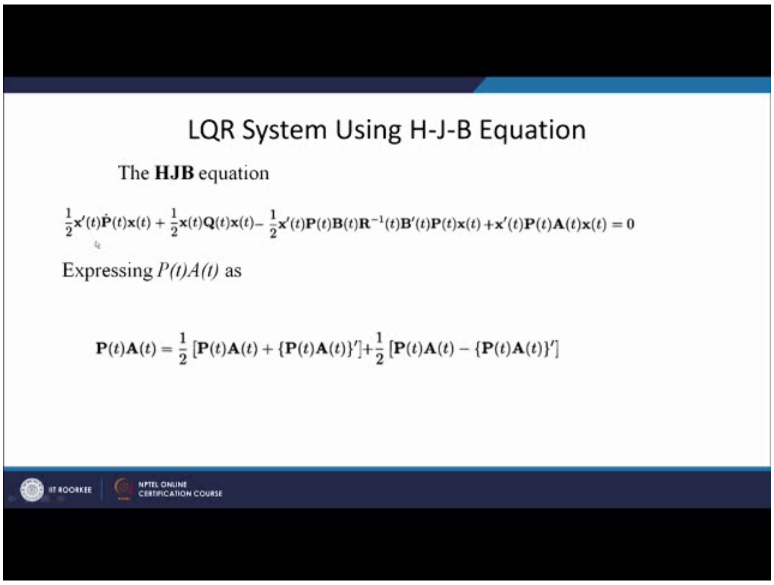

I am getting my equation as HJB equation as half of x prime P dot x plus half of x Q x minus half x prime P B R inverse prime t x x prime P t A t x t. So, now, if we will see the first term first term is quadratic and symmetric second term is quadratic and symmetric third term is also quadratic and symmetric except the last term last term will have my P t A t P is symmetric A may be any of the metric. So, we are using the principle that let if A is any square matrix then a can be written as A 1 plus A 2 where A 1 is symmetric A 2 is Q symmetric then A 1 will be A plus a transpose by 2 and A 2 will be a minus a transpose by 2. So, any matrix I can write in terms of the A symmetric form is giving me the A plus A prime by 2 and A 2 is given as A minus A prime by 2.

So, the matrix P t given here we can break up as half of P t A t transpose of P t A t plus half. So, this is my symmetric term and half P t A t minus transpose of P t a t is my skew symmetric form. So, this P t A t I can represent in this because my all terms are symmetric. So, the only part of this matrix which is useful that is my symmetric part because the skew symmetric part is multiplied with the function sorry x prime half of P t A t minus P t A t prime into x.

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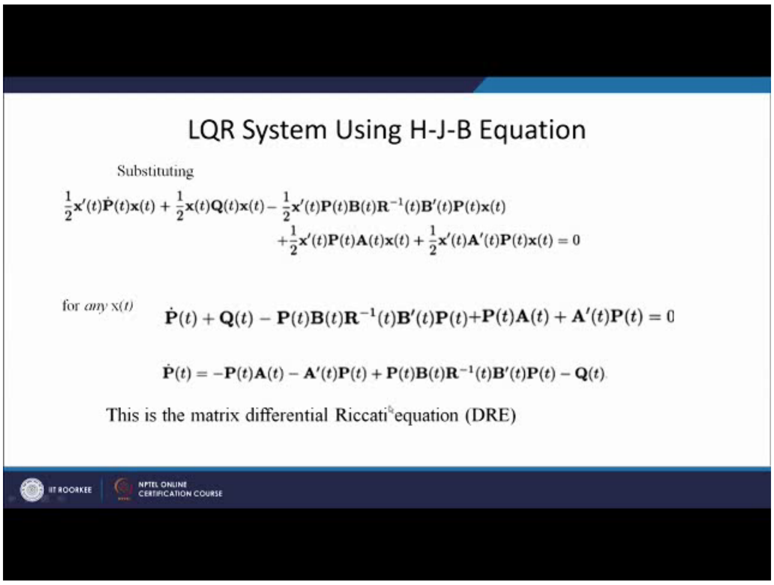

So, this will vanish and we are left only with this symmetric part. So, I can replace P t A t as half of P t A t plus P t A t prime if I will replace this. So, I will get the half of x prime P t A t plus half of x prime transpose of P t A t is a transpose P x. So, now, I can convert my; this whole equation in terms of this equation.

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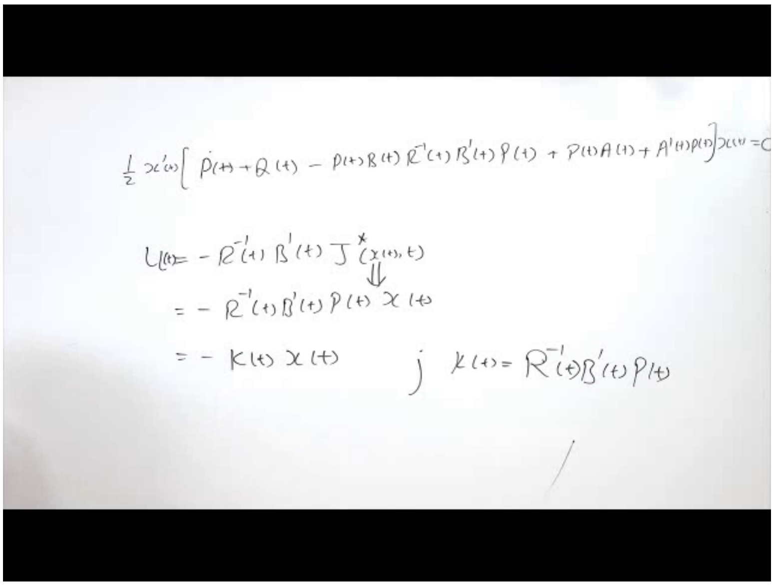

And this equation if you will see can be break up as half of x prime t I have P dot plus Q P dot this will be P t plus Q t minus t sorry P B R inverse B prime P plus P t A t plus a transpose t plus a transpose t P t multiplied with sorry simple x t must be equal to 0. So, for any value of the x t because x t may be arbitrary. So, I can say what is here in bracket. So, this equation to be 0 my; this whole term must be equal to 0.

So, I can write as P dot t plus Q t minus P B R inverse B prime P a plus a prime P equals to 0 for P t I can write simply in terms of as minus P t a t minus a prime P plus P B R inverse B prime P minus Q and this is my well known matrix differential Riccati equation which we have proved also which we have also derived using the calculus of variation approach.

So, once the P is known my u was minus R inverse J x t t this was my u. So, this u will be nothing but my R inverse B transpose t P t x t because this solution is nothing but P t x t and this I can say as minus k t x t. So, I can have the close loop feedback control where my k t is nothing but R inverse B prime P t and P t is nothing but the solution of my matrix Riccati equation. So, using the HJB equation also we can have the close loop feedback control.

So, this is the application of the Hamilton Jacobi Bellman equation to a continuous time system now up to this point we have made our discussion in which we have not considered any constraint to any of our variable neither on the u nor on the x. So, at the end of our course we will consider one problem in which we are going to constrain the u and see how the constrain on the u will change the solution of my problem. So, this constrain optimization problem we use the Pontryagin minimum principle.

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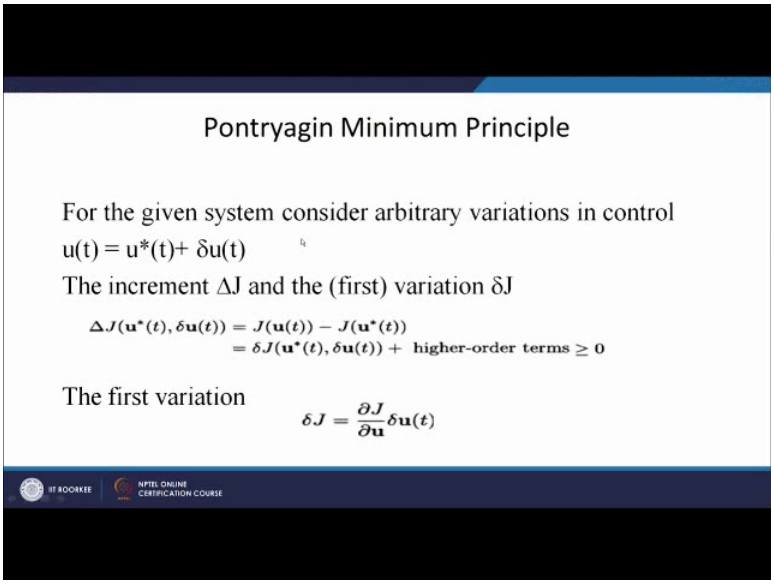

So, we will discuss this principle say in a given system if you will consider any system and consider a variation in the control means by control is varying from its optimal value u star t to delta u t with this variation we can find out actually is the increment in the system or the functional and what is my first variation if we will try to find out this. So, as we know by definition my increment is defined as J at a given value of the control minus J at the optimal control where u t is u star plus delta u. Now this J u t can be expanded as Taylor series.

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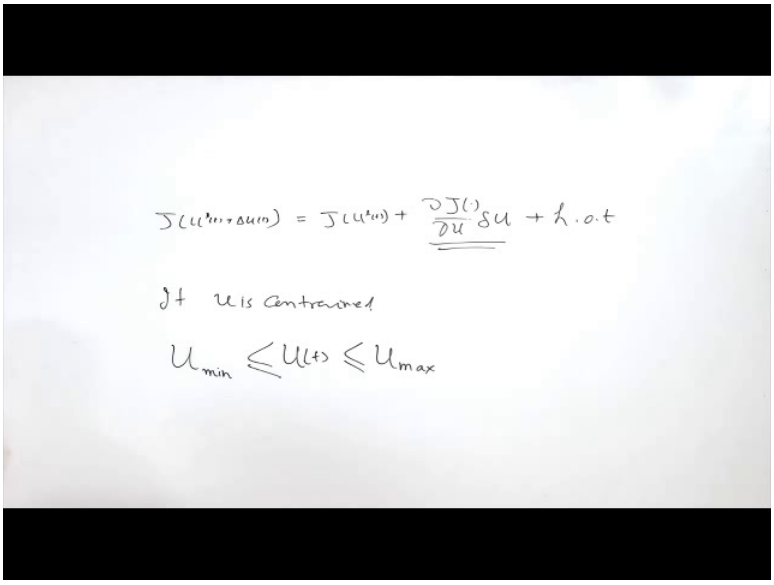

So, what we can write for J u star t plus delta u t because this is the value of my J u t. So, this will be nothing but my J u star of t plus del J by del u delta u the first order term plus higher order term this will appear as my extension and once I will separate J u star u t. So, I will left only with the delta J u star of t plus the higher order term and by the minimization principle if my cos function is minimum then my increment must be greater than or equal to 0.

So, once we will and in this we will write our first variation delta J as del J by del u into delta u t. So, for optimal cost my increment should be greater than 0 and in case of the unconstrain what was our condition.

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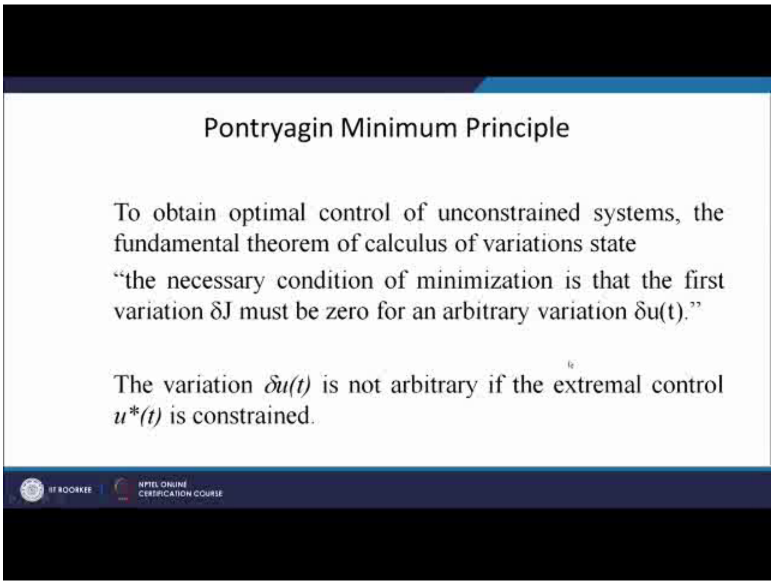

This means my first variation must be vanishes means we considered del J equals to 0 if my u is unconstrained and the sufficient condition is second variation is must be positive definite. So, if I will write the increment this will be the solution of my first variation plus second variation which is positive. So, this means J is always greater than equal to 0. So, my fundamental theorem is at optimal value of the u if u is unconstrained then my first variation will be 0.

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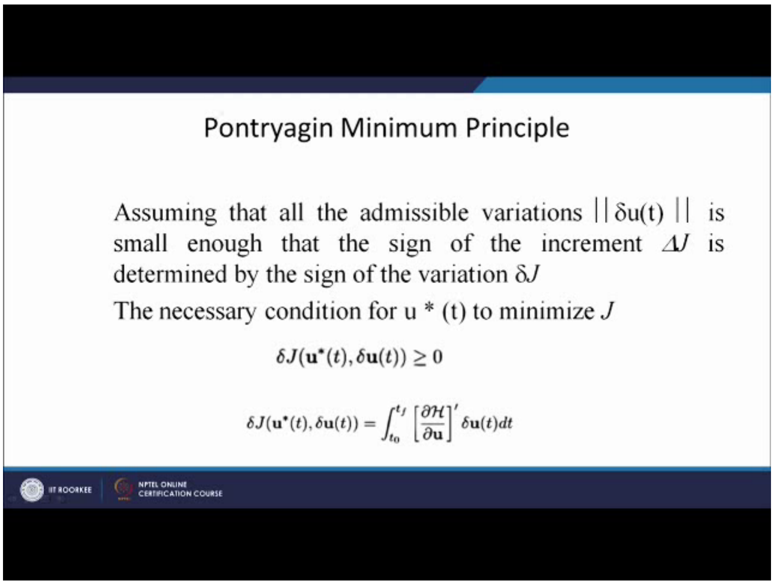

But if u is constrained means if u is constrained this means u t will have its limits with say u minimum to u max.

So, this is if my u is constrained then I cannot say my delta u will be arbitrary. So, delta u t is not arbitrary if external control u is constrained. So, assuming that all admissible variation means delta u is small enough that the sign of the increment delta J is determined by sign of variation delta J this means the sign of this increment. So, what we are saying if my delta u is not large enough then the sign of delta J will be determined by the sign of first variation because all other higher term will be smaller than the first variation. So, if u is constrained sorry u is unconstrained then my first variation is 0 then the sign is decided by the other variations, but if u is constrained then my first variation will not be 0 and the sign of increment is decided by the first variation.

So, this means I can say the necessary condition for u star to be minimized then my first variation must be greater than or maximum equal to 0. So, only 0 condition is not applied if my u is unconstraint and if I will find out that delta J the first variation which is t 0 to t f del x by del u into delta u if I am finding representing my problem in terms of the Hamiltonian that this is del x by del u delta u will be my first variation in delta J as we have seen before we can apply the Legrangin or the Hamiltonian approach and if you are using the Hamiltonian.

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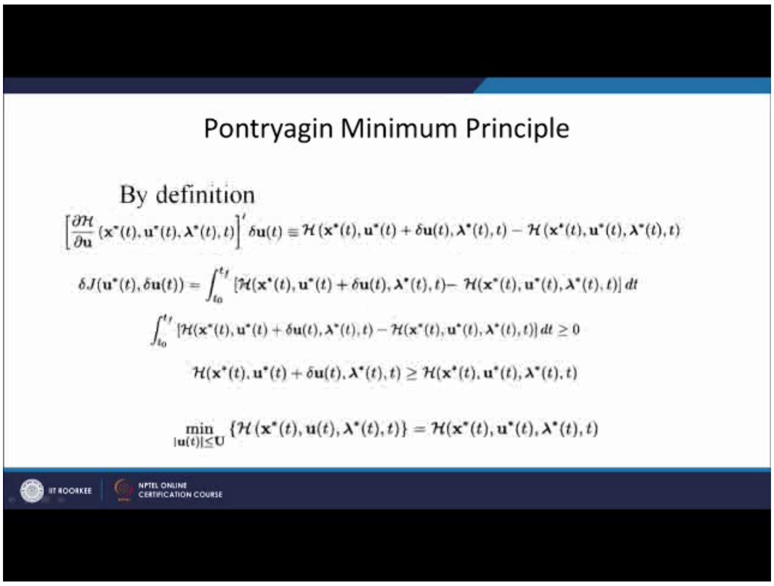

We can write delta J in terms of the Hamiltonian as del x del u transpose multiplied with delta u this means del x by del u delta u can be written as H at the variation point minus H at the optimal point. So, this is the meaning of my del H by del u this means delta J is H at variation point and H at the optimal point and this del J much be greater than equal to 0 this means my right hand side of this must be greater than or equal to 0.

This shows that if I will have H at a variation point this value will always be greater than equal to the value of the H at optimal point. So, this means if I will minimize the H at any given control I can minimize this u within the constrained of u max and u minimum then I will get whatever be my optimal value of the H and this is my this is the Pontryagin minimum principle means the Hamiltonian; we have to minimize with respect to u to get the optimal value of my Hamiltonian, because this Hamiltonian at the u point and this Hamiltonian at the u star point minimization of this Hamiltonian giving me the optimal value of my H.

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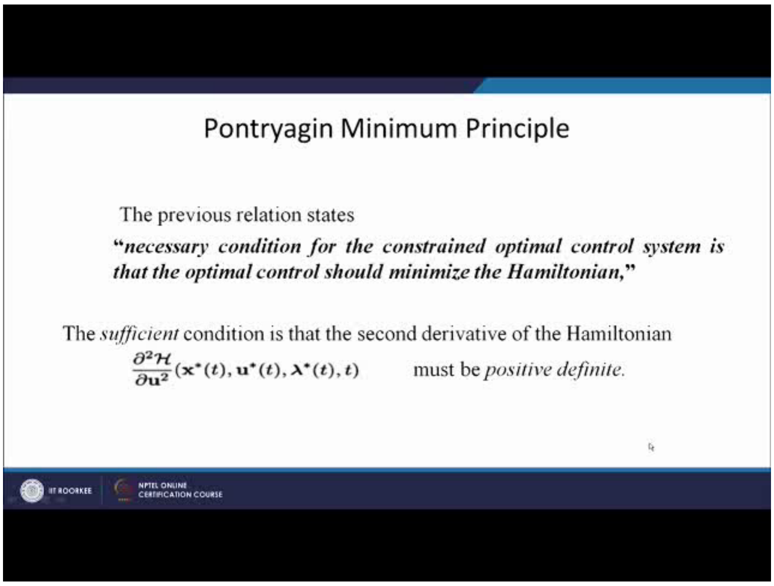

And this is my necessary condition for the constrained optimal control system that the optimal control should minimize the Hamiltonian and the sufficient condition is my second derivative of this H must be positive definite. So, this Pontryagin minimum principle we can directly apply to the constrained system.

In the next class we will see that how the constrained system this Pontryagin minimum principle can be applied to the constrained system and will be used to solve the minimum time control problem.

Thank you very much.