

**Optimal Control**  
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**Lecture - 35**  
**Analytical Solution of Matrix Difference Riccati Equation (Continued)**

So, welcome friends to the session of our discussion which we will continue on the analytical solution of matrix difference Riccati equation. In the previous class we are start over discussion on this in which we are defining  $X(k)$  equal to a matrix  $H$  which is given as  $A^{-1} A^{-1} E Q A^{-1} A^T + Q A^{-1} E$ .

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**Analytical Solution of Matrix DRE**

The Hamiltonian System with the state & co-state equation is given by,

$$\begin{bmatrix} X(k) \\ \lambda(k) \end{bmatrix} = \begin{bmatrix} A^{-1} & A^{-1}E \\ QA^{-1} & A^T + QA^{-1}E \end{bmatrix} \begin{bmatrix} X(k+1) \\ \lambda(k+1) \end{bmatrix} \quad (1)$$

where,

$$E = BR^{-1}B'$$

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And this matrix we are getting from over Hamiltonian system which was defined for  $X(k+1)$  and  $\lambda(k)$ . So, in the previous class we can we have seen that we can find and give name  $H$  to this matrix.

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### Analytical Solution of Matrix DRE


The Hamiltonian System Matrix is defined as,

$$H = \begin{bmatrix} A^{-1} & A^{-1}E \\ QA^{-1} & A^T + QA^{-1}E \end{bmatrix} \quad (5)$$

Consider a matrix of the form,

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad (6)$$

Such that,

$$J^{-1} = -J$$


Then we have shown that with the transformation J as a null matrix of n by n, identity matrix of n by n we can define J as a 2 n by 2 n matrix.

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### Analytical Solution of Matrix DRE

Pre and post multiplying the System Matrix results in,


$$H'JH = J$$

Proof:

$$H'JH = \begin{bmatrix} A^{-T} & A^{-T}Q \\ EA^{-T} & A + EA^{-T}Q \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A^{-1} & A^{-1}E \\ QA^{-1} & A^T + QA^{-1}E \end{bmatrix}$$

$$= \begin{bmatrix} -A^{-T}Q & A^{-T} \\ -A - EA^{-T}Q & EA^{-T} \end{bmatrix} \begin{bmatrix} A^{-1} & A^{-1}E \\ QA^{-1} & A^T + QA^{-1}E \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = J$$

Resulting in,

$$J = H'JH \Rightarrow JH^{-1} = H' \Rightarrow H^{-1} = J^{-1}H' \Rightarrow H^{-1} = -JH'$$



Where J inverse is minus J and we got H inverse at J H inverse J using the H inverse relation we are shown that if mu is an eigenvalue of the matrix H then 1 upon mu is also an eigenvalue of the matrix H.

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### Analytical Solution of Matrix DRE

The inverse of Hamiltonian matrix can be obtained as,

$$\begin{aligned} \mathbf{H}^{-1} &= -\mathbf{J}\mathbf{H}^T \\ &= -\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-T} & \mathbf{A}^{-T}\mathbf{Q} \\ \mathbf{E}\mathbf{A}^{-T} & \mathbf{A} + \mathbf{E}\mathbf{A}^{-T}\mathbf{Q} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \\ &= -\begin{bmatrix} \mathbf{E}\mathbf{A}^{-T} & \mathbf{A} + \mathbf{E}\mathbf{A}^{-T}\mathbf{Q} \\ -\mathbf{A}^{-T} & -\mathbf{A}^{-T}\mathbf{Q} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A} + \mathbf{E}\mathbf{A}^{-T}\mathbf{Q} & -\mathbf{E}\mathbf{A}^{-T} \\ -\mathbf{A}^{-T}\mathbf{Q} & \mathbf{A}^{-T} \end{bmatrix} \end{aligned} \quad (7)$$



So, this means  $\mu$  and  $1/\mu$  are the eigenvalues of matrix  $\mathbf{H}$ .


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### Analytical Solution of Matrix DRE

Comparing equation (9) & (7) we have,

$$\mathbf{H}^{-T} \begin{bmatrix} \mathbf{v}_2 \\ -\mathbf{v}_1 \end{bmatrix} = \mu \begin{bmatrix} \mathbf{v}_2 \\ -\mathbf{v}_1 \end{bmatrix} \quad (10)$$

From eqns. (8) and (10), it is clear that, if ' $\mu$ ' is an eigenvalue of the Hamiltonian Matrix  $\mathbf{H}$ , ' $1/\mu$ ' is also an eigenvalue of the Hamiltonian Matrix.



So, if we are considering a discrete time system and in discrete time system if you will see the eigenvalues lie inside the unit circle and outside the unit circle. If we say let  $\mu$  is the eigenvalue laying outside the unit circle sorry; inside the unit circle then  $1/\mu$  eigenvalues will lie outside the unit circle.



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### Analytical Solution of Matrix DRE

This property of the Hamiltonian matrix results in its ability to diagonalize in the form,

$$D = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix}$$

Where **M** is a **diagonal matrix** with diagonal elements as the **eigenvalues** of the Hamiltonian matrix, which are inside the unit circle.

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So, by this we can arrange these eigenvalues in terms of the diagonal by the similarity transformation. So, this means if **M** is a diagonal matrix with diagonal element as the eigenvalues of the Hamiltonian matrix **H** which are inside the unit circle. So, if **M** will contain the eigenvalues which are inside the unit circle then **M** inverse will contain the eigenvalues which are outside the unit circle.

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### Analytical Solution of Matrix DRE

Let '**W**' be the modal matrix that transforms the Hamiltonian matrix into diagonal form.

$$W^{-1}HW = D$$

The modal matrix of eigenvectors may be defined as,

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

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So, we can consider a modal matrix **W** which is defined as  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  which transform matrix **H** into the diagonal form where elements of the  $W_{11}$   $W_{21}$  this

nothing but one set of the eigenvectors corresponding to the eigenvalue  $\mu$ .  $W^{-1}HW$  is the another set of the eigenvector corresponding to eigenvalues  $1$  upon  $\mu$ .

So,  $W^{-1}HW$  is the transformation which convert sorry; which transform the matrix  $H$  into the diagonal form and in this diagonal form my eigenvalues are arranged in the diagonal as the eigenvalue inside the unit circle and eigenvalues as outside the unit circle.

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### Analytical Solution of Matrix DRE

The states of the system are also transformed using the same modal matrix.

$$\begin{bmatrix} X(k) \\ \lambda(k) \end{bmatrix} = W \begin{bmatrix} Z(k) \\ \Lambda(k) \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} Z(k) \\ \Lambda(k) \end{bmatrix} \quad (11)$$

The transformed Hamiltonian system is given by,

$$\begin{bmatrix} Z(k) \\ \Lambda(k) \end{bmatrix} = W^{-1} \begin{bmatrix} X(k) \\ \lambda(k) \end{bmatrix} = W^{-1}H \begin{bmatrix} X(k+1) \\ \lambda(k+1) \end{bmatrix} = W^{-1}HW \begin{bmatrix} Z(k+1) \\ \Lambda(k+1) \end{bmatrix} = D \begin{bmatrix} Z(k+1) \\ \Lambda(k+1) \end{bmatrix} \quad (12)$$

So, now, my transformation is let us say using the similarity transformation we transform  $X(k)$   $\lambda(k)$  as  $Z(k)$   $\Delta(k)$  multiplying with the modal matrix  $W$ .

So, this means  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  as  $Z(k)$  and  $\Delta(k)$ . So, if I will write the transformed Hamiltonian system in terms of the  $Z(k)$  and  $\Delta(k)$ . So, that will be this  $W^{-1}HX(k)$   $\lambda(k)$  as we have sorry it should be  $X(k)$   $\lambda(k)$  this is by mistake here because we are using the discrete time system. So, this will be  $X(k)$  and  $\lambda(k)$ . So, this will be  $W^{-1}HX(k)$   $\lambda(k)$  here and this  $X(k)$   $\lambda(k)$  we can write as  $H \begin{bmatrix} X(k+1) \\ \lambda(k+1) \end{bmatrix}$  and this  $X(k+1)$   $\lambda(k+1)$  we can represent as  $Z(k+1)$   $\Delta(k+1)$  as  $W \begin{bmatrix} Z(k+1) \\ \Delta(k+1) \end{bmatrix}$  and this  $W^{-1}HW$  is nothing, but my diagonal transformation. So, I am getting  $D \begin{bmatrix} Z(k+1) \\ \Delta(k+1) \end{bmatrix}$  and  $D$  is what? As we have considered before  $D$  is nothing, but  $M^{-1}HM$ .


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### Analytical Solution of Matrix DRE

The solution of the system of difference equation (12) is given by,

$$\begin{bmatrix} Z(k) \\ \Lambda(k) \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} Z(k+1) \\ \Lambda(k+1) \end{bmatrix} \quad (13)$$

$$\begin{bmatrix} Z(k) \\ \Lambda(k) \end{bmatrix} = \begin{bmatrix} M^{(k_f-k)} & 0 \\ 0 & M^{-(k_f-k)} \end{bmatrix} \begin{bmatrix} Z(k_f) \\ \Lambda(k_f) \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} Z(k_f) \\ \Lambda(k) \end{bmatrix} = \begin{bmatrix} M^{-(k_f-k)} & 0 \\ 0 & M^{-(k_f-k)} \end{bmatrix} \begin{bmatrix} Z(k) \\ \Lambda(k_f) \end{bmatrix} \quad (14)$$


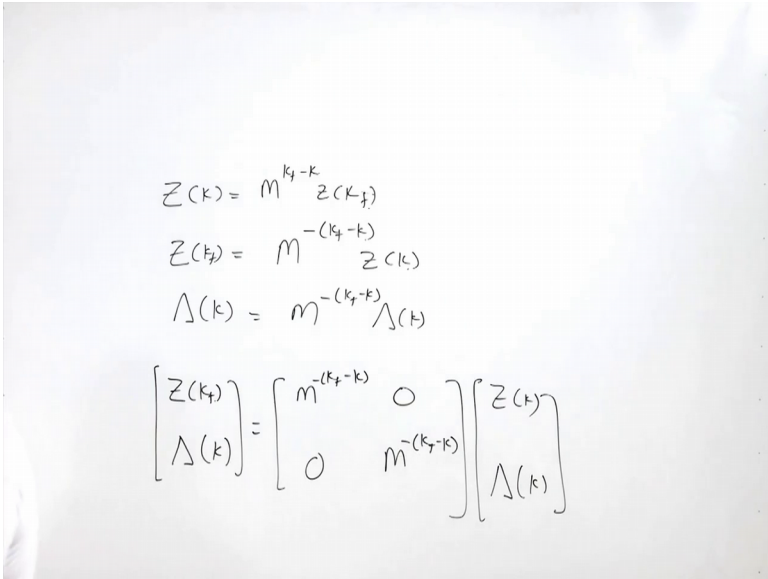
So, we can directly write  $Z(k)$  as  $M^{k_f-k} Z(k_f)$  and  $\Lambda(k)$  as  $M^{-(k_f-k)} \Lambda(k_f)$ . So, say set of the difference equations where  $Z(k)$  and  $\Lambda(k)$  combinedly will have the  $2n$  vectors  $n$  for  $Z$  and  $n$  for  $\Lambda$  and  $M$  inverse is  $2n$  by  $2n$  matrix  $Z(k)$  and  $\Lambda(k)$  again a vector of  $2n$  cross  $1$ . So, this is my difference equations which can be solved utilizing the endpoint condition. So, if I will solve this. So, my solution of the equation will be  $Z(k)$  which is  $M^{k_f-k} Z(k_f)$  and  $\Lambda(k)$  which is  $M^{-(k_f-k)} \Lambda(k_f)$ .

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$$Z(k) = M^{k_f-k} Z(k_f)$$

$$Z(k_f) = M^{-(k_f-k)} Z(k)$$

$$\Lambda(k) = M^{-(k_f-k)} \Lambda(k_f)$$

$$\begin{bmatrix} Z(k_f) \\ \Lambda(k) \end{bmatrix} = \begin{bmatrix} M^{-(k_f-k)} & 0 \\ 0 & M^{-(k_f-k)} \end{bmatrix} \begin{bmatrix} Z(k) \\ \Lambda(k_f) \end{bmatrix}$$


So, this I am writing in terms of the  $Z(k)$  simply by changing the matrix system matrix which is given here and that we can see as we are saying  $Z(k)$  to the power of  $k$ ,  $Z(k)$  which we are getting by  $M$  to the power  $k$  of  $Z(k)$  and this term will be 0.

So,  $Z(k)$  I am representing simply as  $M$  to the power  $k$  of  $Z(k)$  while my second equation is in  $k$  which already is  $M$  to the power  $k$  of  $Z(k)$  this we are getting from the second equation. This  $k$  is 0 and  $M$  to the power  $k$  of  $Z(k)$ . And this now I am representing my system in terms of  $Z(k)$  this will be nothing, but my  $k$  of  $M$  to the power  $k$  of  $Z(k)$  and this side parameter will be now  $Z(k)$ . So, this representation we can transformed into  $Z(k)$  in terms of the  $Z(k)$ .

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### Analytical Solution of Matrix DRE



From the transformation in eqn. (2) and MRE boundary condition (4) we have,

$$\lambda(k_f) = F X(k_f) \tag{15}$$

Substituting eqn. (11) in above eqn.,

$$W_{21} Z(k_f) + W_{22} \Lambda(k_f) = F[W_{11} Z(k_f) + W_{12} \Lambda(k_f)]$$

$$\Lambda(k_f) = -[W_{22} - F W_{12}]^{-1} [W_{21} - F W_{11}] Z(k_f) = T(k_f) Z(k_f) \tag{16}$$

Now, we see my endpoint transformation  $\lambda(k_f)$ ,  $F$  of  $X(k_f)$  say I am saying  $\lambda(k_f)$  as  $F X(k_f)$ .

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$$\begin{aligned} \lambda(k_f) &= F x(k_f) \\ W_{21} z(k_f) + W_{22} \Delta(k_f) &= F (W_{11} z(k_f) + W_{12} \Delta(k_f)) \\ (W_{22} - F W_{12}) \Delta(k_f) &= -(W_{12} - F W_{11}) z(k_f) \\ \Delta(k_f) &= \underbrace{-(W_{22} - F W_{12})^{-1} (W_{12} - F W_{11})}_{T(k_f)} z(k_f) \\ \Delta(k_f) &= T(k_f) z(k_f) \end{aligned}$$

Now from equation 11, if  $k$  is taken as the  $k_f$  I can write  $X(k_f)$  and  $\lambda(k_f)$  in terms of the  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  and  $Z(k_f)$ . So, first is my  $\lambda(k_f)$  is  $W_{21} Z(k_f)$  plus  $W_{22} \Delta(k_f)$ , this is and this equal to my  $F$ , what is  $x(k_f)$ ? That I can see here  $W_{11} Z(k_f)$   $W_{12} \Delta(k_f)$   $W_{11} Z(k_f)$  and  $W_{12} \Delta(k_f)$ . So, I have to represent  $\Delta(k_f)$  in terms of  $Z(k_f)$  as we are representing  $\lambda(k_f)$  in terms of the  $k_f$  we are trying to see can we represent can we have a relation between  $\Delta(k_f)$  and  $Z(k_f)$ .

So, what we are writing  $W_{21} Z(k_f)$   $W_{22} \Delta(k_f)$ ,  $F$  of  $W_{11} Z(k_f)$   $W_{12} \Delta(k_f)$ . So, we are arranging the elements of  $\Delta(k_f)$  one side  $Z(k_f)$  on other side. So, this is  $W_{22}$  plus  $F$   $W_{12}$  sorry minus  $F$   $W_{12}$  multiplied with  $\Delta(k_f)$  and this is equal to  $W_{12}$  minus  $F$   $W_{11}$   $Z(k_f)$  with negative sign. So, we are writing  $W_{22}$  minus  $F$   $W_{12}$   $\Delta(k_f)$  and  $W_{12}$  minus  $F$   $W_{11}$  with  $Z(k_f)$  with negative sign. So,  $\Delta(k_f)$  we can simply write as  $W_{22}$  minus  $F$   $W_{12}$  inverse with negative sign multiplied with  $W_{12}$  minus  $F$   $W_{11}$   $Z(k_f)$  and along with this negative sign this whole we represent as  $T(k_f)$ . So, we are writing a transformation  $\Delta(k_f)$  as  $T(k_f) Z(k_f)$ .

So, just by arranging this elements of  $Z$  and  $\Delta$  we can write  $\Delta(k_f)$  as  $T(k_f) Z(k_f)$  where  $T$  is nothing but minus  $W_{22}$  minus  $F$   $W_{12}$  whole inverse multiplied with the  $W_{12}$  minus  $F$   $W_{11}$ . So, this give me the transformation of  $\Delta(k_f)$  in terms of the  $Z(k_f)$ . Now we are utilizing a relation given by the equation fourteen as  $Z(k_f)$  equal to  $M$  to the power minus  $k_f$   $Z(k_f)$  and  $\Delta(k_f)$  as  $M$  to the power minus  $k_f$  minus  $k_f$   $\Delta(k_f)$ .



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### Analytical Solution of Matrix DRE

From eqn. (13) we have,


$$\Lambda(k) = M^{-(k_f-k)} \Lambda(k_f) \quad (17)$$

Substituting (16) in above eqn. results in,

$$\Lambda(k) = M^{-(k_f-k)} T(k_f) Z(k_f) \quad (18)$$

Substituting (14) in above eqn. results in,

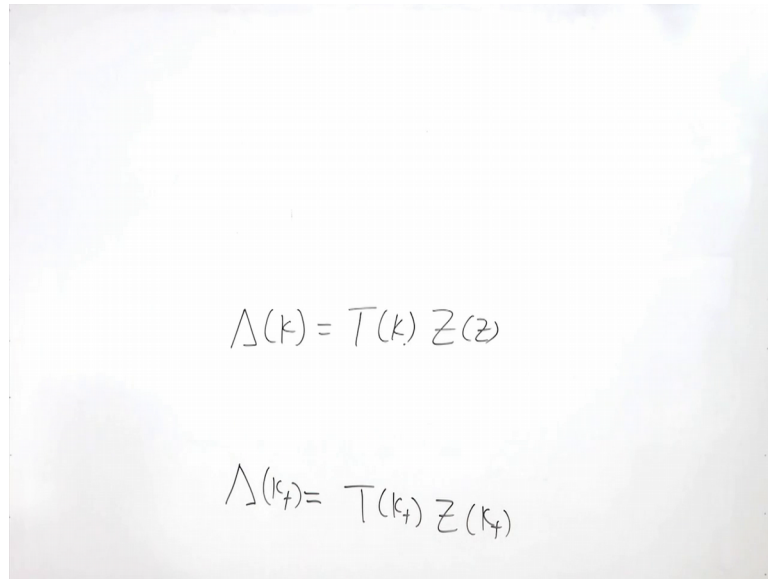
$$\Lambda(k) = M^{-(k_f-k)} T(k_f) M^{-(k_f-k)} Z(k) = T(k) Z(k) \quad (19)$$



From this we are using the relation  $\Lambda(k)$  as  $M$  to the power minus  $k_f - k$  of  $\Lambda(k_f)$  and  $\Lambda(k_f)$  we just have shown that equal to  $T(k_f) Z(k_f)$ . So, this is  $M$  to the power minus  $k_f - k$   $T(k_f) Z(k_f)$  and  $Z(k_f)$  we have taken from our equation fourteen as  $M$  to the power minus  $k_f - k$   $Z(k)$ .

So, by this transformation what actually we are getting? We are getting a relation between  $\Lambda(k)$  and  $Z(k)$ . So, this whole  $M$  to the power minus  $k_f - k$   $T(k_f) M$  to the power minus  $k_f - k$  this whole we are using as  $T(k)$ . So, by the first relation we say by this we have shown that  $\Lambda(k)$  as  $T(k) Z(k)$ .

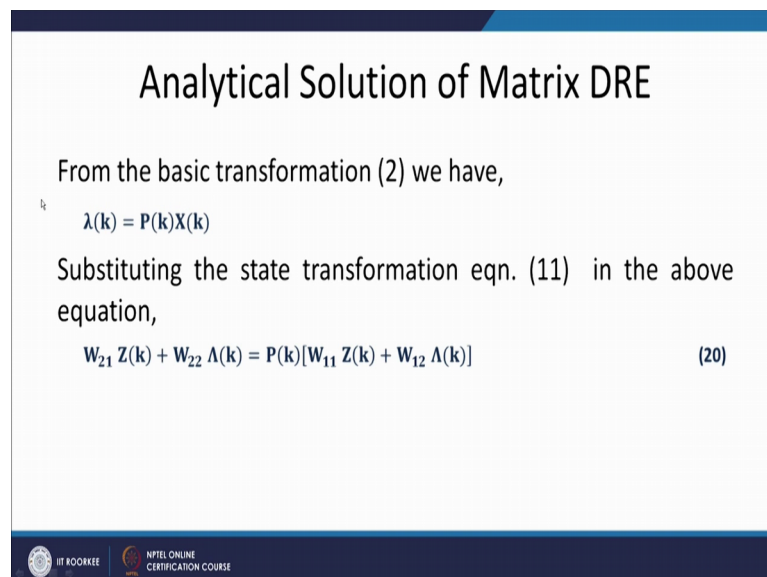
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The image shows a whiteboard with two handwritten equations. The first equation is  $\Lambda(k) = T(k) Z(k)$  and the second equation is  $\Lambda(k_f) = T(k_f) Z(k_f)$ .

And by this new relation we are showing del of k as T of k Z of k where if we will see what is the T of k f in this T of k f is given by a matrix relation given in f and W and T is given n terms of M to the power k f minus k T of k f M to the power minus k f minus k. So, with this 2 transformation now we can write what is the transformation of lambda k equal to P k X k.

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The slide is titled "Analytical Solution of Matrix DRE". It contains the following text:

From the basic transformation (2) we have,

$$\lambda(k) = P(k)X(k)$$

Substituting the state transformation eqn. (11) in the above equation,

$$W_{21} Z(k) + W_{22} \Lambda(k) = P(k)[W_{11} Z(k) + W_{12} \Lambda(k)] \quad (20)$$

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So, now, consider my transformation  $\lambda(k)$  equal to  $P(k) X(k)$ . And again  $\lambda(k)$  and  $X(k)$  will represent in terms of the  $W_{11}$ ,  $W_{12}$ ,  $W_{21}$ ,  $W_{22}$  as we have seen here  $X(k)$  I can represent in  $W_{11}$   $W_{12}$  form and  $\lambda(k)$  in  $W_{21}$   $W_{22}$ .

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$$\begin{aligned}\lambda(k) &= P(k) X(k) \\ W_{21} Z(k) + W_{22} \Delta(k) &= P(k) [W_{11} Z(k) + W_{12} \Delta(k)] \\ W_{21} Z(k) + W_{22} T(k) Z(k) &= P(k) [W_{11} Z(k) + W_{12} T(k) Z(k)] \\ [W_{21} + W_{22} T(k)] &= P(k) [W_{11} + W_{12} T(k)] \\ P(k) &= [W_{21} + W_{22} T(k)] [W_{11} + W_{12} T(k)]^{-1}\end{aligned}$$

So,  $\lambda(k)$  is  $W_{21} Z(k)$  and  $W_{22} \Delta(k)$ . So,  $\lambda(k)$  I am writing as sorry  $W_{21} Z(k)$  plus  $W_{22} \Delta(k)$  this is my  $P(k)$  and  $X(k)$  in terms of  $W_{11}$  and  $W_{12}$ . This  $W_{11} Z(k)$  plus  $W_{12} \Delta(k)$  sorry; this will be not  $W_{12}$  this is  $W_{21}$ . Again we will arrange the elements of  $Z(k)$  and  $\Delta(k)$ . What we will we use the transformation  $\Delta(k)$  equal to?  $\Delta(k)$  equal to  $T(k) Z(k)$ , so we write  $W_{21} Z(k)$  plus  $W_{22} T(k) Z(k)$  and on this side also  $P(k)$  is  $W_{11} Z(k)$  plus  $W_{12} T(k) Z(k)$ , by the whole equation now I am representing in terms of the  $Z(k)$  which can be cancelled out. So, I have  $W_{21}$  plus  $W_{22} T(k)$  as  $P(k)$   $W_{11}$  plus  $W_{12} T(k)$ .

So, I can write my Riccati coefficient as first multiplying the inverse of  $W_{11}$  plus  $W_{12} T(k)$ , we have to post multiply. So, my first term will be  $W_{21}$  plus  $W_{22} T(k)$  and then multiplied with the inverse of  $W_{11}$  plus  $W_{12} T(k)$  inverse.

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## Analytical Solution of Matrix DRE


Substituting (19) in (20) results in,

$$W_{21} Z(k) + W_{22} T(k) Z(k) = P(k)[W_{11} Z(k) + W_{12} T(k) Z(k)] \quad (21)$$

Removing  $Z(k)$  and rearranging the above equation to determine  $P(k)$  results in,

$$P(k) = [W_{21} + W_{22} T(k)][W_{11} + W_{12} T(k)]^{-1}$$

Where,

$$T(k) = M^{-(k_f - k)} [-W_{22} - F W_{12}]^{-1} [W_{21} - F W_{11}] M^{-(k_f - k)}$$


So, with this I get my  $P(k)$  as  $W_{21} + W_{22} T(k)$  multiplied with  $W_{11} + W_{12} T(k)$  inverse and  $T(k)$  is given as  $M$  to the power  $k$  of minus  $k$  this whole multiplication which is representing nothing but my  $T(k)$  of  $M$  to the power minus  $k$  of minus  $k$ . So, this is represented as my  $T(k)$ .

So, for a time varying case, for time varying finite horizon problem I can find my Riccati coefficient  $P(k)$  in terms of the elements of my modal matrix and  $T(k)$  which will depend upon the solution of here sorry; which will depend upon the eigenvalues. So, this my eigenvalue and the eigenvector approach to find out the Riccati coefficient  $P(k)$  which is required to determine the controller gain. So, for time varying finite horizon problem I find out  $P(k)$  equal to this.

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The image shows handwritten notes on a whiteboard. At the top, it says  $k_f \rightarrow \infty$ ;  $T(k) \rightarrow 0$ . Below that, an arrow points from the text "Infinite horizon problem" to the equation  $P(k) = [W_{21}] [W_{11}]^{-1}$ . Further down, another arrow points from the text "For Time Varying Finite horizon problem" to the equation  $P(k) = [W_{21} + W_{22}T(k)] [W_{11} + W_{12}T(k)]^{-1}$ . A hand is visible at the bottom left corner of the whiteboard.

As my  $k_f$  approaches to infinity, so I can see in my  $T(k)$   $k_f$  is approaching to infinity. So, this term is going to be 1 upon  $M$  to the power infinity giving you the infinity and making this term to 0. So, as  $k_f$  approaches to infinity my  $T(k)$  will approach to 0 value. If  $T(k)$  will be 0, so  $P(k)$  is simply represented as  $W_{21}$  and  $W_{11}$  inverse; as  $k$  of approaching to infinity, so my this problem is infinite horizon. So, for infinite horizon problem my  $P(k)$  will be this for finite horizon problem I can find out my  $P(k)$  has given by second equation. So, by this approach analytically we can find out what is the solution of my matrix difference Riccati equation.

So, I stop my discussion here. From the next class we will start our discussion on the dynamic programming. Up to this we have discussed how we can use the calculus of variation to solve a optimal control problem, our approaches for the given system which is represented in the state space, we are trying to find out the optimal value of the  $u$  which is minimizing a given performance index and finding the optimal  $u$  we require the Riccati coefficient  $P$  to be determined and  $P$  can be determined by the solution of my Riccati equation either in continuous time or in discrete time. And for both the cases we have seen that how we can find out the solution for the Riccati equation directly in terms of the differential or the difference equation solved by using the any numerical technique or solved by some analytical approach. From the next class we start our discussion on the dynamic programming.

Thank you very much.