Optimal Control
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Lecture – 33
Matrix Discrete Riccati Equation

So, welcome friends to this session of our discussion. In the previous class we were discussing about the optimal control of the discrete time system and we have taken the case of a linear time varying system which using the Hamiltonian approach we develop the optimal control and optimal control is nothing, but my minus R inverse B prime lambda k plus 1. And once we are deriving the u we lead to the Matrix Difference Riccati Equation which is of the form has given here P k equal to A prime k P k plus 1, I E k P k plus 1 inverse a k plus Q k.

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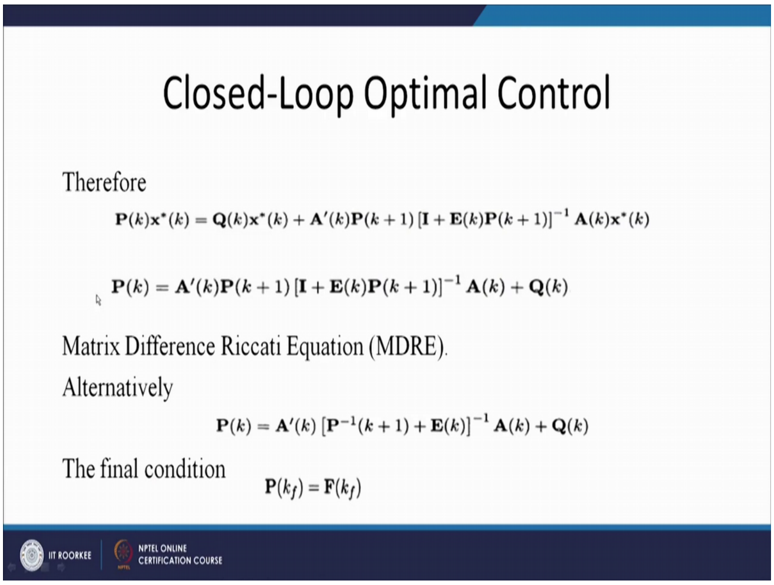

So, this is my discrete Riccati equation and equation is in the matrix form because P is our symmetric positive definite matrix, but the elements of the P is varying with the k, because we are considering the time varying system.

So, it is a matrix differential Riccati equation if P we will take inside this bracket. So, we could the alternatively we can write P k as A prime k P inverse k plus 1 plus E k inverse A k plus Q k and this difference equation we have to solve backwardly with the boundary condition as P k f equal to F of k f.

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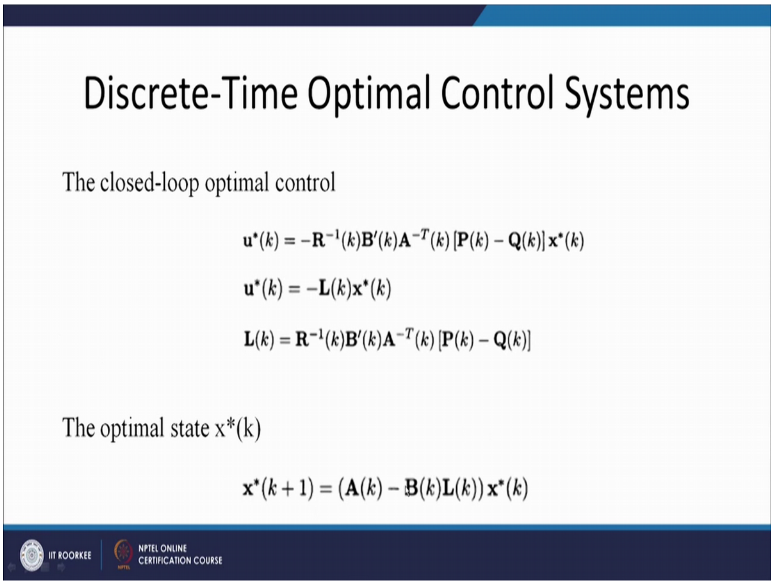

If we will find out the u the u value is minus R inverse B prime inverse of the A transpose which we are writing A to the power minus t the meaning of this is inverse of the A transpose multiplied with P k minus Q k into x k where the R inverse B prime A inverse transpose P k minus Q k we are defining as my L, L is nothing, but I can say my controller gain which I am a using in my closed loop system as A k minus B k L k because here L k x k is nothing, but my u k. So, this is A k plus B k u k u I will replace this. So, I will get the closed loop system equation as x star k plus 1, A k minus B k L k x star k. So, now my new system matrix will be A k minus B k L k.

So, you properly designing the L k I can have my optimal control. In this discussion as we have discussed in the previous class also. To find out the value of the u k or L k we need the inverse transpose of the matrix A. So, for that matrix A should be a nonsingular matrix otherwise the inverse of this matrix we will not be possible. So, to avoid this inverse we can think of another form of the Riccati equation which you will discuss today.

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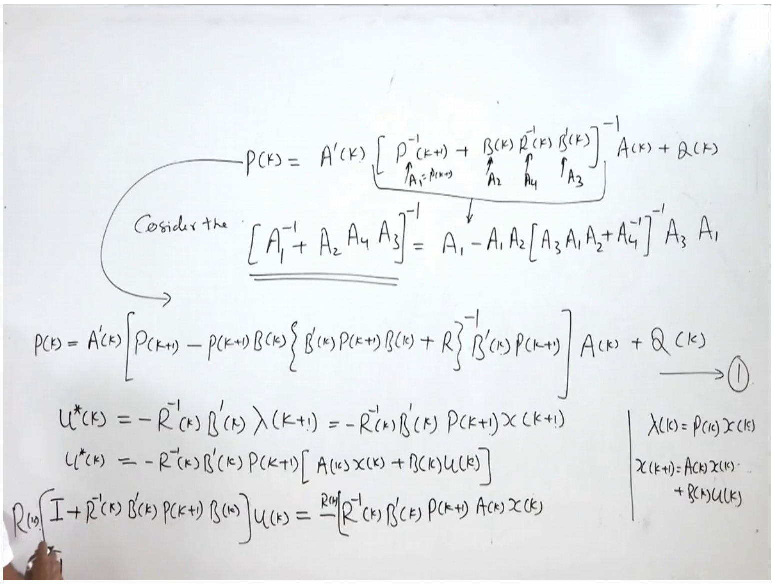

So, as we have given my Riccati equation as A prime k P inverse k plus 1 plus B k R inverse k B prime k whole inverse A k plus. So, this is my discrete matrix Riccati equation.

So, we take the matrix identity consider the matrix identity A 1 inverse plus A 2, A 4, A 3 inverse of the whole equal to A 1 minus A 1 A 2, A 3 A 1 A 2 plus A 4 inverse whole inverse A 3 A 1. So, if we will consider the four matrix A 1, A 2, A 3, A 4 and we will find out the value of this identity which is A 1 inverse plus A 2 A 4 A 3 inverse of the whole matrix this is equal to A 1 minus A 1 A 2, A 3 A 1 A 2 plus A 4 inverse, inverse of the whole A 3 A 1.

So, in by Riccati equation I consider this P 1 matrix to be the A 1, B I take as my A 2, R inverse I take my A 4 and B prime k I take my A 3. So, this particular term this can be expanded using this matrix identity. So, if I will explain this what I will get? So, I will get the P k which is A prime k I explain this matrix using this identity. So, this is my a sorry A 1 I am taking as P k plus 1. So, this is, this is the A 1 inverse we are taking. So, I am writing this as the A 1 A 1 is nothing, but my P k plus 1 minus A 1 A 2, A 1 is P k plus 1 A 2 is my B k then I will have the matrix A 3 A 1 A 2, A 3 B prime k A 1 P k plus 1. So, A 3 A 1 A 2 is nothing but my B k plus A 4 inverse A 4 is my R inverse. So, this I can simply write as my R and the inverse of this matrix then A 3 A 1 A 3, B prime k A 1, P k plus 1. So, this is the expansion of this whole matrix this inverse, this is multiplied with A k plus sorry; this is my Q k A k plus Q k.

So, this difference equation is converted into this form if I will use this identity. So, this equation I can write in this particular form and let us say this is my equation 1. We will utilize this equation later on, we are given with u star k as minus R inverse B transpose lambda k plus 1 my; if I will take del h by del u equal to 0 then I will had get the value of my optimal control as R inverse B prime lambda k plus 1. Lambda k equal to and we have considered lambda k equal to P k x k. So, for this I can write minus R inverse B prime in place of lambda k plus 1 I will have P k plus 1 x of k plus 1.

And my given system is x k plus 1 as A k x k plus B k u k. So, x k plus 1 I will replace by A k x k plus B k u k. So, my u star k is nothing but minus R inverse B transpose P k plus 1 and for this I will have A k x k plus B k u k. So, now, from this I will collect all the u k term at one side. So, I can write this as I plus R inverse B prime P k plus 1 B k because this multiplied with B k u k I am taking on this side. So, this is I plus R inverse k B prime k P k plus 1 B k into u k which is nothing but equal to my minus R inverse P prime k P k plus 1 A k x of k.

Now, in this equation I am pre multiplying with R k on this side and I am also multiplying this with R k. So, with this free multiplication I can multiply this R k in this equation and this R k in this equation, so what we will get?

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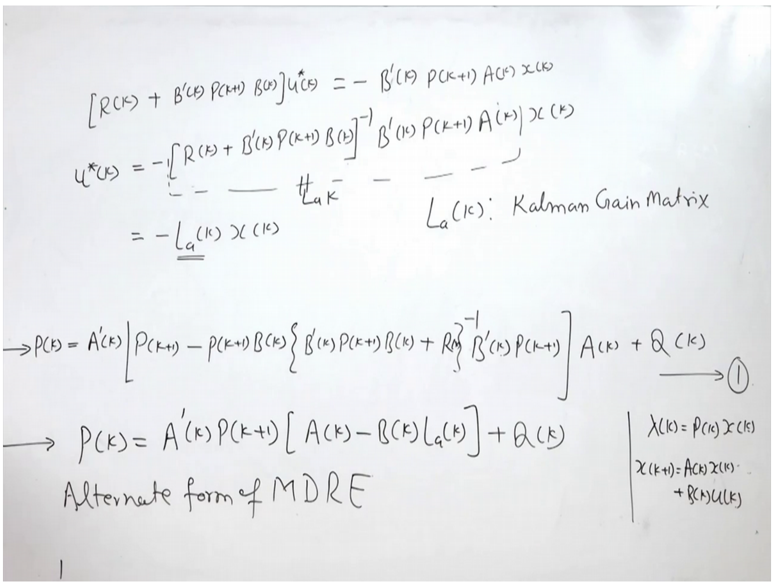

So, I am multiplying this into, so I am getting R k because R k is multiplied here this R k will be the unity matrix R k plus B prime k P k plus 1 B k. So, this is my left hand side R k B prime P k plus 1, B k the whole multiplied with u k and on this side I have minus R inverse will become unity here B prime k P k plus 1 A k x k.

So, a naturally this u is my optimal control. So, I can write my u star k as just minus R k plus B prime k P k plus 1 B k whole inverse B prime k P k plus 1 A k and this is my x k. So, now, this whole I will can represent as my L k and I can write this as nothing, but my minus L A k x k. So, what actually we have done? We have determined the feedback loop. So, we are feeding back our state x of k multiplying with the L k to determine my optimal value of the u. So, this L a k is known as my Kalman Gain Matrix. So, as we have seen in the previous form of the Riccati equation we will having the u, u in which we have to determine the L taking the inverse of the a transpose say in this case low a transpose is required, what we required the inverse of the R k plus B prime k P k plus 1 B k inverse.

And if we will analyze this term in L k R k is a positive definite matrix which inverse is always possible. So, inverse of the R k exist and this positive definite matrix is added with B prime k P k plus 1 B k, B prime k B k always gives as squared term and P k pus 1 is a again positive semi definite matrix sorry positive definite matrix having symmetric. So, P is the asymmetric positive definite matrix R is a positive definite matrix maybe symmetric. So, this means inverse of this term will always exist. So, in this case if this exist, so I can directly find out I can find out the value of La k. As we have seen this is my to determine the La I have to solve equation number 1 which is giving me the matrix difference Riccati equation. Now this P k we can also represent in terms of the La. So, if we will simplify this P k. So, P k can also be represented as, you can simplify this in the form, it should be plus Q k the last term.

So, if we will explain this term and write this particular term which will be coming out by the extension of this bracket we can say this is the R k plus B prime P k plus 1 B k B prime P k plus 1 A k. So, this nothing but giving you the La k, so this P k can alternatively be written in the form of La in my this equation. So, this will be my alternate form of matrix difference Riccati equation. So, normally we utilize this form to determine the value of the P k and to find out the value of u k because in this case the Kalman Gain Matrix which we are finding does not require the inverse of the a transpose matrix and what the universe exist in the La k we have shown that this inverse always exist. As my R is a positive definite matrix, P k plus 1 is also a symmetric positive definite matrix. So, this inverse term always exist.

So, why this we will have the two forms we have discussed 1 can also developed some another form of difference Riccati equation. So, in a time varying case we have 1 form of the matrix Riccati as P k equal to A prime k P inverse k plus 1 E k minus E k the whole inverse A k plus Q k, but this over u k require transpose of the sorry; inverse of the A transpose matrix my second form is P k equal to A prime k P k plus 1 A k minus B k La k plus Q k in this case La k does not require the inverse of the A transpose and the inverse of matrix R k plus B prime P B this inverse we have shown that always exist.

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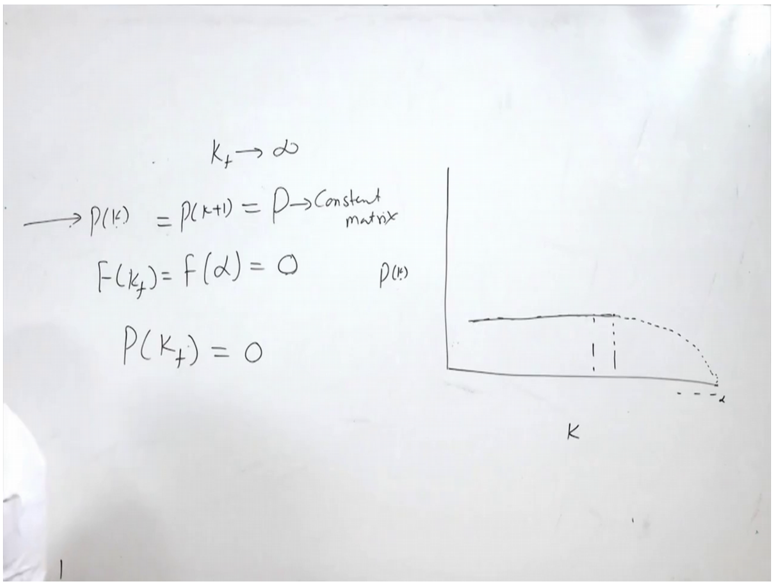

Now if you will consider this, this was the time varying case as my k f will approaches to infinity. So, as k f will approaches to infinity my P k plus 1 is nothing, but its previous value of P k this means we are a starting evaluating the P k if this is my k from infinity very large value if I will say infinity, I am starting from this 0 value and as k f is infinity my F of t f which is F of sorry; the F of k f this is nothing, but infinity and this will be 0 this my P k f is nothing but 0.

So, we are a starting evaluating the P k from a 0 value and in backward and this solution after sometime this will going to be a constant value. So, we can say that my P k is equal to my P k plus 1 or this is equivalent to sub matrix P which is a constant matrix.

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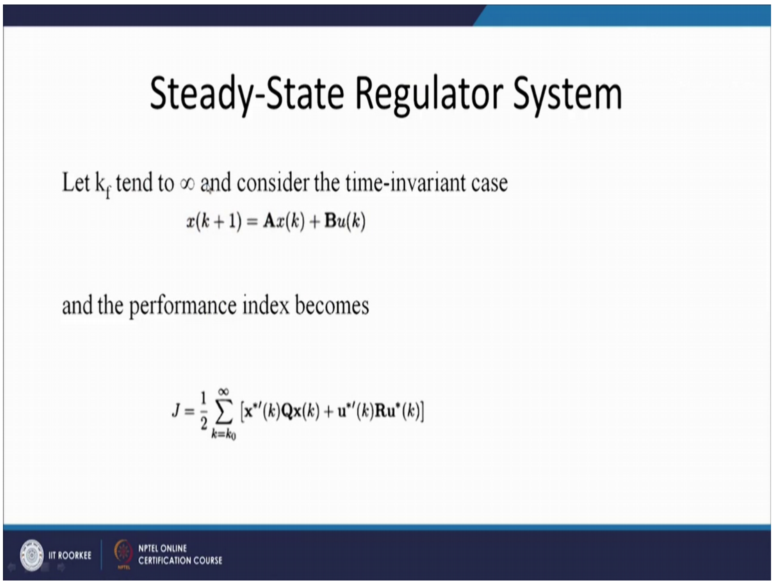

So, if we will consider k f as approaching to infinity and consider a time-invariant case this means matrix A and B both are the constant matrix and my performance index has because k f is infinity, so there is no meaning of the terminal cost terminal cost will be 0 and we have only half of summation k 0 to infinity as x prime Q x plus u prime R u, where Q and R, Q is positive definite R is positive definite, but the now these two matrix will also be the constant matrix.

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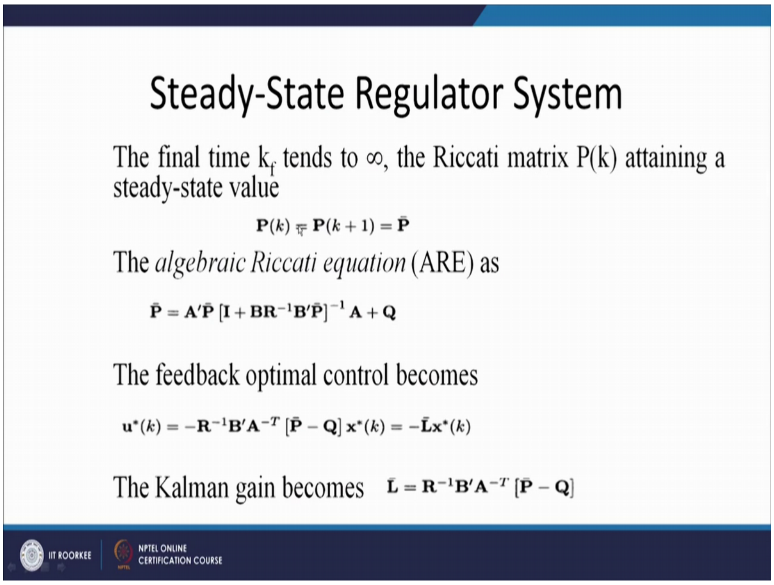

So, my A B Q R are the constant matrix my P k and P k plus 1 R equal to a constant matrix say P bar. So, my whole equation is converted which was the difference equation before because this term we will have the P k plus 1. So, this P k plus 1 and P k both are same, so this is converted into the algebraic equation and this we column I discrete algebraic Riccati equation or simple algebraic Riccati equation for a discrete time system. So, we can write the P as A transpose P I plus P B R inverse B prime P by which we can also find out the u star as R inverse B prime A minus transpose P minus Q which will be the value of the L. So, my Kalman Gain here becomes R inverse B prime a transpose P minus Q. Again in this case as we have seen we required the inverse of the A transpose, so whatever be our alternate from which we have developed for the discrete system in which there is no inversion of the matrix A is required that can directly we converted into the a steady state form.

So, I stop my discussion here today. In the next class we will see how we can solve a time varying matrix difference equation using the analytical approach.

Thank you very much.