Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture - 35
Analytical Solution of Matrix Difference Riccati Equation (Continued)

So, welcome friends to the session of our discussion which we will continue on the analytical solution of matrix difference Riccati equation. In the previous class we are start over discussion on this in which we are defining X k lambda k equal to a matrix H which is given as A inverse A inverse E Q A inverse A transpose plus Q A inverse E.

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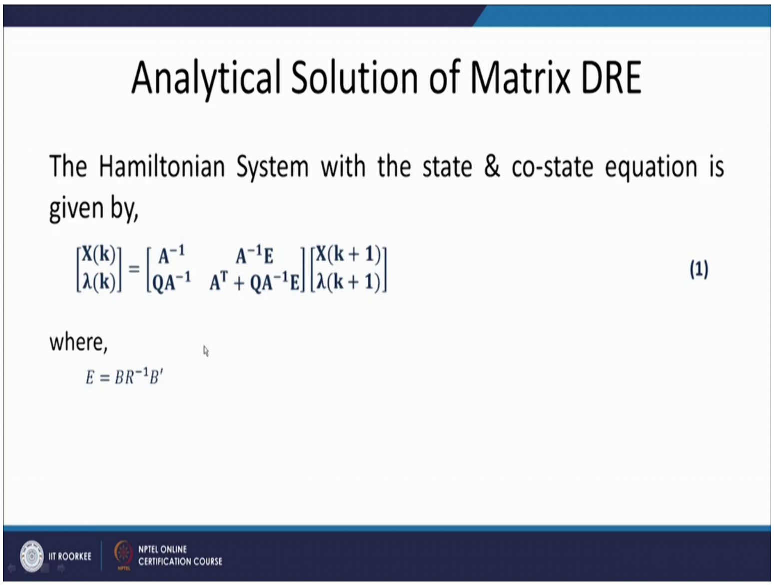

And this matrix we are getting from over Hamiltonian system which was defined for X k plus 1 and lambda k. So, in the previous class we can we have seen that we can find and give name H to this matrix.

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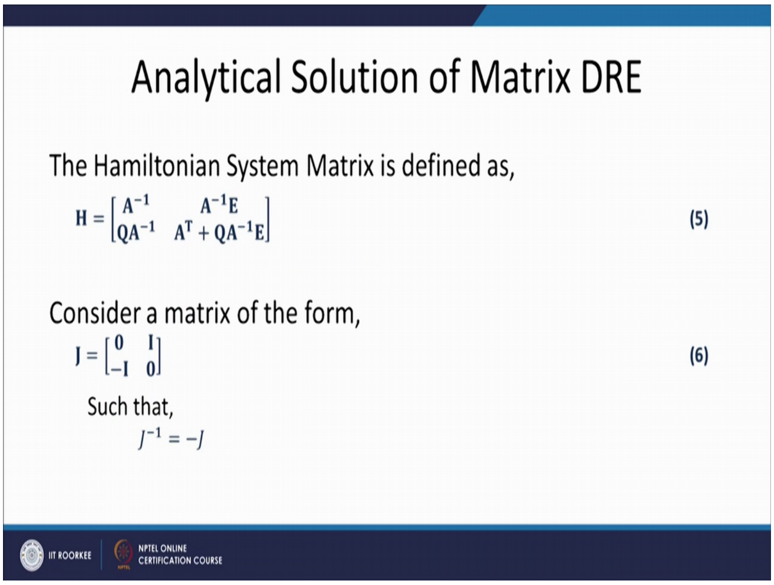

Then we have shown that with the transformation J as a null matrix of n by n, identity matrix of n by n we can define J as a 2 n by 2 n matrix.

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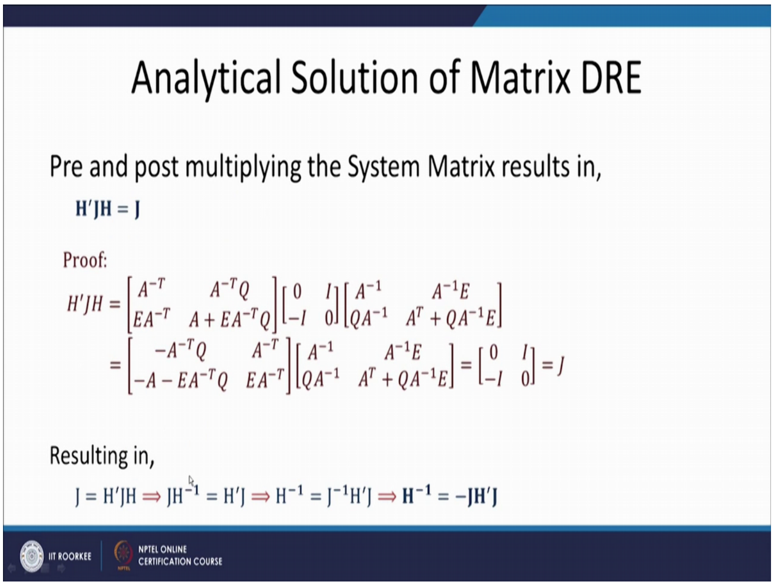

Where J inverse is minus J and we got H inverse at J H inverse J using the H inverse relation we are shown that if mu is an eigenvalue of the matrix H then 1 upon mu is also an eigenvalue of the matrix H.

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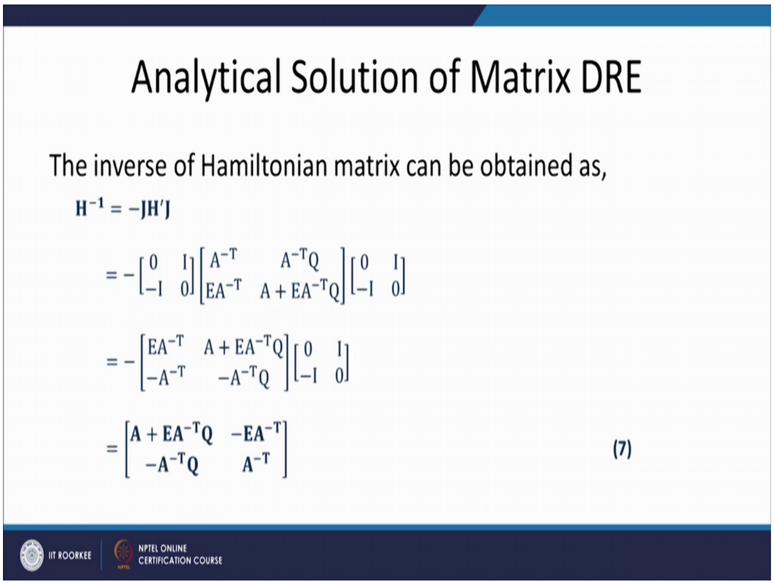

So, this means mu and 1 upon mu are the eigenvalues of matrix H.

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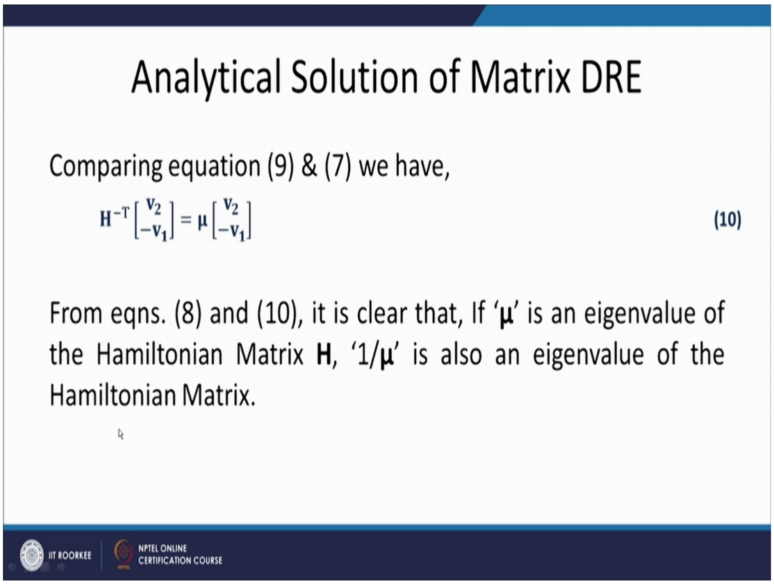

So, if we are considering a discrete time system and in discrete time system if you will see the eigenvalues lie inside the unit circle and outside the unit circle. If we say let mu is the eigenvalue laying outside the unit circle sorry; inside the unit circle then 1 upon mu eigenvalues will lie outside the unit circle.

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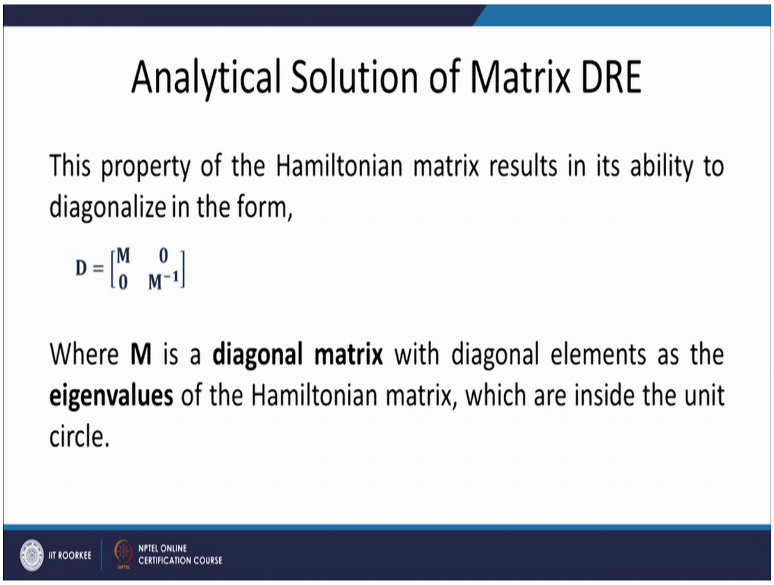

So, by this we can arrange these eigenvalues in terms of the diagonal by the similarity transformation. So, this means if M is a diagonal matrix with diagonal element as the eigenvalues of the Hamiltonian matrix H which are inside the unit circle. So, if M will contain the eigenvalues which are inside the unit circle then M inverse will contain the eigenvalues which are outside the unit circle.

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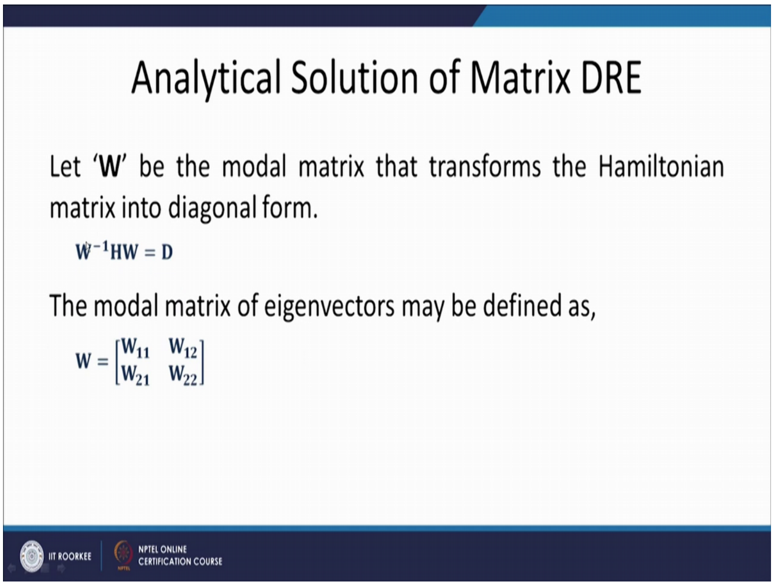

So, we can consider a modal matrix W which is defined as W 11, W 12, W 21, W 22 which transform matrix H into the diagonal form where elements of the W 11 W 21 this nothing but one set of the eigenvectors corresponding to the eigenvalue mu W 12 W 22 is the another set of the eigenvector corresponding to eigenvalues 1 upon mu.

So, W inverse H W is the transformation which convert sorry; which transform the matrix H into the diagonal form and in this diagonal form my eigenvalues are arranged in the diagonal as the eigenvalue inside the unit circle and eigenvalues as outside the unit circle.

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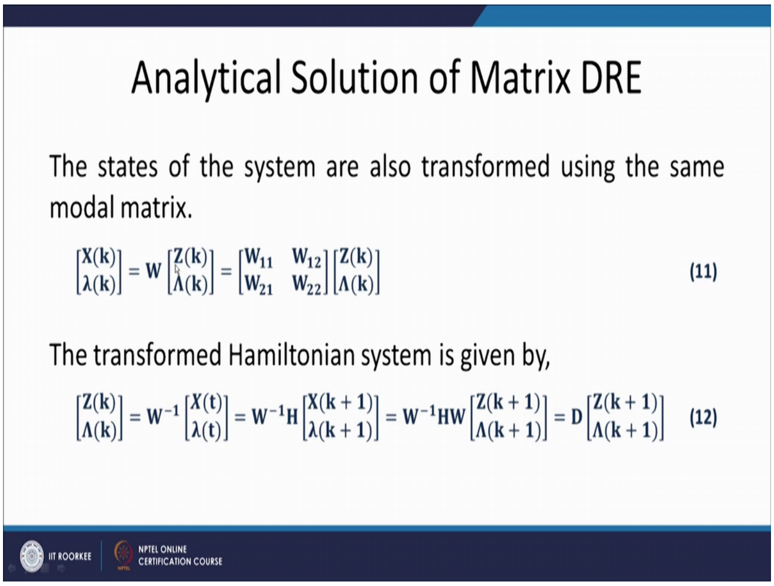

So, now, my transformation is let us say using the similarity transformation we transform X k lambda k as Z k del k multiplying with the modal matrix W.

So, this means W 11, W 12, W 21, W 22 as Z k and del k. So, if I will write the transformed Hamiltonian system in terms of the Z k and del k. So, that will be this W inverse X t lambda t, X t lambda t as we have sorry it should be X k lambda k this is by mistake here because we are using the discrete time system. So, this will be X k and lambda k. So, this will be W inverse X k lambda k here and this X k lambda k we can write as H of X of X k plus 1 lambda of k plus 1 and this X k plus 1 lambda k plus 1 we can represent as Z k plus 1 del k plus 1 as W of Z k plus 1 del k plus 1 and this W inverse H W is nothing, but my diagonal transformation. So, I am getting D Z k plus 1 del k plus 1 and D is what? As we have considered before D is nothing, but M 0 0 M inverse.

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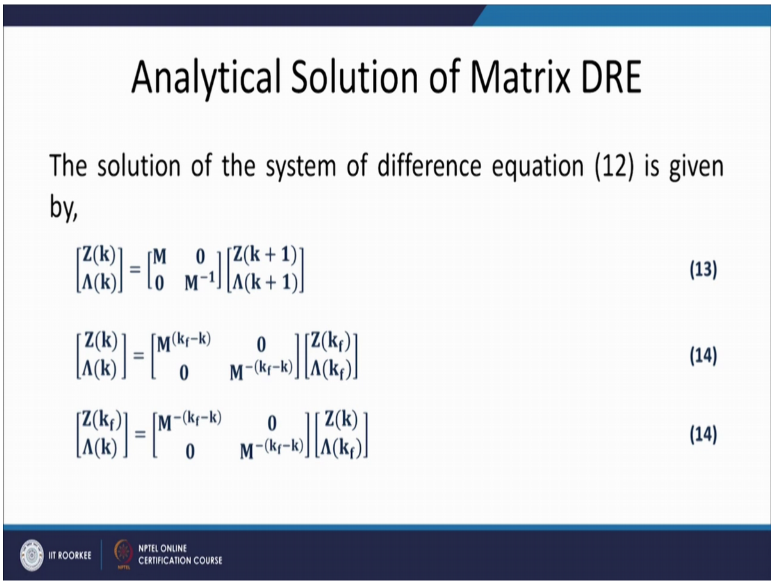

So, we can directly right Z k del k as M 0 0 M inverse Z k plus 1 del k plus 1. So, say set of the differential equations where Z k del k combinedly will have the 2 n vectors n for Z and n for del and 0 0 M inverse is 2 n by 2 n matrix Z k plus 1 del k plus 1 again a vector of 2 n cross 1. So, this is my difference equations which can be solved utilizing the endpoint condition. So, if I will solve this. So, my solution of the equation will be Z k del k which is M to the power k of minus k 0 0 M to the power minus k of minus k Z k f del k f.

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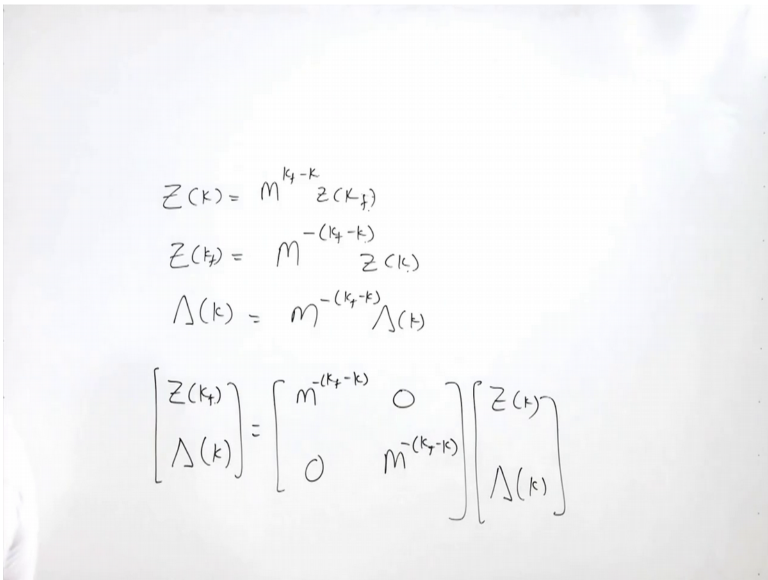

So, this I am writing in terms of the Z k f del k f simply by changing the matrix system matrix which is given here and that we can see as we are saying Z k M to the power of minus k, Z k f which we are getting by M to the power k of minus k Z k f and this term will be 0.

So, Z k f I am representing simply as M to the power minus k of minus k Z k while my second equation is in del k which already is M to the power minus k of minus k del k this we are getting from the second equation. This del k is 0 and M to the power minus k of minus k del k. And this now I am representing my system in terms of Z of k f del of k this will be nothing, but my k of minus k 0 0 M to the power minus k of minus k and this side parameter will be now Z k and del k. So, this representation we can transformed into Z k f del k in terms of the Z k del k f.

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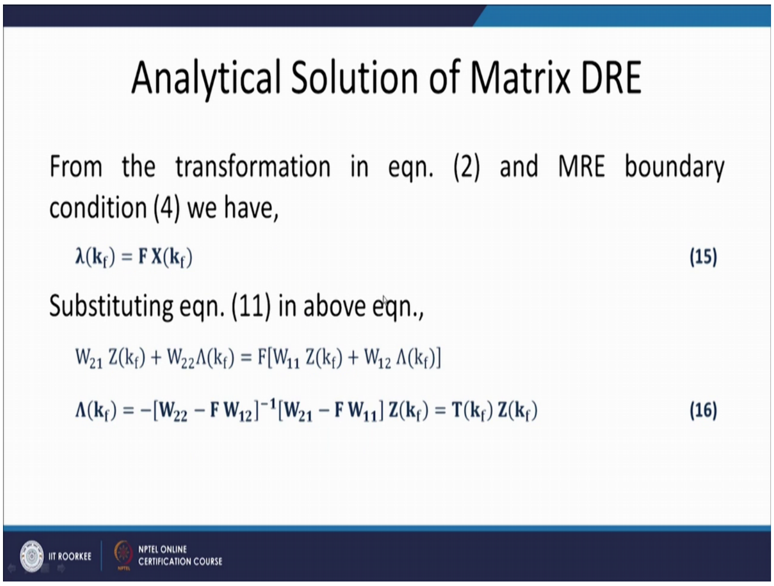

Now, we see my endpoint transformation lambda of k f, F of x of k f say I am saying lambda of k f as F x of k f.

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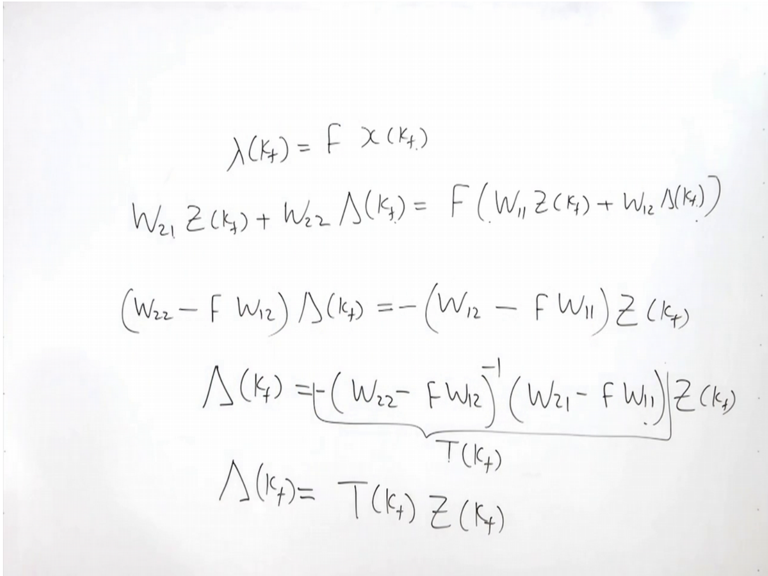

Now from equation 11, if k is taken as the k f I can write X k f and lambda k f in terms of the W 11 W 12 W 21 W 22 and Z k f del k f. So, first is my lambda of k f is W 21 Z of k f plus W 22 del of k f, this is and this equal to my f, what is x of k f? That I can see here W 11 Z of k f W 12 del of k f W 11 Z of k f and W 12 del of k f. So, I have to represent del of k f in terms of Z of k f as we are representing lambda k f in terms of the k f we are trying to see can we represent can we have a relation between del of k f and Z of k f.

So, what we are writing W 21 Z of k f W 22 del of k f, F of W 11 Z of k f W 12 del of k f. So, we are arranging the elements of del of k f one side Z of k f on other side. So, this is W 22 plus f W 12 sorry minus f W 12 multiplied with del of k f and this is equal to W 12 minus F W 11 Z of k f with negative sign. So, we are writing W 22 minus F W 12 del of k f and W 12 minus F W 11 with Z of k f with negative sign. So, del of k f we can simply write as W 22 minus F W 12 inverse with negative sign multiplied with W 21 minus F W 11 Z of k f and along with this negative sign this whole we represent as T of k f. So, we are writing a transformation del k f as T k f Z of k f.

So, just by arranging this elements of Z and del we can write del of k f as T of k f Z of k f where T is nothing but minus W 22 minus F W 1 whole inverse multiplied with the W 21 minus F W 11. So, this give me the transformation of del of k f in terms of the Z of k f. Now we are utilizing a relation given by the equation fourteen as Z of k f equal to M to the power minus k f Z of k f and del of k as M to the power minus k f minus k del of k f.

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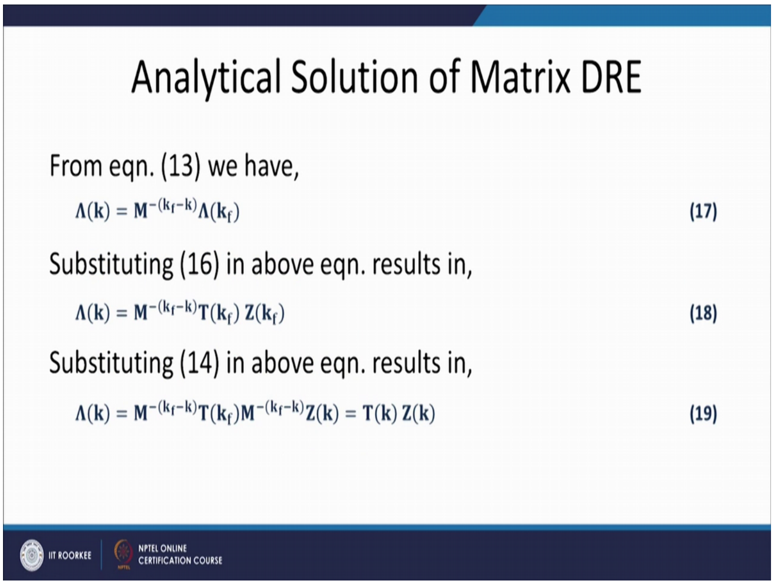

From this we are using the relation del of k as M to the power minus k of minus k del of k f and del of k f we just have shown that equal to T of k f Z of k f. So, this is M to the power minus k f minus k T of k f Z of k f and Z of k f we have taken from our equation fourteen as M to the power minus k f minus k Z of k.

So, by this transformation what actually we are getting? We are getting a relation between del of k and Z of k. So, this whole M to the power minus k f minus k T k f M to the power minus k f minus k this whole we are using as T of k. So, by the first relation we say by this we have shown that del of k f as T of k f, Z of k f.

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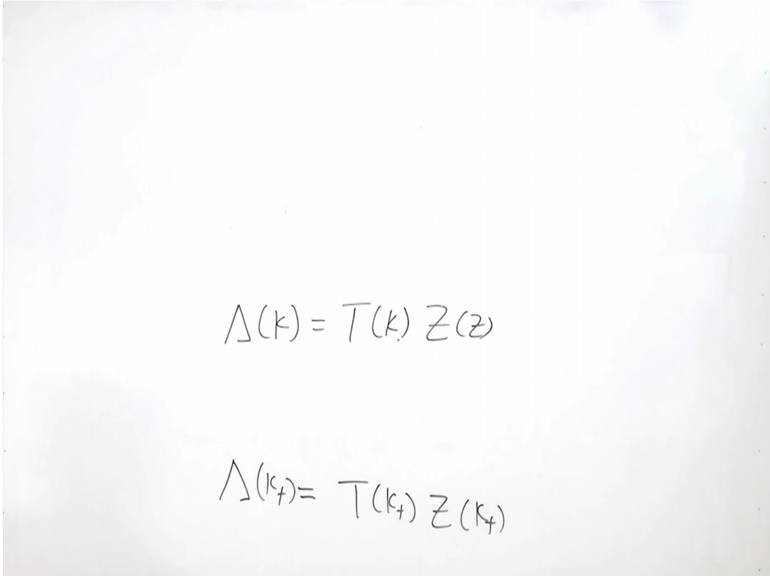

And by this new relation we are showing del of k as T of k Z of k where if we will see what is the T of k f in this T of k f is given by a matrix relation given in f and W and T is given n terms of M to the power k f minus k T of k f M to the power minus k f minus k. So, with this 2 transformation now we can write what is the transformation of lambda k equal to P k X k.

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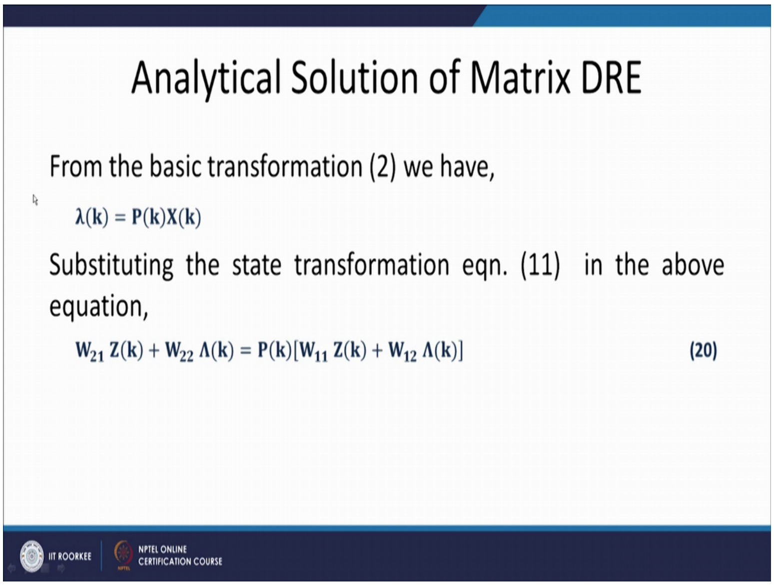

So, now, consider my transformation lambda k equal to P k x of k. And again lambda and X I will represent in terms of the W 11, W 12, W 21, W 22 as we have seen here X k I can represent in W 11 W 12 form and lambda k in W 21 W 22.

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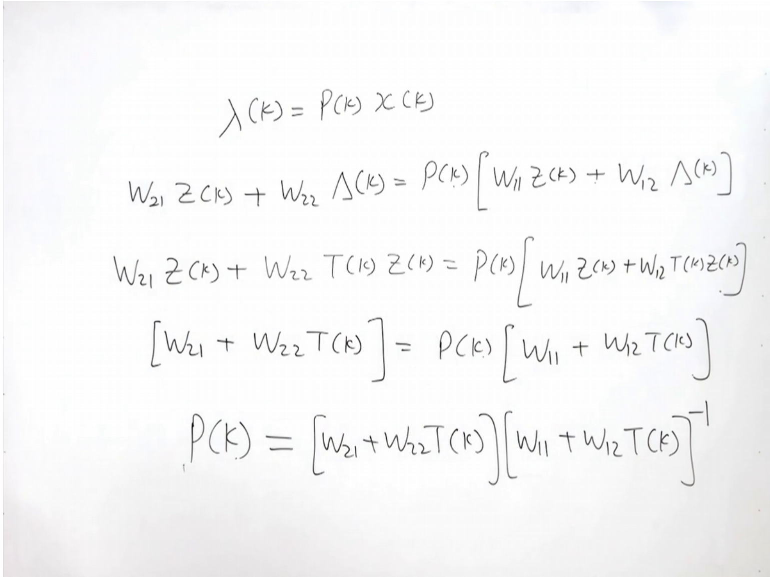

So, lambda k is W 21 Z k and W 22 del k. So, lambda k I am writing as sorry W 12 Z of k plus W 22 del of k this is my P k and x k in terms of W 11 and W 12. This W 11 Z k plus W 12 del k sorry; this will be not W 12 this is W 21. Again we will arrange the elements of Z k and del k. What we will we use the transformation del k equal to? Del k equal to T k Z k, so we write W 21 Z k plus W 22 T k Z k and on this side also P k is W 11 Z k plus W 12 T k Z k, by the whole equation now I am representing in terms of the Z k which can be cancelled out. So, I have W 21 plus W 22 T k as P k W 11 plus W 12 T of k.

So, I can write my Ricatti coefficient as first multiplying the inverse of W 11 plus W 12 T k, we have to post multiply. So, my first term will be W 21 plus W 22 T of k and then multiplied with the inverse of W 11 plus W 12 T of k inverse.

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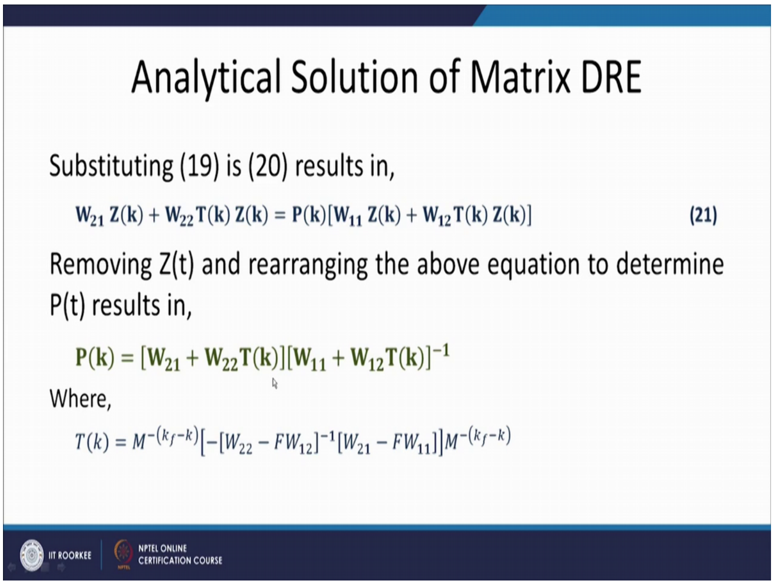

So, with this I get my P k as W 21 plus W 22 T k multiplied with W 11 plus W 12 T k inverse and T k is given as M to the power k of minus k this whole multiplication which is representing nothing but my T of k f M to the power minus k f minus k. So, this is represented as my T k.

So, for a time varying case, for time varying finite horizon problem I can find my Riccati coefficient P k in terms of the elements of my modal matrix and T k which will depend upon the solution of here sorry; which will depend upon the eigenvalues. So, this my eigenvalue and the eigenvector approach to find out the Riccati coefficient P k which is required to determine the controller gain. So, for time varying finite horizon problem I find out P k equal to this.

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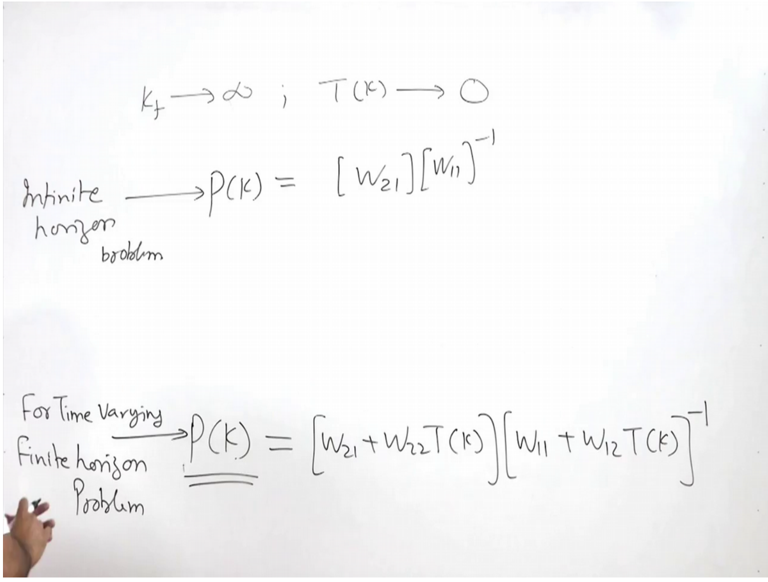

As my k f approaches to infinity, so I can see in my T k k f is approaching to infinity. So, this term is going to be 1 upon M to the power infinity giving you the infinity and making this term to 0. So, as k f approaches to infinity my T k will approach to 0 value. If T k will be 0, so P k is simply represented as W 21 and W 11 inverse; as k of approaching to infinity, so my this problem is infinite horizon. So, for infinite horizon problem my P k will be this for finite horizon problem I can find out my P k has given by second equation. So, by this approach analytically we can find out what is the solution of my matrix difference Riccati equation.

So, I stop my discussion here. From the next class we will start our discussion on the dynamic programming. Up to this we have discussed how we can use the calculus of variation to solve a optimal control problem, our approaches for the given system which is represented in the state space, we are trying to find out the optimal value of the u which is minimizing a given performance index and finding the optimal u we require the Riccati coefficient P to be determined and P can be determined by the solution of my Riccati equation either in continuous time or in discrete time. And for both the cases we have seen that how we can find out the solution for the Riccati equation directly in terms of the differential or the difference equation solved by using the any numerical technique or solved by some analytical approach. From the next class we start our discussion on the dynamic programming.

Thank you very much.