Optimal Control

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Lecture - 32
Discrete-Time Optimal Control Systems (Continued)

So, welcome friends to this session which again we will continue for the discrete time optimal control system. In the previous class we have seen that how a Lagrangian we can represent in the Hamiltonian form and all discrete EL equation we can write in terms of the Hamiltonian form.

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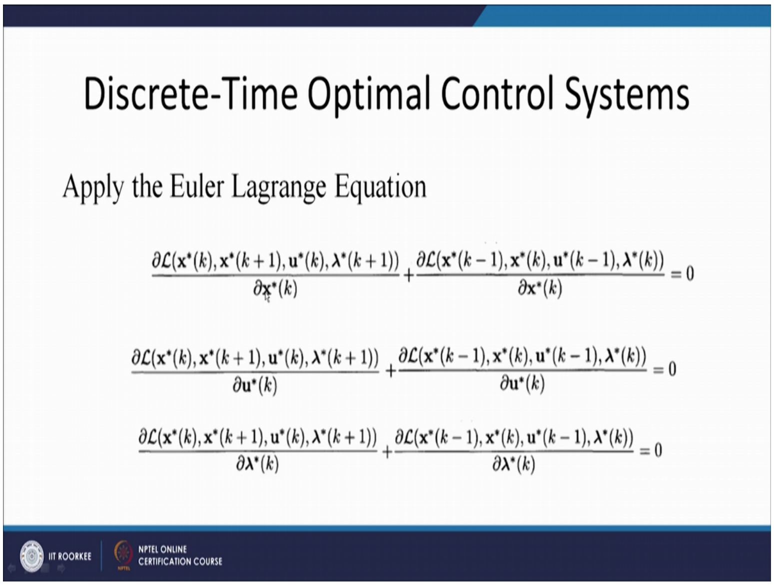

So, these three EL equation which is with respect to x k which will give me the co state equation, optimal control and the state equation.

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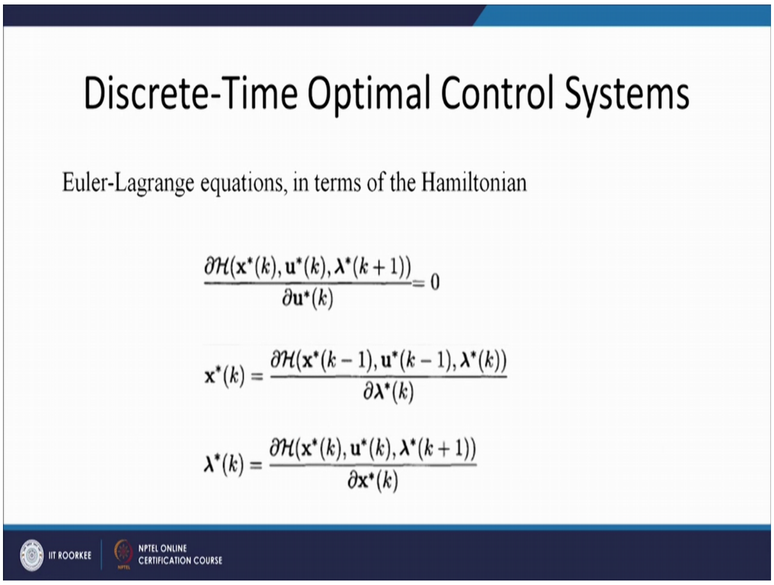

If we will write these in terms of the Hamiltonian we get del H by del u equal to 0 del H by del x equal to lambda k x and del H which is at point k minus 1 u k minus 1 and lambda k give me the x k.

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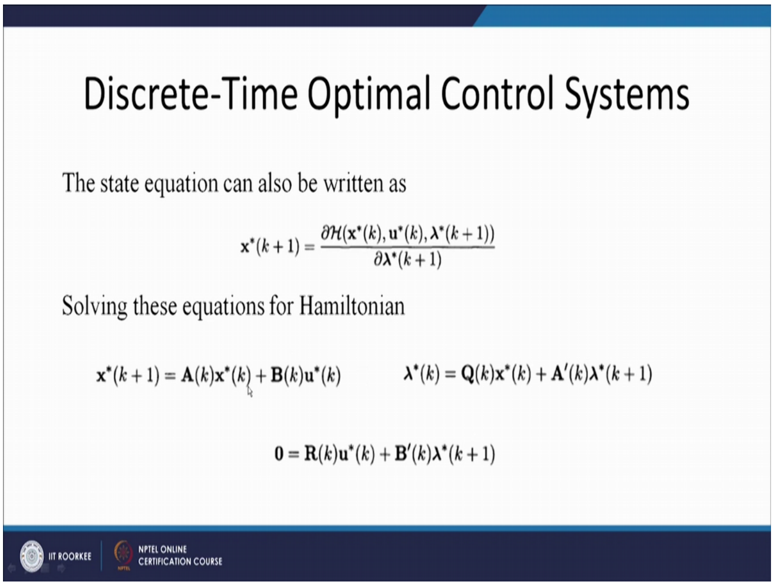

So, to get the proper form we convert this second equation as x star of k plus 1. So, we get all functions of H in terms of the x star k u star k and lambda star of k plus 1.

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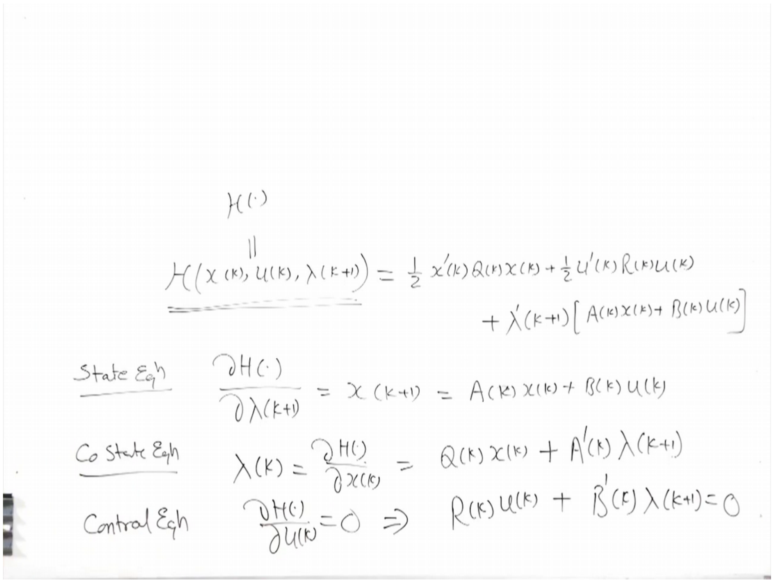

So, we have defined the Hamiltonian as H half of x prime Q x half of u prime R u lambda prime k plus 1 as A k x k plus B k u k. So, by this basically we have to find out our control equation. So, if a state equation and co state equation. So, if I will write a state equation how we write this? Del H, so I am writing this as H dot del H dot then I have to differentiate this with respect to lambda k plus 1 this means what the x k plus 1 x star k plus 1 we have taken as del H by lambda star k plus 1.

So, this means I have to differentiate this with respect to my lambda k plus 1 and this give me nothing, but my x of k plus 1. So, if I will differentiate this with respect to lambda k plus 1 first two term are independent of lambda, this is lambda prime k plus 1. So, nothing but by this differentiation I will get A k x k plus B k u k which is nothing, but my a state equation. Co state equation how I will get my co state equation is this I will write this is lambda star k. So, this is lambda k is nothing, but my del H by del x of k this is my del H by del x of k. So, with respect to x k I have to differentiate. So, my first term is the function of this. So, this will give me is a quadratic term Q k x of k, second term independent of the x and in the last term I get only the first term which is the function of x.

So, this give me plus A transpose k lambda k plus 1. So, this will be my co state equation and the third is my control equation which is sorry; del H by del u del H by del u of k is equal to 0. So, this means I have to differentiate this with respect to u k, I will get from here. So, this implies if I will differentiate this with respect to u. So, this give me R k u k I am in the second term my this term is the function of u, this give me plus I am differentiating this B prime k lambda k plus 1 equal to 0. So, by this I get my state equation co state equation as well as my control equation. So, I get my a state equation as x star k plus 1 as A k x k plus B k u k my co state equation as lambda star k as Q k x k plus a prime k lambda prime k plus 1 and control equation as R k u k plus B prime k lambda k plus 1 equal to 0.

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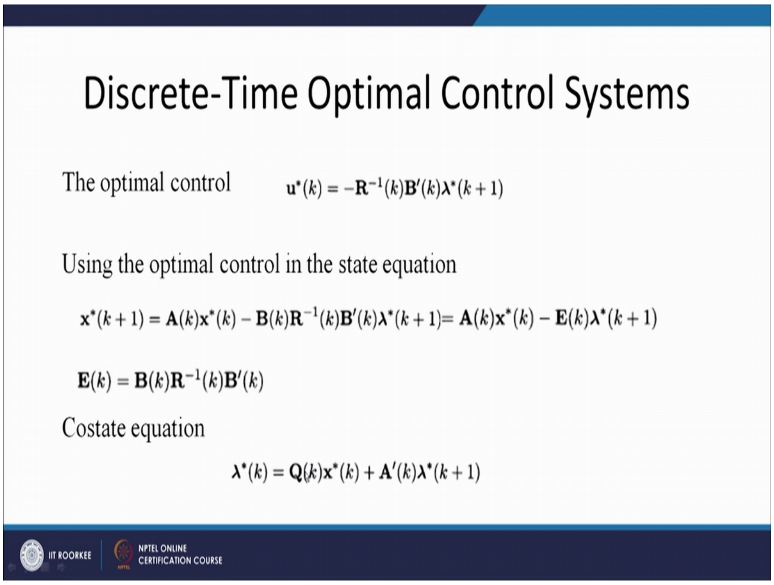

So, directly I can write the value of u from this as minus R inverse B prime lambda k plus 1 and this u star k if I will place in my state equation this will be x star k plus 1 as a k x k B k in place of u k I am writing R inverse B prime lambda k plus 1 and I am defining this B prime R inverse sorry, B k R inverse B prime as E. So, if E is defined as. So, my lambda x star of k plus 1 is A k x k minus E k lambda k plus 1.

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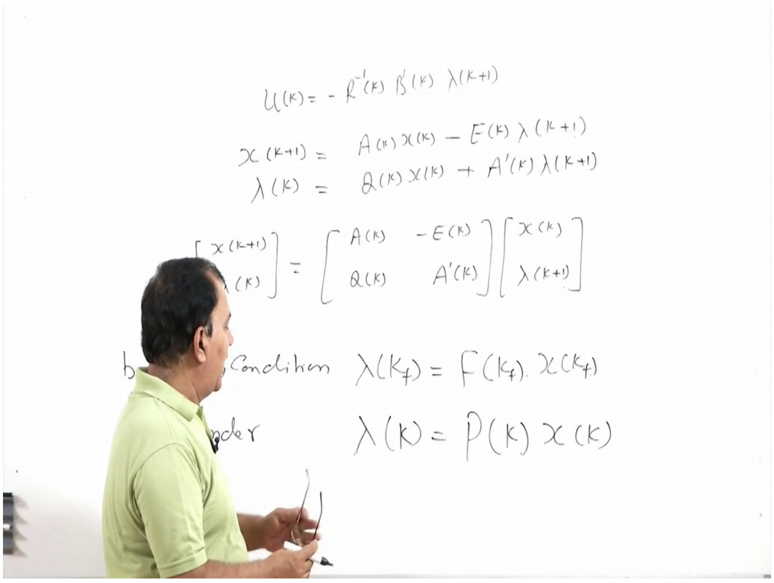

So, we have taken u k as minus R inverse B prime k lambda k plus 1, this we have taken as the u and we are writing x star k plus 1 as A k x k minus E k lambda k plus 1, this we are writing from my state equation. So, just for simplicity I am dropping this star from all the writing where E is nothing, but my B k R inverse k x k B k and my co state equation I will take it as such which is my lambda k, Q k x of k plus A prime k lambda k plus 1 this is my co state equation which we are taking directly. Now by combining these equation I can write my Hamiltonian system as x k plus 1m lambda k as A k minus E k Q k A prime k and this will be x k and this will be lambda k plus 1.

So, now if we will analyze this equation x is at the instant k plus 1 and lambda is at the instant of k this we are expressing in terms of the x k and lambda k plus 1.

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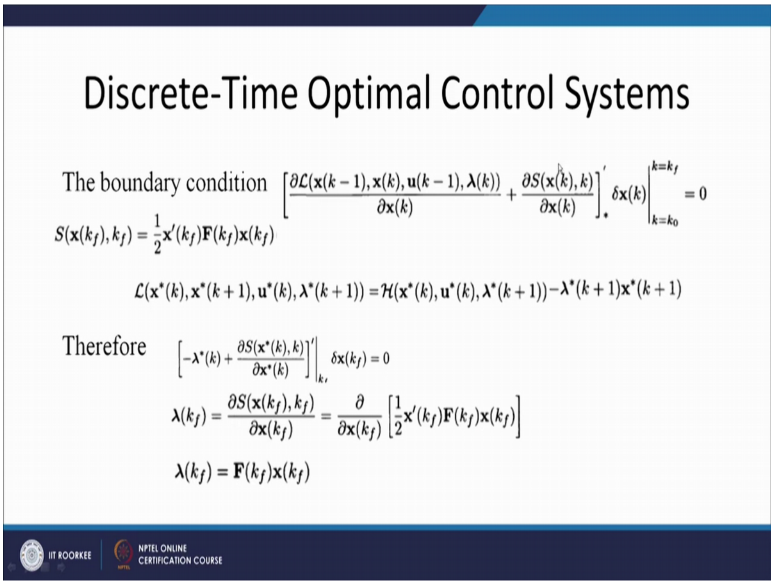

So, this is my Hamiltonian system and what will be my boundary condition? Our boundary condition is del L at x k minus 1 x k u k minus 1 lambda k differentiated with respect to x k and del S by del x k at the point k 0 to k f this must be equal to 0. k 0 is known to us, so delta x of k 0 will be 0. So, that term will vanishes and we are left only with the k equal to k f term. Now L k is H of x k u k lambda k plus 1 and we have to differentiate L of k minus 1. So, this means my H will be a function of x of k minus 1, u of k minus 1 and lambda k this means the difference once I will differentiate this with respect to x k.

So, my, this term will give me the 0 value. So, I am left and what I have here at k minus 1 point this is lambda star k x of k and this give me only minus lambda star and this term is similar del S by del x of k delta x of k equal to 0 S is nothing, but my half of x prime f of k f x of k f and if I will differentiate this. So, at k f point my lambda of k f equal to del S of x of k f k f by delta x of k f, if this S of k f I will differentiate with this. So, I will get nothing, but lambda k f equal to f of k f x of k f. So, my boundary condition for this is lambda k f is F of k f x of k f. So, this is my boundary condition. So, if you recall this is similar like we have got in the continuous time system.

So, from here we can have a relation between lambda and x we can assume a relation between lambda and x. So, I can write, so we can say let us consider the transformation has lambda t equal to some unknown matrix sorry; lambda k equal to some unknown matrix P k x k, which can transfer my lambda into x.

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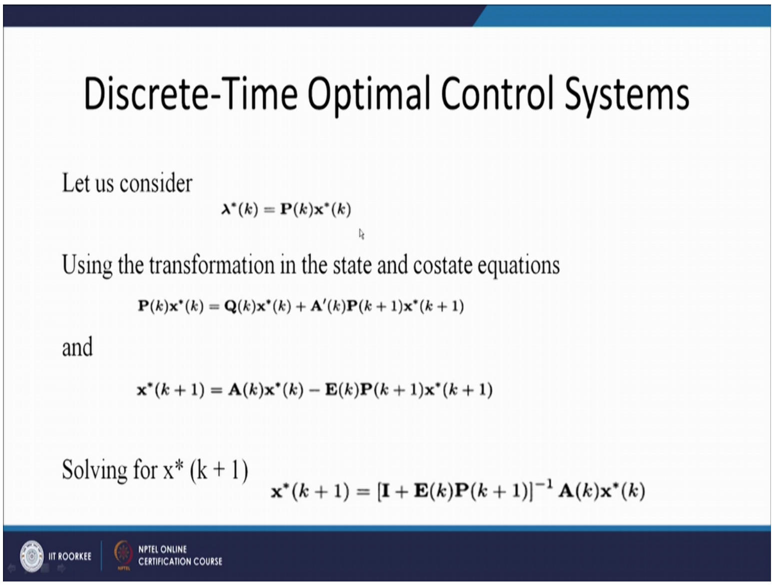

So, we can consider as my lambda star k equal to P k x k and this will lead further to a Riccati equation. So, I stop my this session here with the consideration that we can consider lambda k as P k x k and this can be further leads to a matrix difference Riccati equation.

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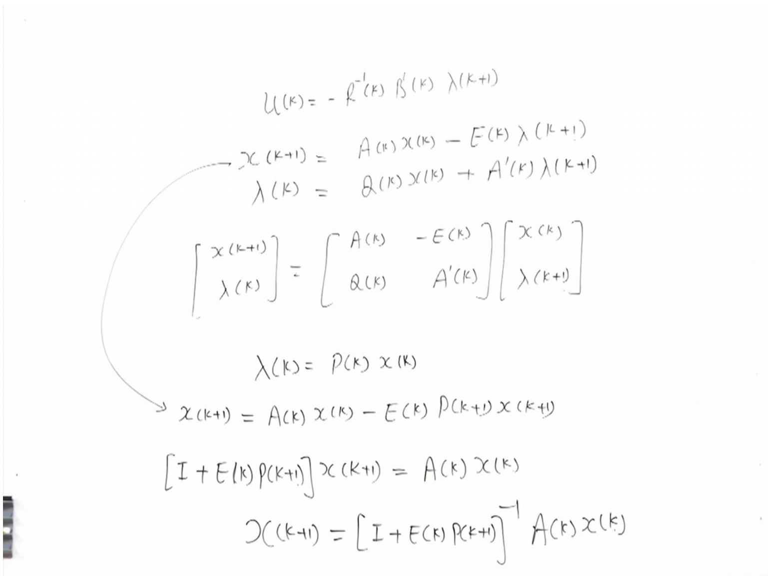

So, we have considered lambda k as P k x k. So, now, my state equation if I will write this as x of k plus 1 as A k x k minus E k and lambda k plus 1, this lambda k plus 1 I am writing from here as P k plus 1 x of k plus 1. So, in my state equation lambda k plus 1 I have replaced by P k plus 1 x of k plus 1. So, by this I can solve this equation as I plus E k, P k plus 1 x of k plus 1 as A k x k.

So, if I will pre multiply with the inverse of this I can write simply my x k plus 1 as I plus E k P k plus 1 inverse A k x k. So, this is my x of k plus 1. So, this means I am writing this x star k plus 1 as I plus E k P k inverse A k x k.

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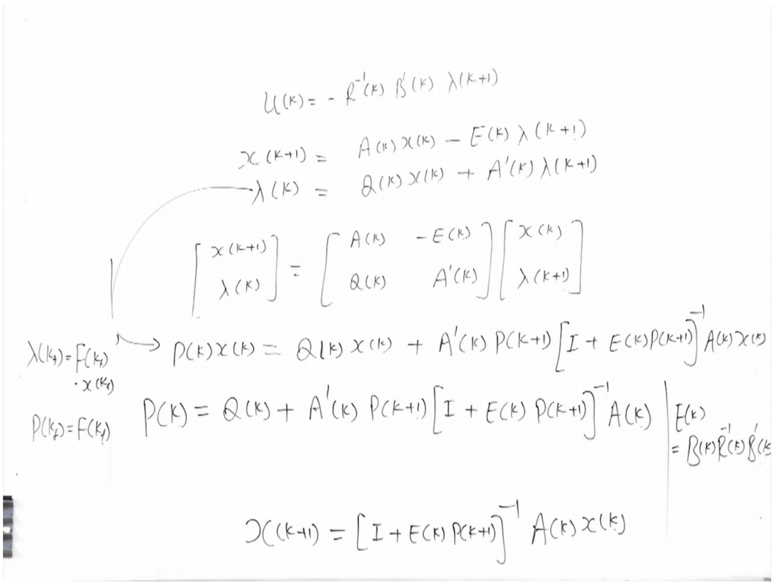

So, I will keep this last equation and I will consider now my this state equation sorry co state equation which give me lambda k, lambda k I will take as P k x of k in place of lambda I am writing plus sorry; this equals to my Q k keep as such x k plus A prime k. What is my lambda k plus 1? P k plus 1 into x of k plus 1, this is x k plus 1 because this lambda k plus 1 I am writing as P k plus 1 x of k plus 1 and this x of k plus 1 I am replacing by this expression I E k P k plus 1 inverse A k x k, so this x k plus 1 I am replacing with I plus E k P k plus 1 inverse A k x k.

So, what is the advantage here is my all terms are in terms of the x k now. So, I can eliminate x k from this equation to get P k Q k plus A transpose k P k plus 1 I plus E k P k plus 1 whole inverse A k, x k is eliminated. So, now, my this equation is in terms of the P k, Q k, P k plus 1, E k and E k is nothing but my what is my E k B R inverse B prime.

So, this means I can solve this equation in terms of the P k because all other terms are known.

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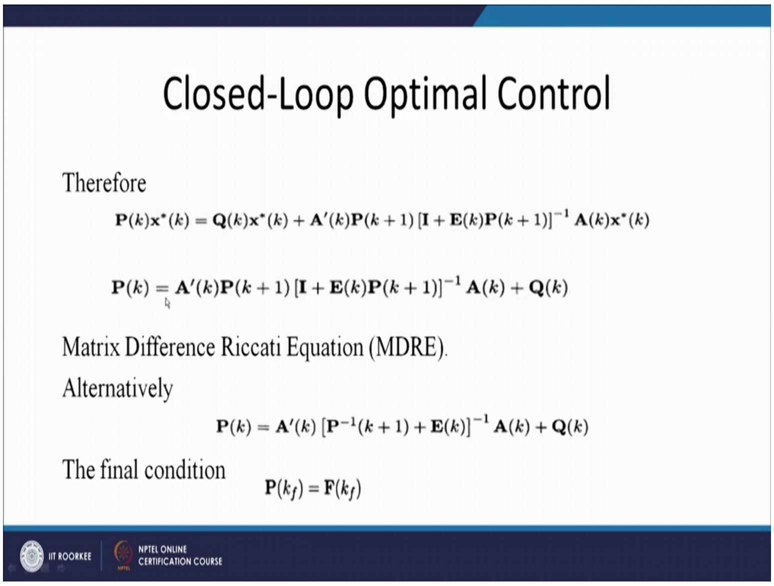

So, I have my equation is P k equal to a prime k P k plus 1 A prime k P k plus 1 I plus E k P k inverse A k plus Q k and this is call my matrix difference Riccati equation and this Riccati equation because we know we have the lambda t f as F of t f, x of t f. So, by this I can say my P of t f is nothing but F of t f.

I have my terminal condition this is a difference equation which can be solved for P k with the terminal condition as sorry; this I have written in the continuous form, but here we will have it discrete form which will be as lambda k f as sorry P of sorry this is F of k f multiplied with x of k f and this will give us P of k f as F of k f. So, this is my Riccati equation which can be solved with the terminal condition as P of k f as F of k f. And alternatively this equation I can also write in the form as if I will take this P k inside this bracket, so this is the same equation as Q k plus A prime k P k I am taking inside P inverse k plus 1 plus E k whole inverse A k.

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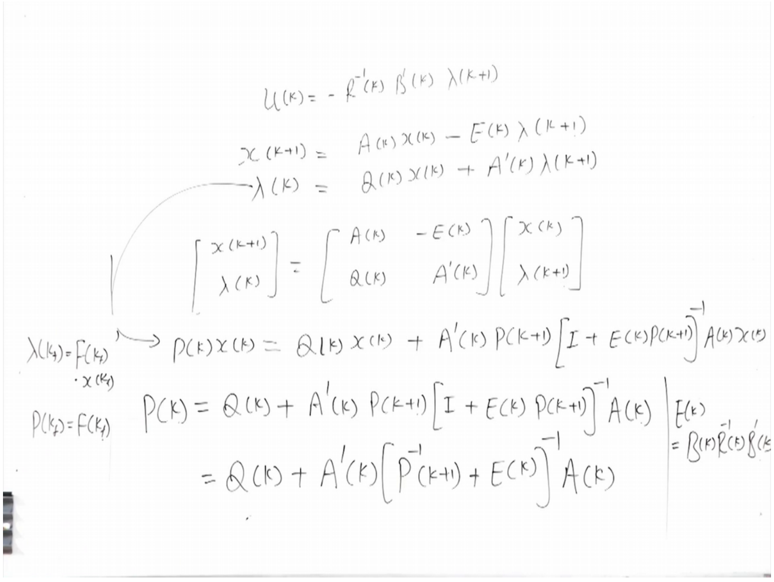

So, any form we can take we have P k as A prime P k plus 1 I plus E k P k inverse A k plus Q k or P k as A prime k P inverse k plus 1 E k inverse A k plus Q k. So, this is my difference Riccati equation which can be solve for P k using the boundary condition as P k f equal to F of k f.

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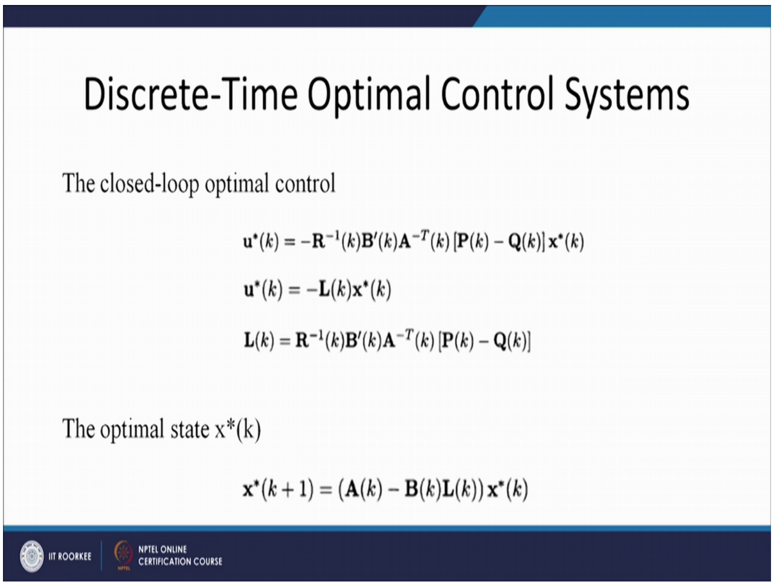

So, once P k is known I have to find out my control low what will be my u k. So, my u k is R inverse B prime lambda k plus 1.

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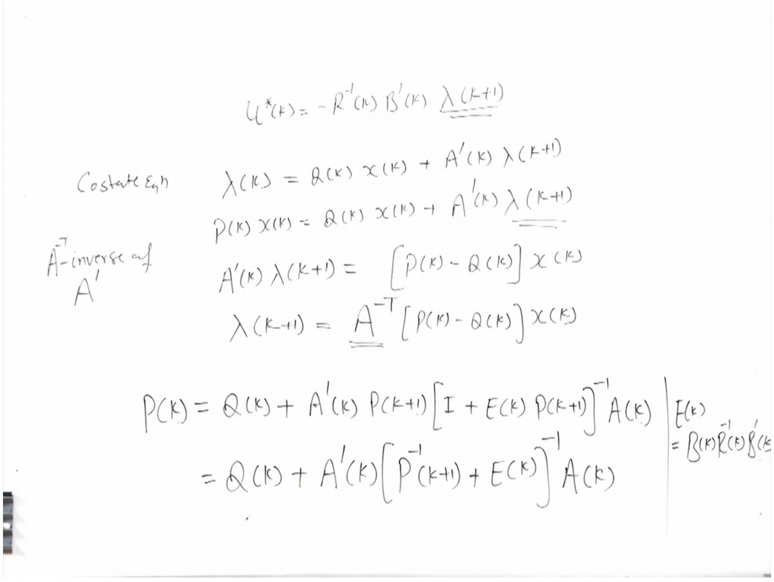

So, now we have to see how we can represent u in terms of the P. Say we know my co state equation is lambda k as Q k x k plus A transpose k lambda k plus 1 and lambda k if I will replace as P k x k this is Q k x k plus A transpose k lambda k plus 1. So, now, I can write this lambda k plus 1 in terms of the x k if I will right A transpose k lambda k plus 1 as P k minus Q k x k and lambda k plus 1 is nothing, but, so this I will write as A minus t P k minus Q k x k.

So, this A minus t is what, what is this? This is the inverse of a transpose. So, this is inverse of A prime. So, this is my lambda k plus 1, I can write my u as u star k as R inverse B prime k and lambda k plus 1 I am replacing as inverse of A transpose multiplied with P k minus Q k into x k, this means I am representing my u k in terms of the x k so that I can utilize my closed loop control. So, I can have the state feedback as x of k this R inverse B prime A minus sorry transpose of the A inverse P k minus Q k this I can define as my L. So, this is define my L k, I can simply defined u k in terms of my L k x k where L k will be my controller gain which is given as R inverse B transpose inverse of the A transpose P k minus Q k.

So, by this I can design my optimal L k which is giving me the optimal value of the u for a closed loop control system and if I will use u equal to minus L k. So, my closed loop system equation will be x star k plus 1 A k minus B k L k x star k. So, by this I can get my closed loop control. Say in this form of the Riccati equation where I have to use P inverse k plus 1 or and in u I have to use the inverse of the A transpose. So, for this if the inverse of the A transpose exist this means my matrix A should be a nonsingular matrix. So, if A is a singular matrix then inverse of the A will not be possible. So, this problem we can eliminate utilizing the alternate form of my Riccati equation which we will discuss in the next class. So, today I stop my discussion at this point.

Thank you very much.