Optimal Control
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Lecture - 34
Analytical Solution of Matrix Difference Riccati Equation

So, welcome friends to this session of our discussion and in this session we will discuss the analytical solution of the matrix difference Riccati equation. The expression for the matrix difference Riccati equation we have developed in the previous class, we have seen the two forms and objective is to determine the value of the P, if P k is known to me I can find out the La k and this La is nothing but my Kalman Gain which I can utilize along with the a state feedback to get my optimal control u.

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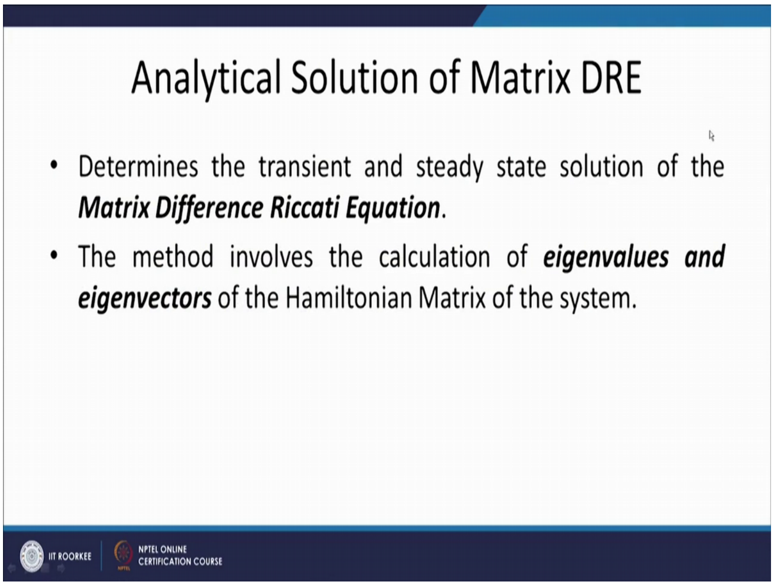

So, in analytical solution of the matrix difference Riccati equation it determine the transient and the steady state solution of the matrix difference Riccati equation this method involves the calculation of eigenvalues and eigenvectors.

As we have determine this for a continuous time systems in the similar manner we can is the design sorry; we can find out the value of the P using the eigenvalues and the eigenvectors, but there my system was the continuous, but here the system will be the discrete in nature.

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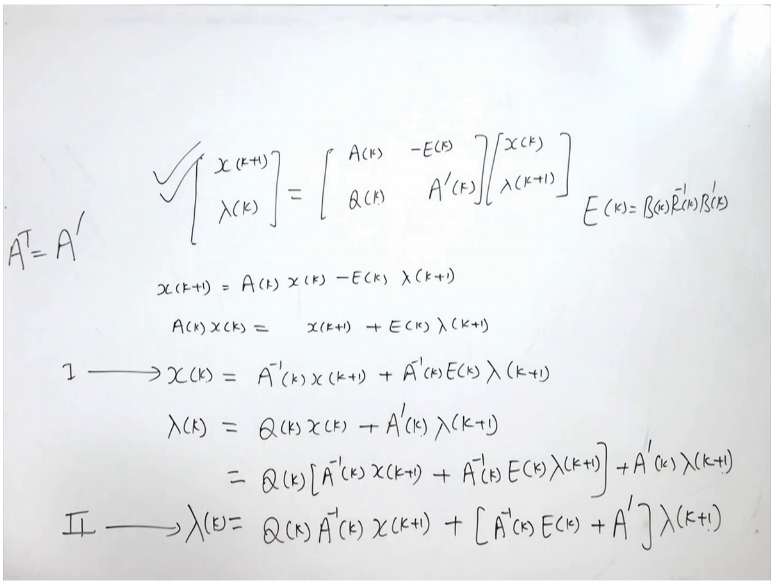

Now the Hamiltonian system with state and the costate equation as you know we have developed the Hamiltonian system in terms of the x k plus 1 and lambda k which was nothing but my A k minus E k Q k A transpose k x k and lambda k plus 1 but in this case I have to find this in terms of the x k lambda k.

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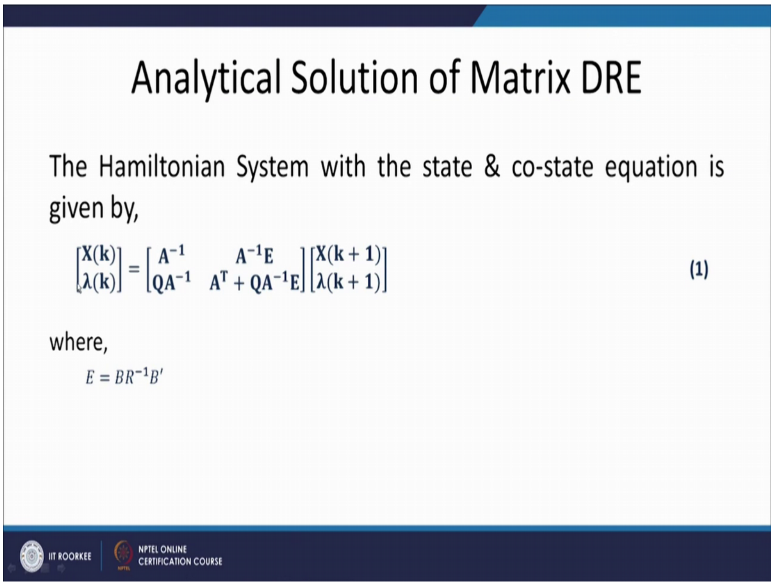

So, this means this equation I want in terms of the x k we have derived at the our Hamiltonian system we can represent as x of k plus 1 lambda of k as A k minus E k Q k A prime k x k lambda k plus 1 where E k is B R inverse B prime.

So, now this set we have to convert in terms of the x k and lambda k. So, if we will write the first equation this is x of k plus 1 as A k x k minus E k lambda k plus 1 my objective is to represent this in terms of the x k. So, I can write A k x k as x of k plus 1 plus E k lambda of k plus 1 or simply I can write my equation as x k equal to A inverse x of k plus 1 I am pre multiplying this with A inverse. So, this is A inverse k E k lambda k plus 1. So, now, this x k I am represented in terms of the x of k plus 1 lambda of k plus 1; similarly the second equation which already given me in terms of the lambda k, but this is Q k x of k plus A transpose k lambda k plus 1.

So, this lambda k I have to represent in terms of x of k plus 1 lambda of k plus 1, this is already is the lambda of k plus 1. So, I can substitute this x k at this place. So, what I will get? Q k A inverse k x of k plus 1 plus A inverse k E k lambda of k plus 1 and plus A transpose k lambda of k plus 1. So, if I will explain this, this is nothing, but my Q k A inverse x of k plus 1 and this I can include into this A inverse k E k plus A transpose lambda k plus 1. So, from these two equations, one is and second is I can write my required form as x k lambda k.

So, what is your x k? A inverse of k x of plus 1 A inverse of k lambda of k plus 1. So, in this just for writing purpose the k has been dropped out, but this is A k inverse A k inverse E k this is Q k A inverse k A transpose of k Q k A inverse of k E k and lambda k plus 1. So, here A transpose we are writing as A prime. So, by this I can write my Hamiltonian system which we have derived in this form as x k plus 1 lambda k in terms of the x k lambda k and this we will take as my matrix H.

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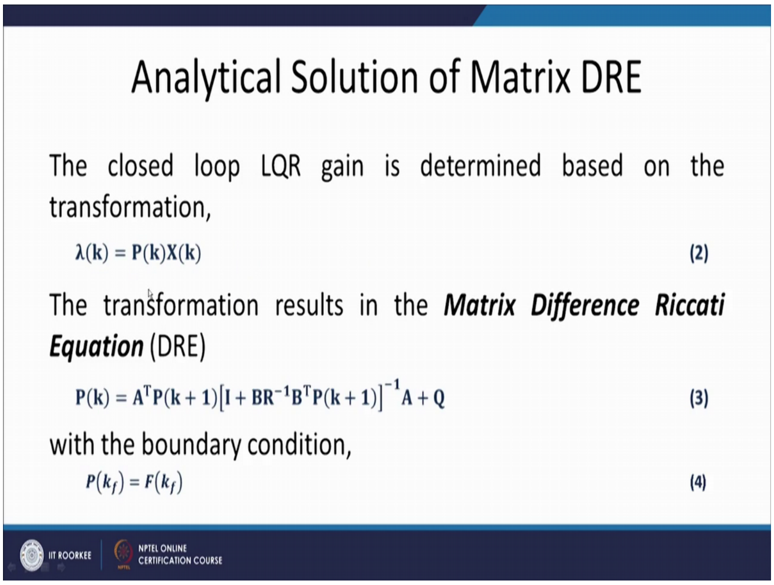

And we also know for the closed loop LQR gain we determined using the transformation as lambda k equal to P k x k this already we have seen and the matrix difference Riccati equation as A transpose P k plus 1 I plus B R inverse B transpose P k plus 1 inverse plus multiplied with A plus Q. This is my matrix difference Riccati equation which we have derived in the previous lecture and with my boundary condition is P k f equal to F of k f. Here my objective is to find out this P k using some analytical approach.

So, to determine this P k I considered by matrix H. So, this matrix which we have written here giving the relation between x k lambda k to the x k plus 1 lambda k plus 1; so, this is A inverse, A inverse E, Q A inverse, A transpose Q A inverse E we are taking as my matrix H.

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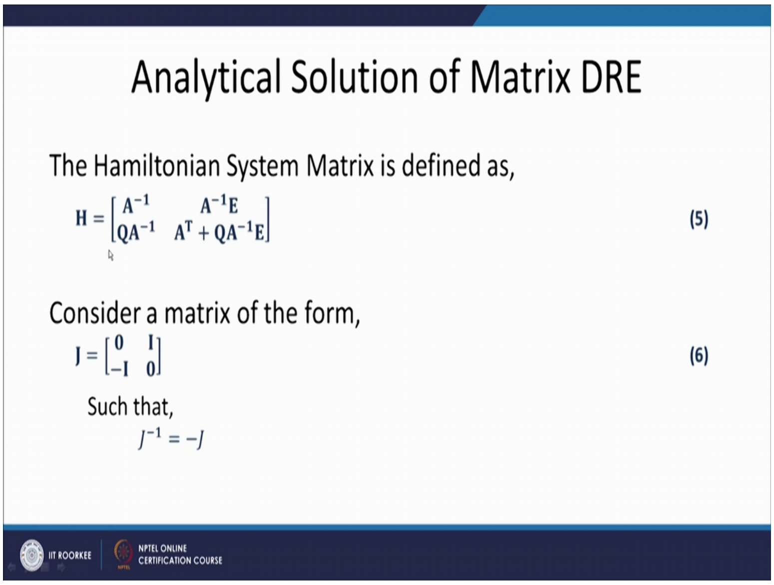

Now consider a transformation matrix which I am taking as a J which is 0, 0 the diagonal element is 0 and the off diagonal elements are the unity matrix. So, the size of my matrix H is 2 n by 2 n because this will have the vector lambda k sorry x k and lambda k. I have the n cross 1 a states as my x and n number of the lambda value. So, total size of the matrix H is 2 n by 2 n. So, I have to consider the J matrix also by 2 by n by 2 n. So, this 0 is nothing, but representing a null matrix of n by n this is representing a null matrix of n by n and I representing a identity matrix of n by n.

So, I have the 2 n rows and the 2 n columns given here. So, if I will take the inverse of the J this. So, naturally I can very easily prove this is nothing, but equal to minus of J; minus the of J means only these two elements will interchange the here sign.

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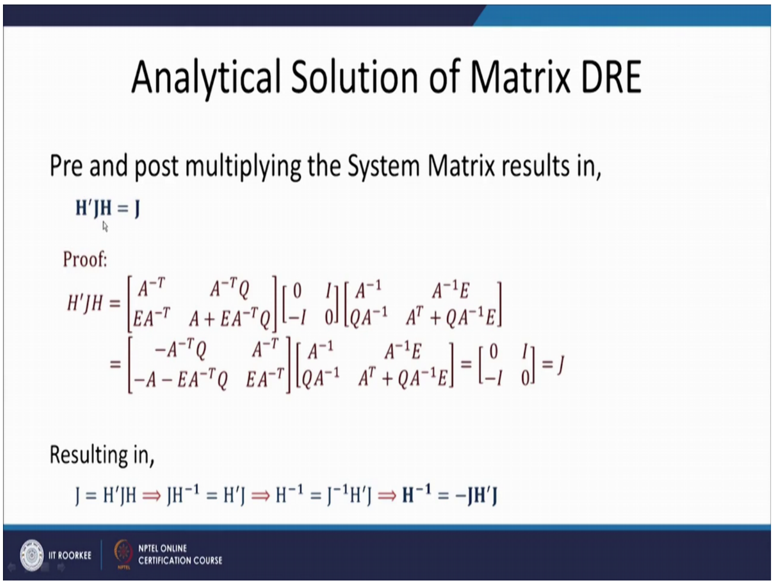

So, I have J inverse as minus J. So, I can easily prove further H transpose J H is nothing but equal to J, which is given here we have the H transpose means what the H matrix we got here we are taking the transpose of this matrix. So, what is the transpose of this matrix?

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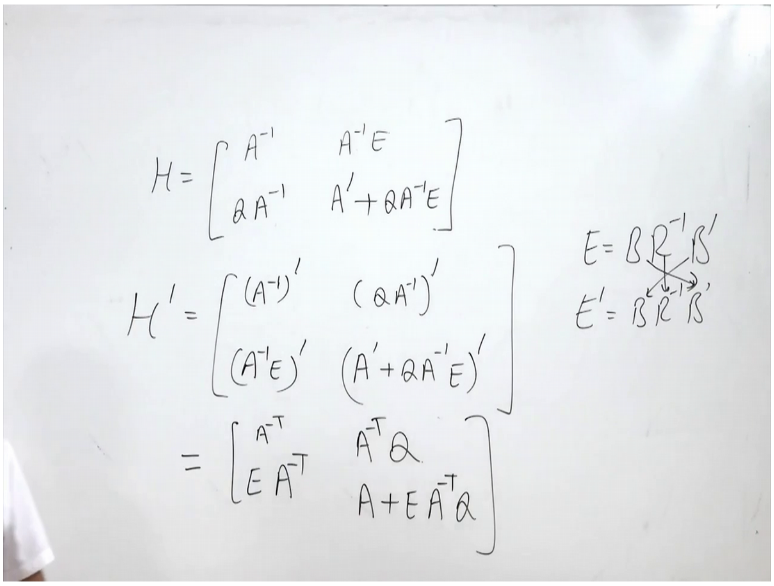

So, we are taking the matrix H as A inverse A inverse E I am dropping the k again A inverse A inverse E, Q A inverse this is A transpose plus Q A inverse P. I have to take the transpose of this matrix. So, what will be this? My first element over is A inverse transpose then Q A inverse transpose A inverse E transpose because this element will come here all the elements and A transpose plus Q A inverse the transpose of the whole.

So, what the first element I will get as my we are proving this H prime A minus transpose because this is I am writing as A minus T. So, this is A minus T means transpose of the A inverse matrix. So, we are taking the A inverse and then taking the transpose. I will write this, this is transpose of the A inverse, Q transpose Q we have taken as a symmetric positive semi definite matrix because this is symmetric. So, I can take this as the my a minus transpose Q this term this E, E is P transpose sorry; B R inverse B transpose if I will take the transpose of this. So, this is B R inverse B transpose is this this remain same and this will be B. So, this is again E inverse of the A transpose if I will explain this A transpose will be A plus, E remain the E A to the power minus T and Q remain the Q.

So, I get this matrix E to the power sorry; inverse of the A transpose inverse of the A transpose multiplied with Q E multiplied with inverse of the A transpose A plus E A to the power minus T Q. So, I am getting this as my H transpose multiplied with J. So, I am trying to find H prime g H; H prime J H you multiply this. So, we will get the same matrix as we are getting the J.

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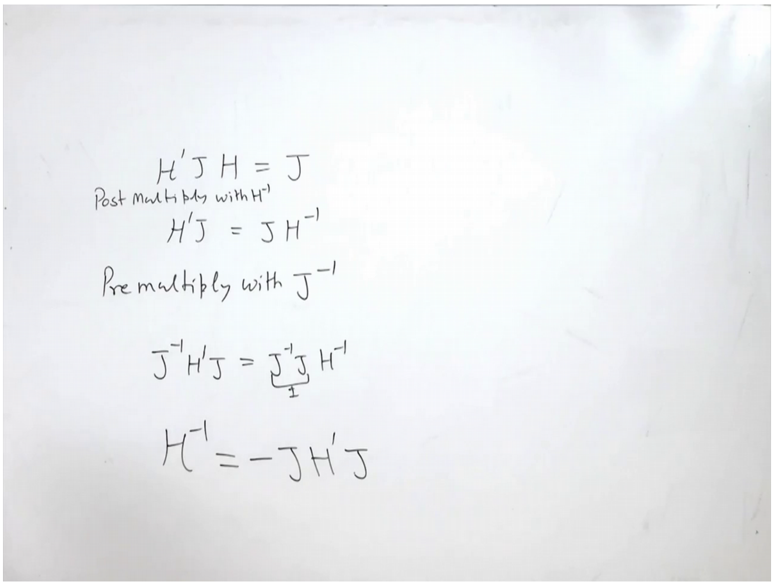

So, this means H prime J H we have taken as the J, we are post multiplying this H inverse. So, if I will multiply this H inverse multiply this H inverse, so this is J H inverse is H prime J, and this say I will write here post multiply with H inverse. Now here pre multiply with J inverse. So, what we will get? J inverse H prime H transpose J as J inverse J H inverse this is nothing, but my I, so I can write my H inverse as J inverse is minus J minus J H prime J.

So, I can easily find out the inverse of the H if I have the transformation J multiply with the H prime and multiply with J. So, I can have my inverse of matrix H which is this matrix here is inverse is nothing but J H prime J. So, H inverse simply by taking the value of the J H transpose and J I will get the inverse of matrix H. Why we need this matrix which will be clear just now.

Now as we said earlier this analytical approach is based on the eigenvalue and the eigenvector approach. We consider v to be the eigenvector of matrix H this means my eigenvector identity H v equal to mu v will exist where mu will be the eigenvalue of matrix H.

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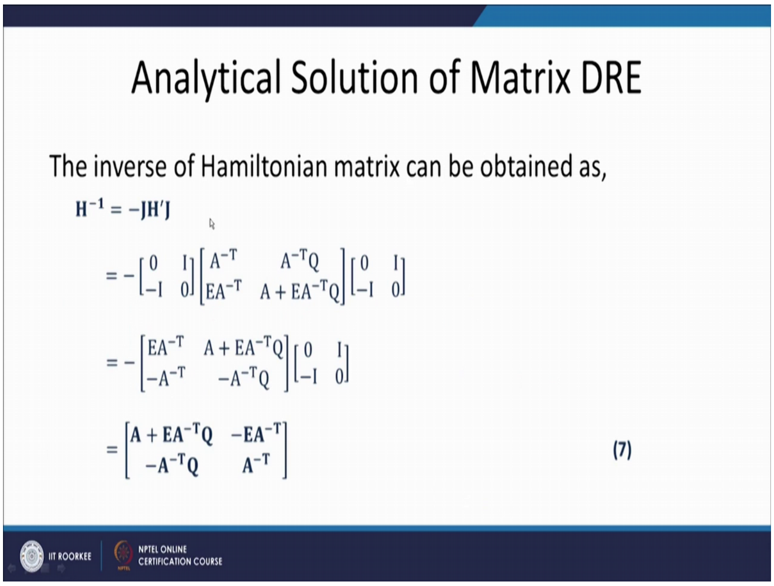

Then this is my H v 1 v 2 which is the a structure of the v, v I have considered v is my eigenvector corresponding to the eigenvalue mu v have the equation H v equal to mu v this has to be satisfied if v is the eigenvector of mu v, I am writing this H multiplied with the v as mu v 1 v 2.

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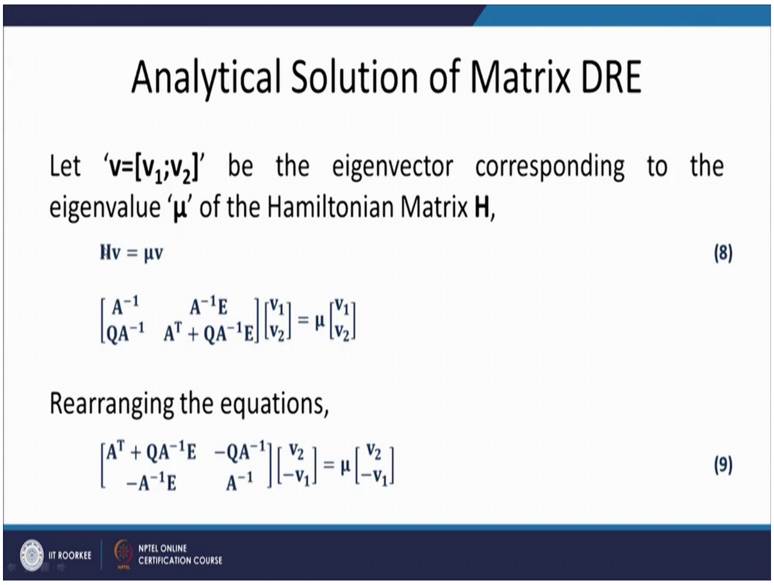

In the next step you can follow yourself we are re-writing the same equation in a different form how, we are writing this as v 2 minus v 1. So, v 2 remain the same, so this by writing the equation independently then I can rearrange the whole equation as A transpose plus Q A inverse E minus Q A inverse A inverse E A inverse v 2 v 1, here also v 2 v 1. Now if we will see what is this matrix this matrix is same as my H transpose matrix. So, this matrix is nothing, but the inverse of my H transpose.

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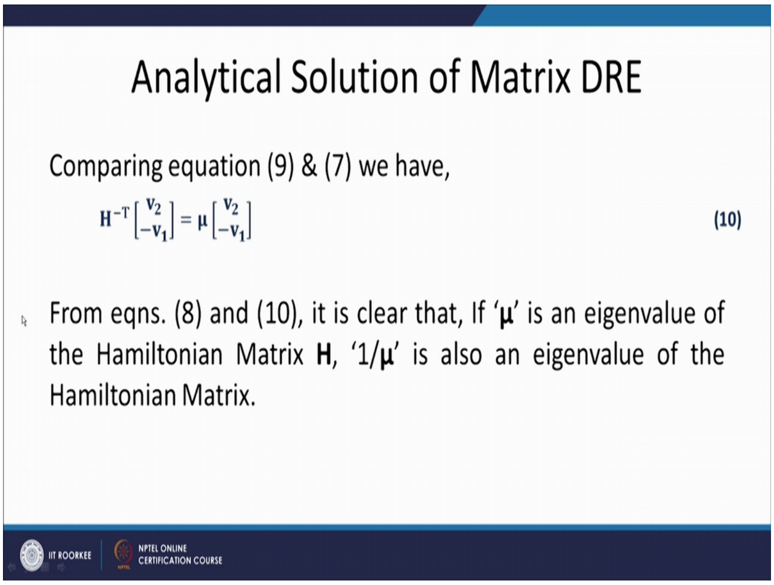

So, this is my H inverse matrix if I will take the transpose of the H inverse, so we determine the H inverse as A plus transpose of the A inverse Q my next element is minus E A to the power minus T, next is minus A to the power minus T Q and my last term is A minus T.

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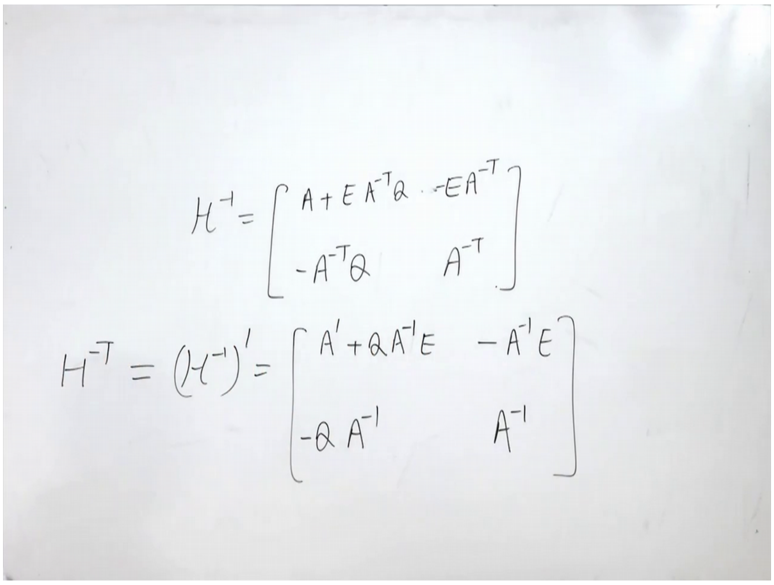

So, this is the H inverse which we have determined using the minus J H prime J what we are trying to find out what is this matrix. If I will take the transpose of this matrix this means transpose of H inverse which I am writing as H to the power minus T. So, what we will be the transpose? Transpose of the first element if I will transpose this, this will be A transpose plus Q transpose remain the Q transpose of the A to the power minus T is inverse of the T transpose. So, this is again I will transpose nothing but the A inverse E remain the same similarly here I will write this is A inverse E, if I will write Q remain the same minus Q A inverse and this is A inverse. And by rearranging this v 1, v 2 and v 1 I am getting A transpose plus Q A inverse E minus Q A inverse sorry; I did a mistake because we are transposing this, so these two term will interchange.

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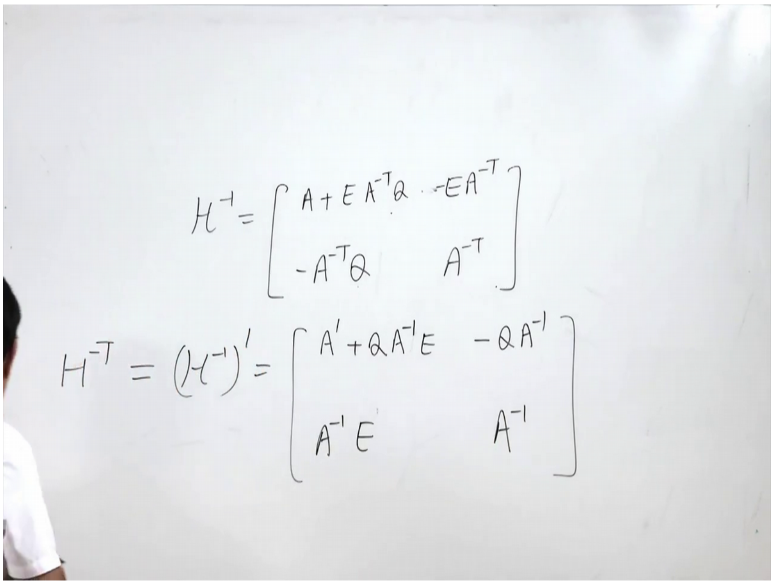

So, in this terms say we are transposing. So, I will get the first term is this in the first row my second element will be the transpose of minus A T A to the power minus T Q which will be minus Q A inverse and this term will be nothing but the transpose of this term, and the transpose of this term is A inverse E and this remain the same.

So, my second term is minus Q A inverse this is minus A inverse E, A inverse. So, this is nothing, but my inverse of the H transpose. So, I am getting this as transpose of the H inverse multiplied v 2 minus v 1 mu v 2 minus v 1. So, what is the meaning of this?

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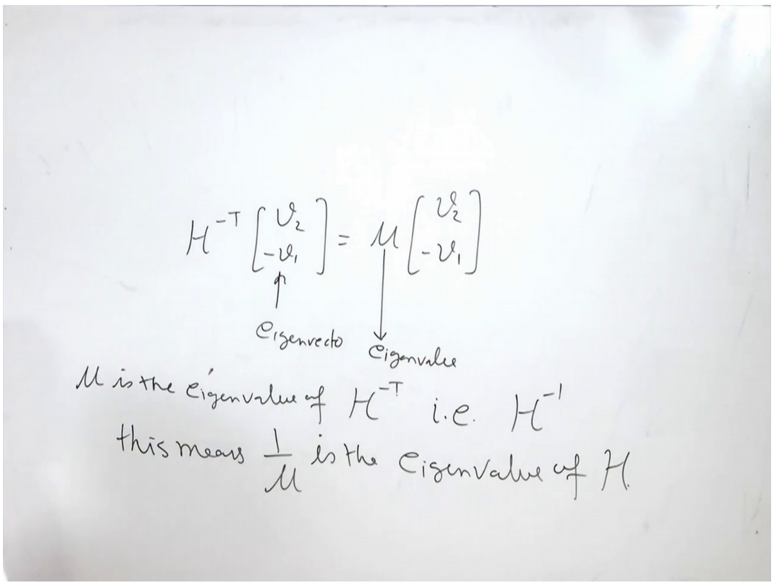

So, this means, so if I will consider this is the eigenvector corresponding to the eigenvalue mu. So, what I can say, mu is the eigenvalue of inverse of the H transpose and corresponding to mu eigenvalue my eigenvector is v 2 minus v 1. So, mu is the eigenvalue of H to the power minus T means this is the transpose of the H inverse, this means mu is also the eigenvalue of H inverse because if a matrix have the eigenvalue its transpose we will also have the same eigenvalues.

So, what the eigenvalues of the H inverse, the same eigenvalues of the transpose of the H inverse we will have. So, mu is the eigenvalue of H inverse and the H in sorry; transpose of the H inverse and the H inverse. So, if mu is 1 eigenvalue of the H inverse this means 1 upon mu is the eigenvalue of H. So, by this observation we can say that if mu is the eigenvalue of H 1 upon mu is also an eigenvalue of H. So, we can say if mu is an eigenvalue of the Hamiltonian matrix H then 1 upon mu is also an eigenvalue of the Hamiltonian matrix H.

What is the significance of this? Say we are my Hamiltonian matrix the Hamiltonian matrix is a discrete matrix. So, if eigenvalues which lie inside the unit circle that are my stable eigenvalues and the eigenvalues which lie outside the unit circle this means those eigenvalues are my unstable eigenvalues. So, if mu is less than 1 I have a stable eigenvalues and if so 1 upon mu which naturally will be greater than 1 means unstable eigenvalues. So, this is my matrix H will have a set of a stable eigenvalues and a set of unstable eigenvalues.

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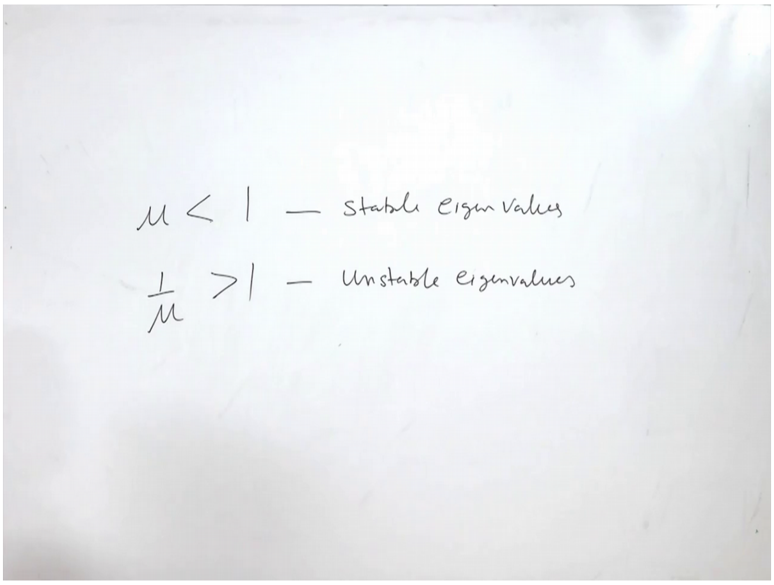

So, this property of my Hamiltonian matrix, due to this property we can convert my Hamiltonian matrix into the diagonal form such that it will have 1 set of the eigenvalues and another set of the eigenvalue. So, 1 set it is the M let us say the mu and M inverse let us say the 1 upon mu. So, I can diagonalize the Hamiltonian matrix in such a way that it will have all eigenvalues which are less than 1 or inside the unit circle as the one set and the another set the eigenvalues which are outside the unit circle or greater than 1 they all can be placed in the diagonal form.

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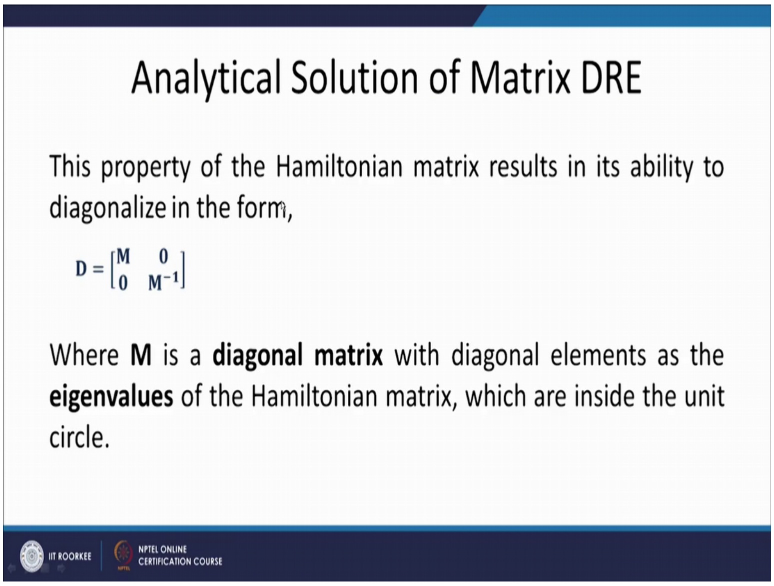

So, I stop my discussion here for this and in the next class we will continue to find out the final value of the P in terms of its eigenvalue and the eigenvector analysis as we are doing.

Thank you very much.