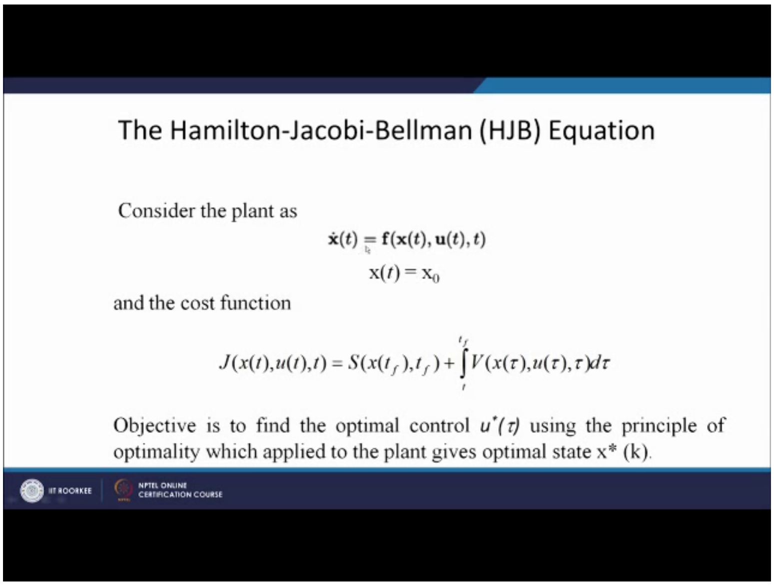
Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture - 37
The Hamilton-Jacobi-Bellman (HJB) Equation

Welcome friends to the session of our discussion. In today session we will discuss about the application of the dynamic programming on continuous time system; in the previous class we have seen that how the dynamic programming approach can be applied to a discrete system.

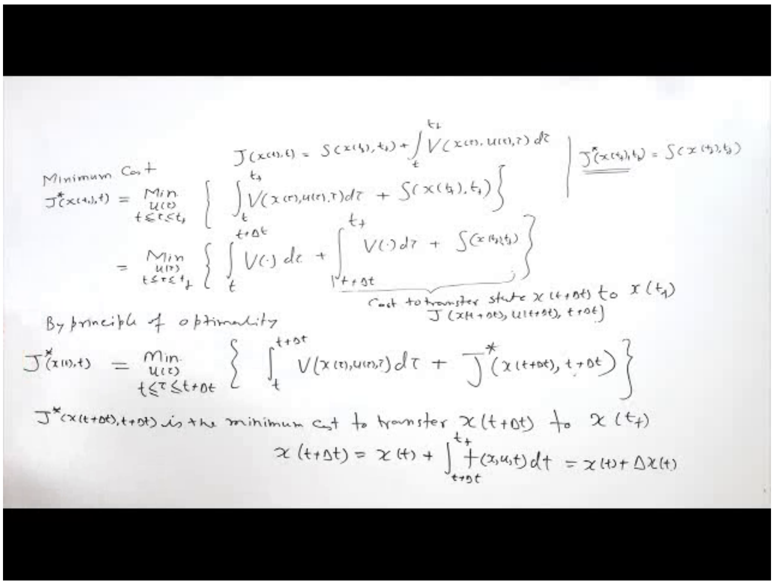
This dynamic approach is basically based on our principle of optimality which we have discussed in the previous class we say the optimal policy as the property that whatever the previously stated decisions the remaining decisions was must constitute an optimal policy with regard to the state resulting from the previous decisions.

(Refer Slide Time: 01:04)



Let us consider a continuous plant given as x dot t as f x u t general plant we are considering maybe linear or non-linear with initial condition x t as x 0 t is my variable parameter which varies from t to t f. So, this means the initial point we are considering as t and t can take any value.

(Refer Slide Time: 01:50)



Now, we define a Cos function as J x t u t t as s of x t f t f which defines the terminal cost plus the integral value as V x tau u tau d tau here tau is the lies between my t to t f. So, here what the J x t u t t represents this represents the cost of transferring the state x t to t f in a time interval of t to t f.

So, this J x t representing my cost to transfer the state at its initial value, let us say x t to t f value now the objective here is again to find out the optimal value of the u star using the principle of optimality which applied to the plant and gives the optimal states. So, as we have defined my J which we are saying as x t t my terminal cost x of t f t f plus integral t to t f V, we are seeing the tau here using the variable u tau tau and theta. So, this is my Cos function and J x t t to represent the cost to transfer the state from x t to x t f.

So, if I will write the boundary point as J x of t f t f which is nothing but my s x of t f t f. So, this is my terminal point at terminal point this is the only cost which is involved. So, I can see this will be my optimal cost at the terminal state x t f because x t f is my final state beyond this I am not transferring. So, at this stage terminal point whatever be the cost that I can say will be my optimal cost now this J x of t, I can write by breaking the integral or see let us say what is my minimum cost minimum cost means what is the optimal value of the J x t t this I can write as I have to minimize u tau where tau is varying between t to t f with let us say I write in this form t to t f V x tau u tau d tau plus s of x of t f t f.

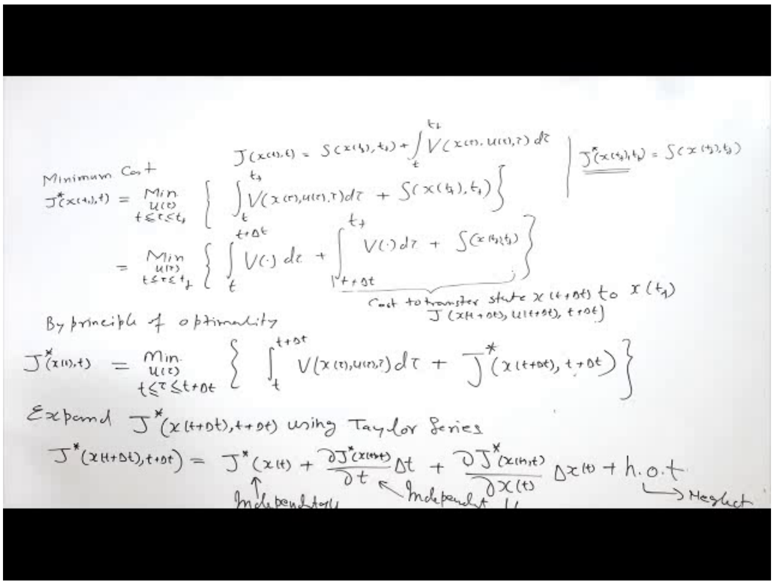
This means I have to minimize this objective function with respect to u tau because my objective is to find out the optimal value of the u. So, that I will get this function get minimized and I will get the minimized cost as J star. So, this I can write as u tau t plus sorry t f and this integral I break from t to t plus delta t V d tau plus t plus delta t to t f t tau plus my s this is x of t f t f point.

Now, what this represent here second tau this second term is the cost to transfer the state from x t plus delta t this is the cost to transfer state x t plus delta t to x of t f. So, this is my cost to transfer the state from this point x t plus delta t to x of t f and if I will say this is nothing but my J x t plus delta t u t plus delta t and t plus delta t. So, this means if I will apply the principle of optimality by the principle of optimality if my; this cost is minimum I can assume my this cost to be the minimum. So, I can simply write as my J star x t t as minimum u tau and I am minimizing the u tau only for t to t plus delta t and let beyond t plus delta t to t f I already have my minimum cost. So, I can write this as t to t plus delta t V I am writing the full x tau u tau tau d tau plus this I can write as J star x t plus delta t t plus sorry t plus delta t.

So, at this stage this is I already have minimized with respect to u. So, this J star is independent for my u, but this x will be as x plus 1, we have considered as the f x u t. So, this is my f. So, this means we still my whole expression will depend on u tau between the time interval t to t plus delta t. So, this is my minimum cost. So, we can say J star x t plus delta t t plus delta t is the minimum cos to transfer x from t plus delta t to x of t f. So, this will be my minimum cost and what is my x t plus delta t this is nothing but my x t the initial cost which is given as the x 0 we have considered plus whatever be the state resulting from t plus delta t to t f point if I will solve my f x u t d t and this I can simply write as x t plus delta x of t. So, means my initial state to determine this is varying which we can find out by this. So, this is the meaning of my J star s t plus delta t d t.

Now, we can expand now expand J star x t plus delta t t plus delta t because the variation at the point t using Taylor series. So, we can expand this using the Taylor series what actually we will get by this J x t plus delta t t plus delta t if I will expand.

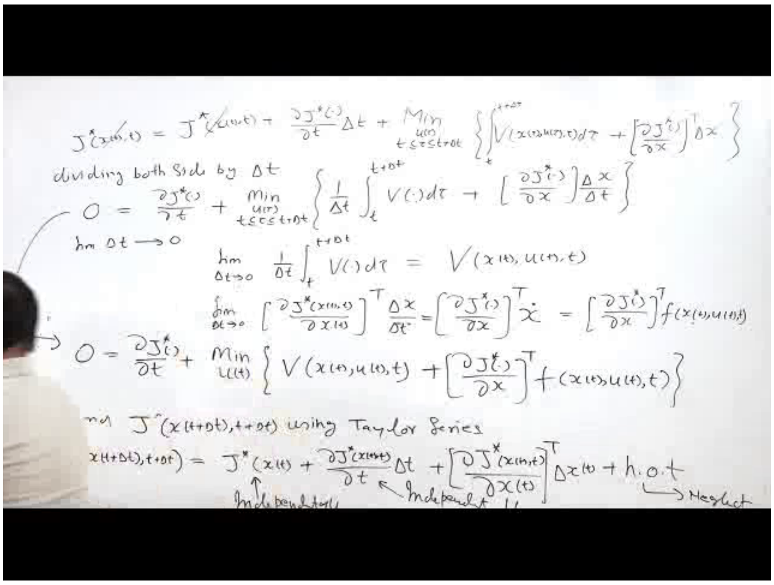
(Refer Slide Time: 13:07)



So, what I am getting J star of x t plus del J star x t t by del t delta t because now my I have the variation in the t which is varying from t to t plus delta t. So, I have the term del J star by del t delta t plus. Now second is x t t by delta x of t delta x t plus my higher order terms and higher order terms will also be the function of your del t and del x, but if you will consider the variation in delta t to be very small. So, from this we can neglect this higher order term these terms we neglect.

So, in this expression this J star x t plus delta t plus delta t we can replace by this whole term and this whole term if you will see this is my minimum which is not dependent on my u. So, this term will be independent of u similarly del J star x t t also independent; independent from independent of u.

(Refer Slide Time: 15:36)



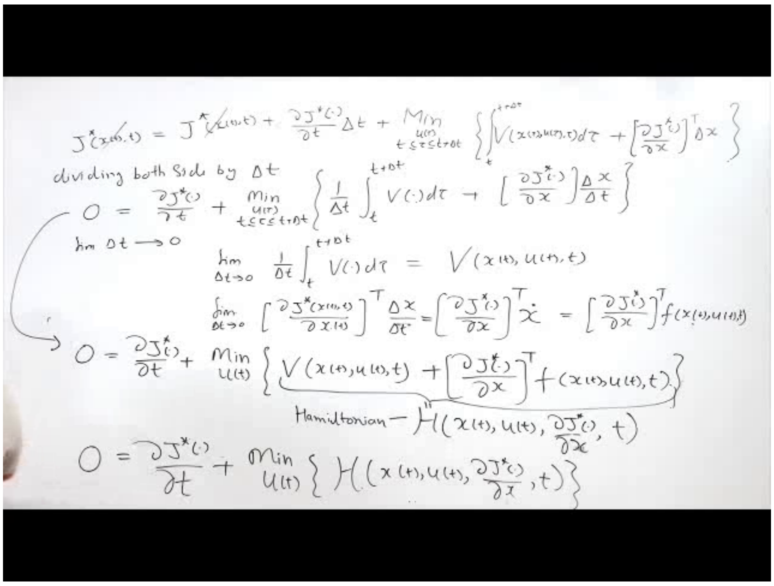
So, these 2 terms we get. So, for J star x t t what we are writing say in this expression if I will replace them this is independent of u this is independent of u. So, these 2 terms I can take it outside from the minimization. So, if I will write here. So, these 2 term I can write outside from my minimization I am writing. So, this is J star x t t plus del J star having the same meaning by del t delta t plus minimization of u tau where tau is within the interval of t to t plus delta t I have V sorry integral from t to t plus delta t V x tau u tau d tau plus my second term del J star by del x say if we will make this. So, this is nothing but because we have to multi x is my vector and this will be also be del J by del x 1 del J by del x 2. So, I have to take to transpose to this. So, actually this term will be del J by del x whole transpose multiplied with del x. So, this is transpose delta x and the higher terms we already have been neglected.

So, in this equation these 2 we cancelled out so this will be 0. And now dividing both sides by delta t. So, what actually we will get this side will be 0 J x will be cancelled out del J star by del t dividing del t. So, this will be 1 plus minimum u tau and tau is within my range t to t plus delta t. So, I will say this is 1 by delta t integral t to delta t I am writing this simply as V dot d tau plus this I am writing as del J dot by del x which will be del x t into delta x by delta t as we are saying we are considering delta t to be very small. So, if we will limit delta t approaching to 0.

So, my terms will be limit delta t approaching to 0 my first term is 1 upon delta t t to t plus delta t V dot d tau with this limit. So, this is nothing but my V x t. So, this is my V x t u t t we can write for 1 by delta t integral V dot t and limit delta t tends to 0 del J star this is my x t t by delta x t the actual is this. So, this we are basically dropped out in this transpose del x by sorry del t which can be written as with this limit to this. So, this is nothing but your del J star by del x transpose and this is nothing but my x dot which is d x by d t or simply I can write this as del J star by del x transpose and x dot is nothing but my f x t u t t. So, my; this term is replaced by this and this term can be replaced by the second term.

So, I can write this equation as 0 del J star by del t plus minimum. So, now, this whole term is I can take this simply as my u t. So, my first term is replaced be V x t u t t and my second term is del J star by del x transpose f x t u t t. So, I can write.

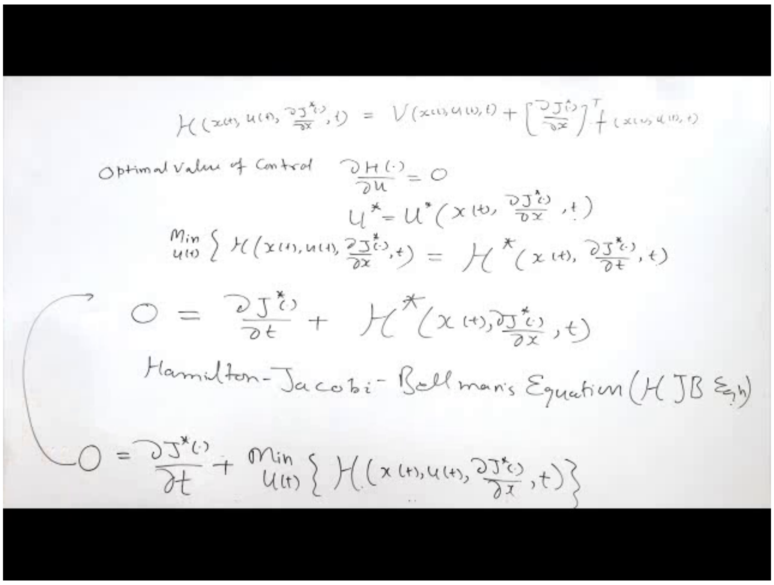
(Refer Slide Time: 23:18)



So, now I will define this whole as my this is equal to I will define this as my Hamiltonian which will be the function of my x t u t this del J star sorry this is del x and naturally t. So, this Hamiltonian is the function of x u del J by del x and t. So, I can write this equation simply as del J by del t plus minimum of u t which is nothing but my h function of x t u t del J by del x and t.

So, this means in the minimization of this I have to minimize the h with respect to u. So, what we have done.

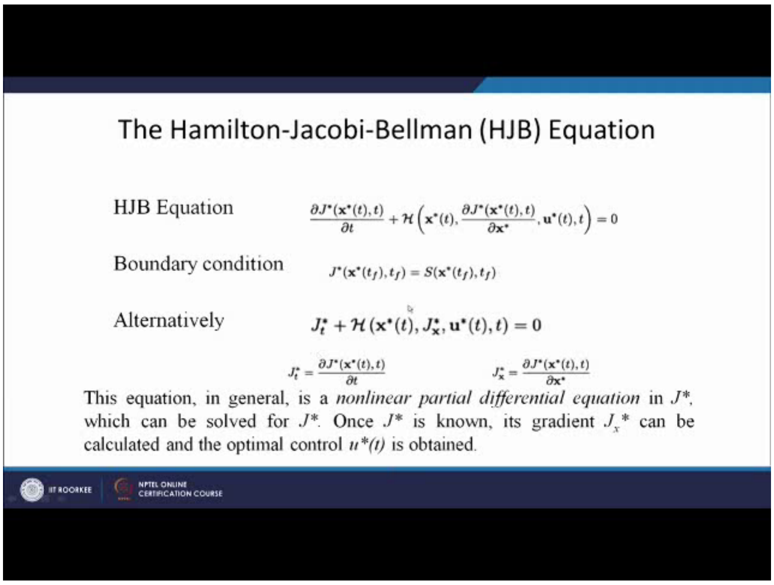
(Refer Slide Time: 25:02)



We have defined the Hamiltonian which is a function of x u del J star by del x t as my V x t u t t plus del J star by del x transpose multiplied with f x t u t t and to satisfy this equation I have to get the minimum value of the h. So, for optimal control for optimal value of control we have del h by del u equal to 0 if we will apply this. So, we will this give me the optimal control u star and we have this u star will be the function of x t because we are determining the u in terms of the x del J by del x and t. So, naturally this; my optimal u is independent of the u term because u V still determining in terms of the x del J by del x and t form.

So, I can write this minimum of if I will write u t as h x t u t del J by del x t and I am substituting u into this. So, I will get nothing but my h star which will be the function of x t del J star by del t t. So, I can simply write this equation as del J star by del t plus the minimum value of the h which is the function of my x t sorry del J star by del x t and this equation is called my Hamilton Jacobi Bellman's equation and more precisely we say HJB equation.

(Refer Slide Time: 29:05)



So, we got del J star x star t t by del t plus h x star del J star by del x t u star t because u is replaced by the optimal value of u equals to 0 this is nothing but my HJB equation and what was the boundary condition J star x star t f t f equals to s x t f t f. So, this means I have to solve this equation subjected to the boundary condition given by J star x of t f t f and this terminal cost is known to me.

So, with the known terminal condition I have to solve this HJB equation, this equation in more general form we also write as J t star plus h equal to 0 where J t is del J by del t and J x here is del J star by del f star it is a non-linear partial differential equation in J star which can be solved for the J star this means we can find out the optimal cost once J star is known del J by del x means that radiant can be calculated and by this we can also find out the can find out the optimal value of the control.

So, we have the application of the HJB equation to solve the optimal control problem. So, this lecture I stop it here. And in the next lecture, we will discuss about how we can implement the HJB equation to determine the optimal control value.

Thank you very much.