**Optimal Control**
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**Lecture – 31**
**Discrete-Time Optimal Control Systems (Continued)**

So, welcome friends of this session of our discussion. Today we will discuss the discrete time optimal control system which we are discussing from the previous class.

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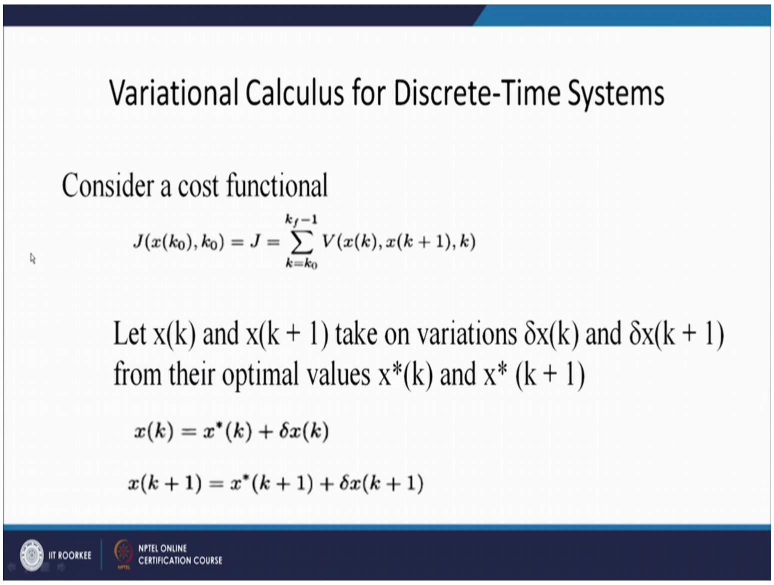


So, in the previous class we have seen that variation of calculus can also be applied to the discrete system we are minimizing a functional which is given as the V x k, x k plus 1, k plus 1 variation at the optimal point is x star k plus delta k and at x k plus 1 is x star k plus 1 delta k plus 1.

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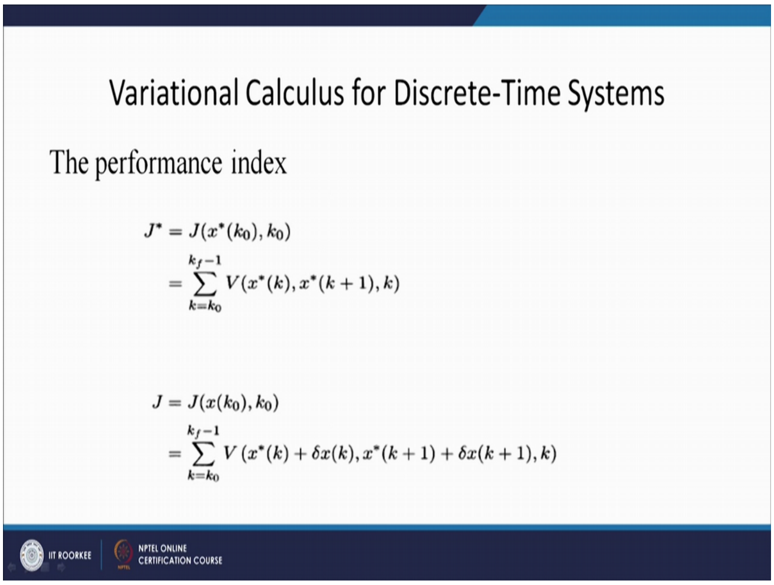


So, we define J star and J star at the optimal point and J star at the variation point x star k delta x k.

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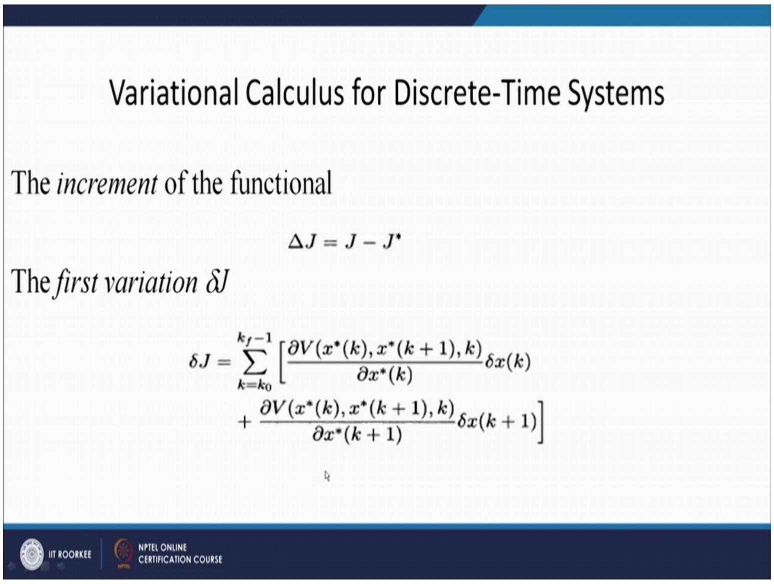


We find out the increment and from the increment we define our first variation as delta J in the variation we got the two terms delta x k and delta x k plus 1.

So, this second term we convert in the form of the delta x k.

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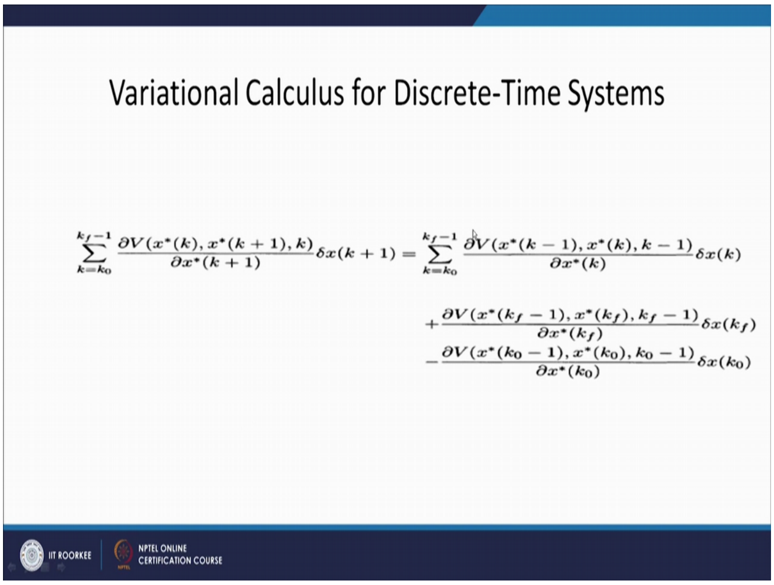


In this conversion we got the first term as delta V x star k minus 1 which is the coefficient of delta x k plus the boundary condition which is given as delta V x star k minus 1 x star k, k minus 1 delta x k at the point equal to k 0 and k equal to k f.

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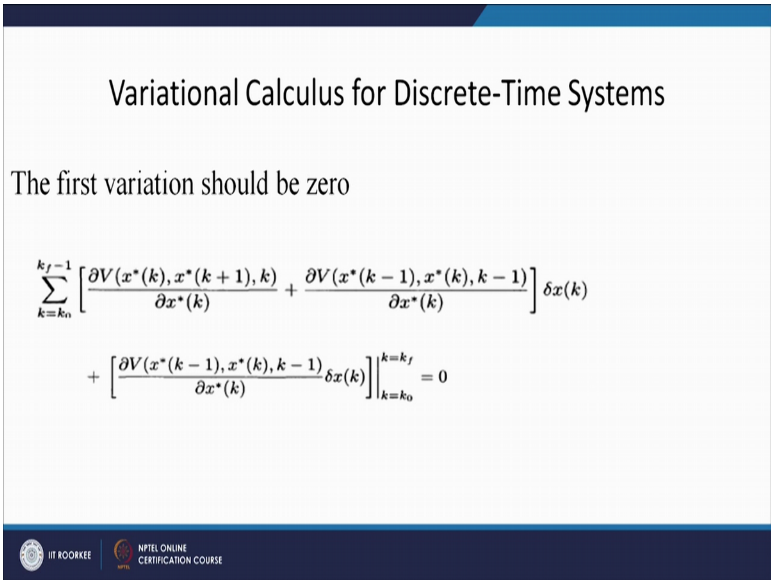


So, k equal to k 0 is my initial condition and k equal to k f is my final condition. So, in normal case we are given with the initial condition.

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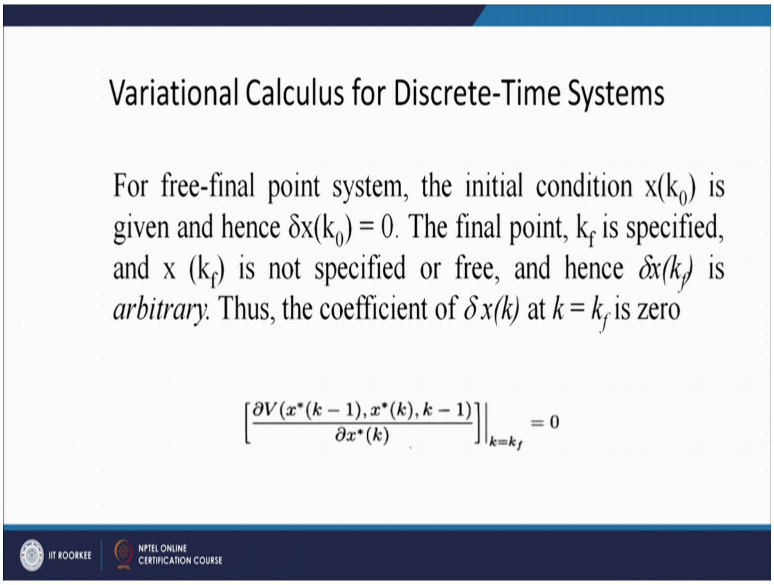


So, delta x k 0 will be 0 and we will left with the boundary condition as delta V x star k minus 1 x star k, k minus 1 by delta x k at the point k equal t k f and this must be equal to 0. So, this is my terminal condition and this gives me the EL equation in the discrete form.

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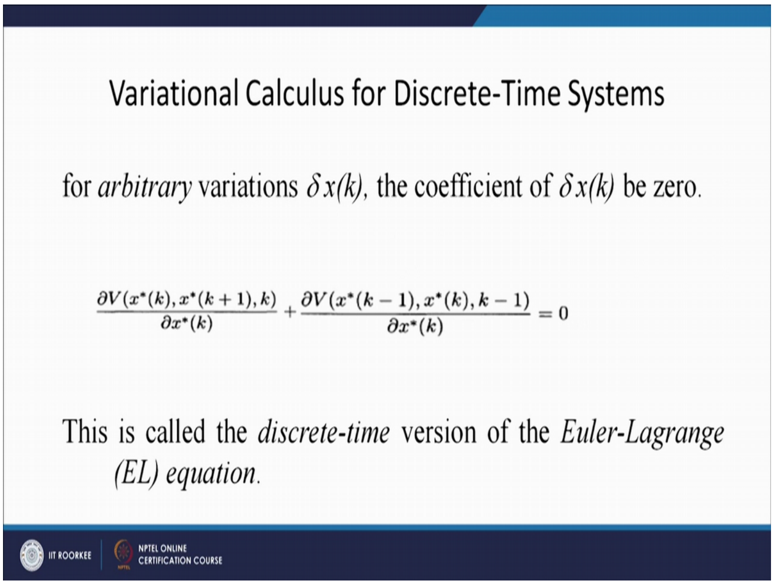

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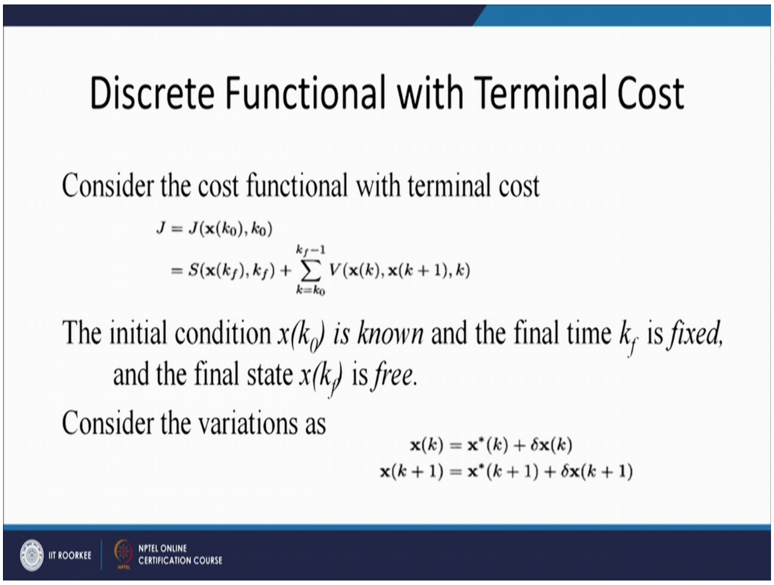


So, this we have done in the previous class. So, today we will see in that discussion we have not considered the terminal cost. So, if we will consider the terminal cost also with my performance index J. So, my J is now S x of k f k f where k f is our terminal point. So, this I my terminal cost and this is my summation cost. Again in the similar manner we considered the variation at the optimal point of x star k, x star k plus 1 has x star k delta x k for x k, for x k plus 1 this is x star k plus 1 plus delta x of k plus 1.

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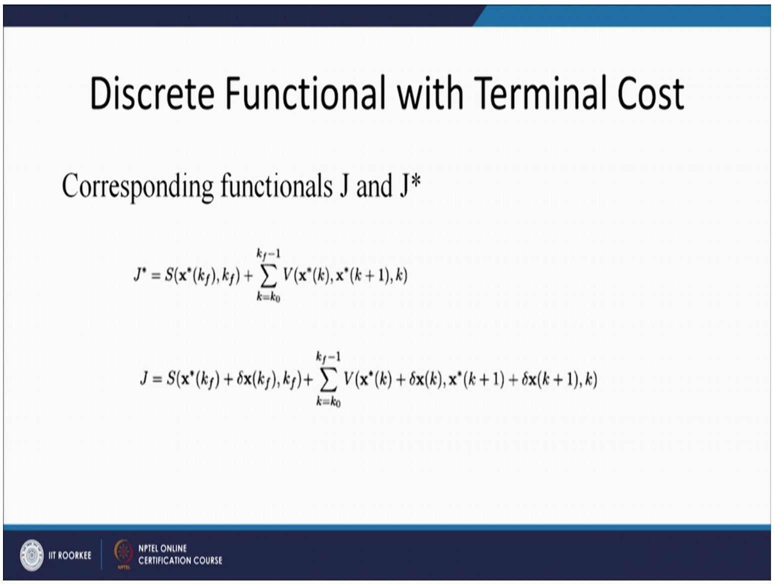

Our approach is similar, again we define J star in terms of the terminal cost and J at the point x star k f plus delta x of k f at the variational point.

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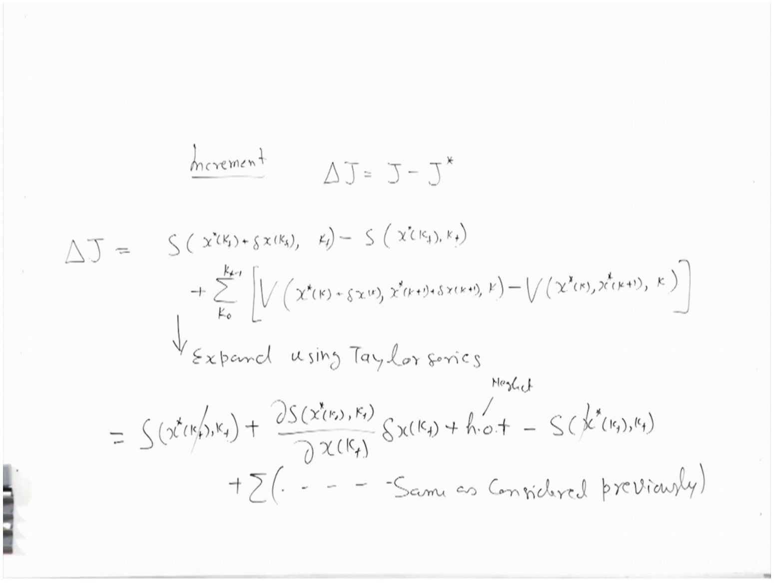


So, after defining J star and J my next step is to find out the increment. So, next we will find out what is the increment and how we define the increment? Delta J f, J minus J star has we know. So, we can write delta J as J will have my first term. So, I am subtracting J star from J. So, this is S x star k f plus delta k f minus this. So, I will write these two term together S of k of k f plus delta x of k f, k f minus S of x of k f. So, this is the first two term I am subtracting J minus J star and second will be the summation subtraction plus this is from k 0 two k f minus 1 V. So, we can say put the star here because this is at the optimal point, this is at the variation point x star k plus delta x of k x star k plus 1 plus delta x k plus 1 and k.

So, this is my summation term of the J and minus V x star of k x star of k plus 1 and k. So, if we will see my summation term this is similar as we have considered in the previous case. So, once we will expand this with Taylor series, expand using the Taylor series. So, I will get from the terminal cost S x star of k f k f plus del S by delta x of k f, delta x of k f minus plus higher order term minus S of x star of k f k f and plus this summation term. So, this summation expansion is same as we have considered in the previous case, same as considered previously.

So, this we will take up same as we have done before this has will cancelled out this higher order term we neglect. So, what actually we will left? We will left say if we will see from the summation I will get the first the summation term same as we have considered.

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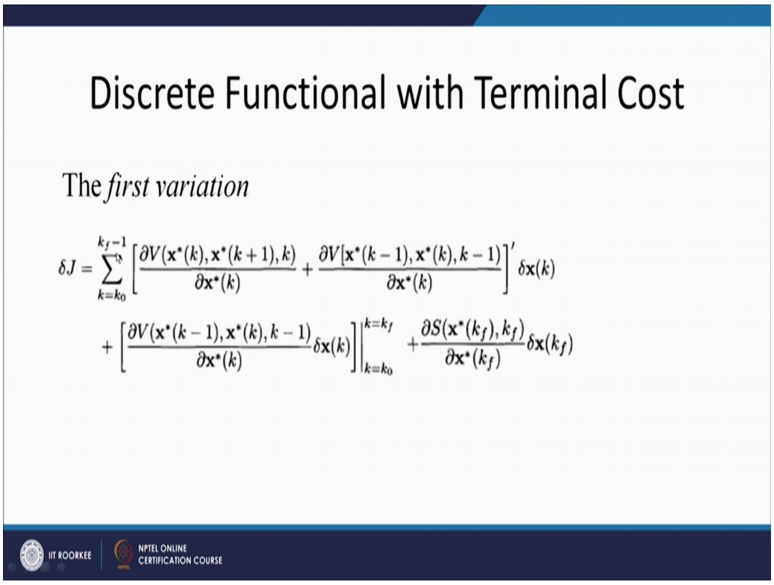


In the previous case plus the boundary condition for k 0 to k f plus this is additional term which we will appear in the first variation. Now I can as my initial condition is given or it is already defined k equal to k 0 and x of k 0. So, delta x of k 0 will be 0. So, this is left only with the k f term. So, this term we will left with the k f.

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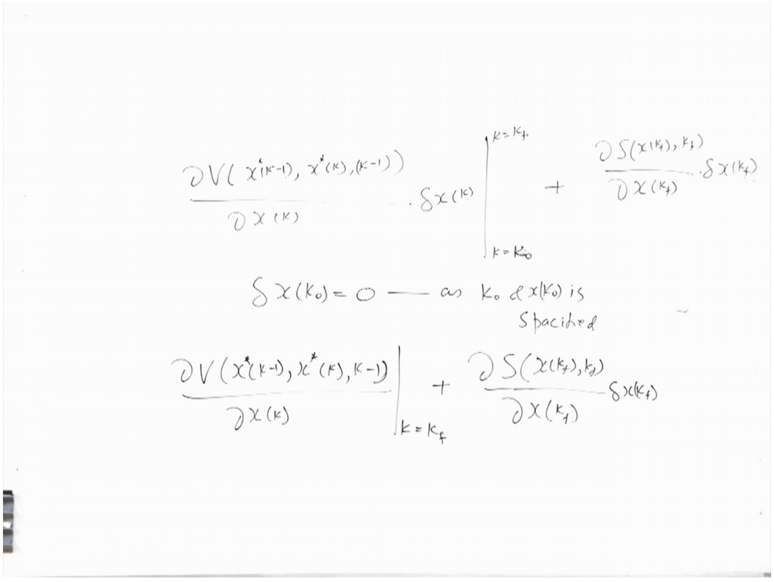

So, what we are saying del V sorry; x of k minus 1 x star of k, k minus 1 delta x of k. So, we are considering this term with delta x k and we are evaluating this as k equal to k 0 and k equal to k f plus we are getting del S k f by delta x of k f into delta x of sorry; delta x of k f.

So, in this term at k equal to k 0 my delta x k 0 is 0 as k 0 and x of k 0 is specified. So, x of k 0 will be 0. So, I am left only with the x of k f. So, delta V I can write this as x of k minus 1 a star we are placing here x star k, k minus 1 by delta x of k. So, this is valuated only at k equal to k f point plus this delta S. So, now, at k equal to k f I am evaluating this, I am also evaluating at k equal to k f. So, I can club these two to get my boundary condition has del V x star k minus 1 x star k, k minus 1 delta x k plus delta S by delta x and the whole we are evaluating at k equal to k f point.

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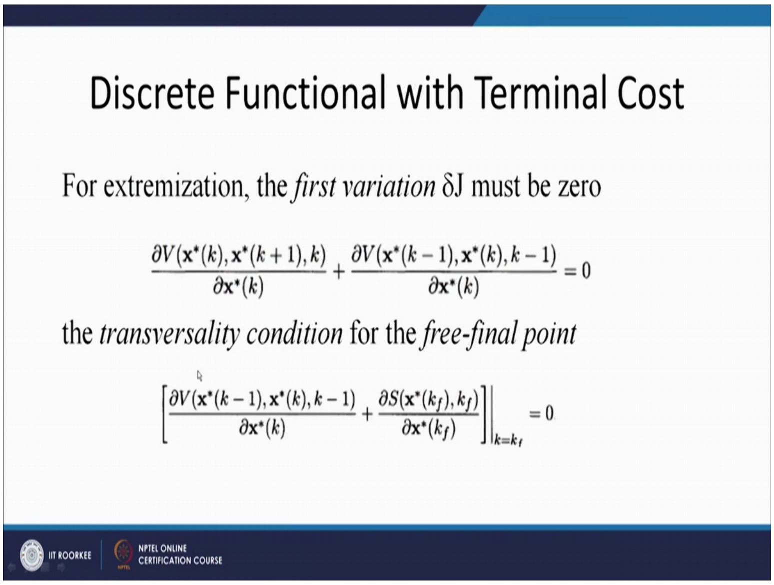


So, with this again we will we have the EL equation in the discrete form plus my boundary condition will be given by as this second equation. So, even if the terminal cause my additional term which is appearing that is del S by del x which is given. Now this concert with the terminal cost, now we can apply to our optimal control problem.

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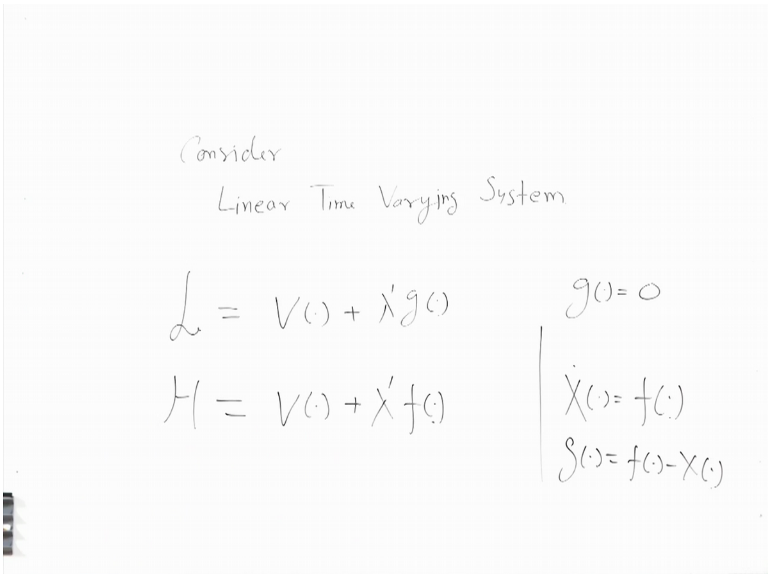


So, we take its application to a linear time varying system. So, we are considering linear time varying system. So, this linear time varying system can consider as x of k plus 1 equal to A k x k plus B k u k. So, varies standard equation and we are given with the initial condition as x k equal to k 0 given as the x of k 0 and the performance index of the plant we can take as J equal to J x of k 0 u of k 0 k 0 because we evaluate performance index as we have seen before at the initial point this directly give. This is equal to the terminal cost which is defined as half of x prime k f, f of k f, x of k f plus half of now, summation of x prime q x plus u prime r u at the k th instant where k is varying from 0 to k f minus 1.

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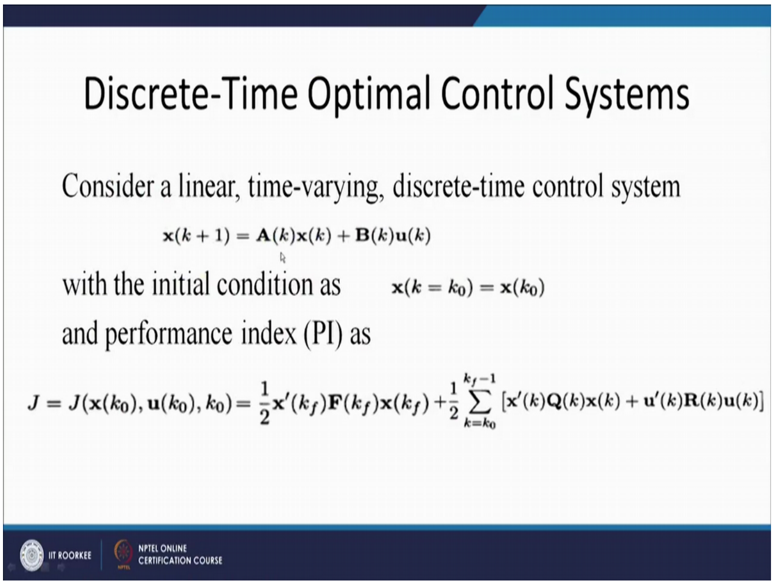


So, for this linear time varying discrete system my objective is to find the optimal u which will minimize the performance index given as J. So, whatever we approach first we will define a augmented cost functional.

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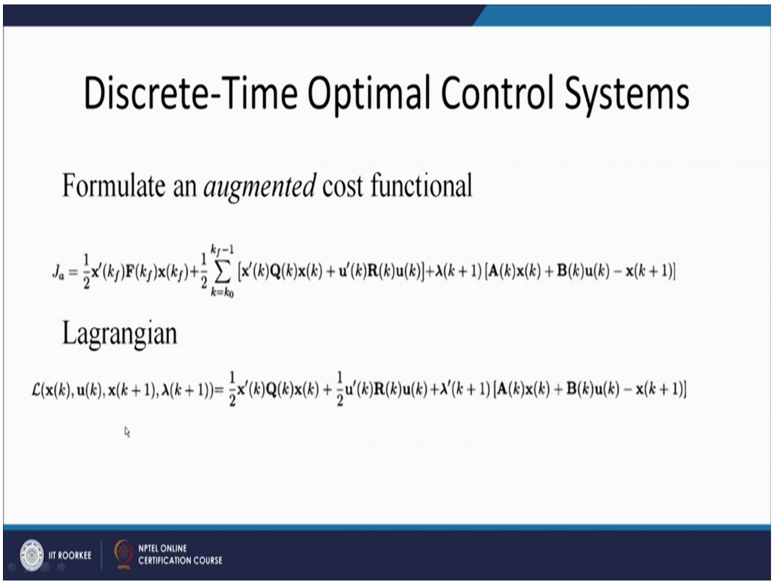


So, the solution of this problem we will start from our Lagrangian approach and then we will convert the Lagrangian into the Hamiltonian and then find out the final equations which we will have. So, what is my Lagrangian? Lagrangian we defined as V plus lambda prime g where g we will take as 0. So, by this we are writing first the augmented cost functional which is include the terminal cost plus half of. So, this I am augmented x prime q x plus u prime r u plus lambda k plus 1 A k x k plus B k u k minus x of k plus 1.

So, this term represent nothing, but my g. So, g equal to 0. So, means my this equation A k x k plus B k u k minus x k plus 1 this giving me the 0. So, if we will see my augmented cost functional is same as the cost functional we have considered before given as J, so the minimization of the J is same as the minimization of the J. In this here we are defining my Lagrangian as half of x prime Q x plus half of u prime R u plus lambda prime k plus 1 A k x k plus B k u k minus x of k plus 1. So, this is defining my Lagrangian now in this the question maybe what should be my Lagrangian function this lambda prime. Lambda prime we can take it as lambda prime k or lambda prime k plus 1. So, we have taken intentionally this as lambda prime k plus 1, the reason will be clear later on because this made our further calculation or the further expression to be to simplify. We are considering lambda as a function of k plus 1 in place of the k, if Lagrangian is defined where Lagrangian is a function of x k u k and lambda k plus 1. So, by this we can directly write our Euler equation as we had write before.

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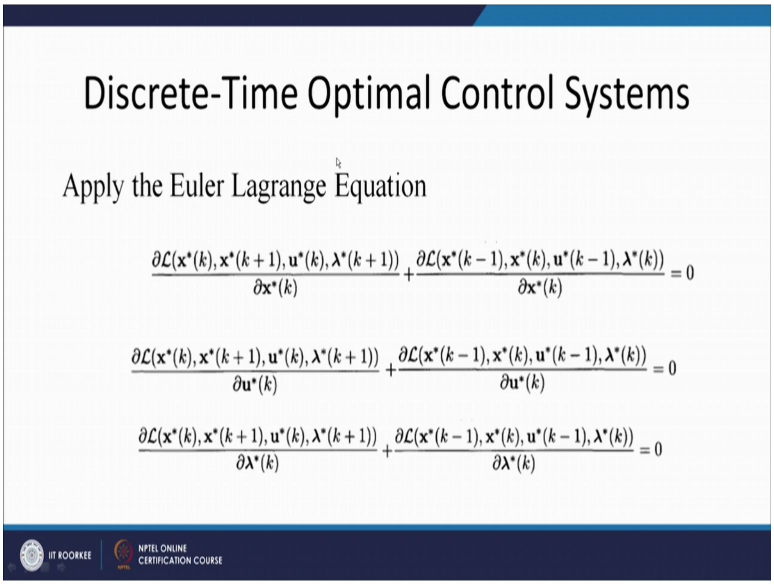


We can follow the same approach as I can define my Lagrangian at x point I can define the optimal Lagrangian Lagrangian at the variational point and then find out the increment. So, this is this approach is similar as we have done before we have find out the first variation is del V by del x k where V is the function of x star k x star k plus 1 and k. So, the two another variables we are adding into this which are as my u k and lambda k, lambda k plus 1.

So, my Lagrangian equation I can write simply as del L by del x k del L by del u k del L by del lambda prime k. So, these three will be my EL equations with respect to my state, my control and the third is my nothing, but the co state which we will have and my boundary condition will be del L k plus 1 x of k plus 1 u of k plus 1. So, del L is a function of this because we have considered my final condition as del V x star k plus 1 x star k at k minus 1 point.

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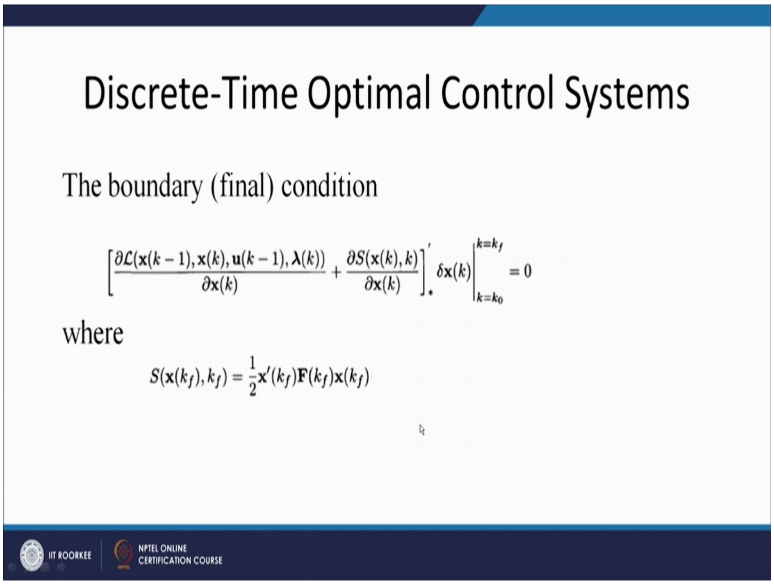


So, naturally once I will write my final boundary condition that will be del L x of k plus 1 u of k plus 1 as the lambda k plus 1. So, here it is coming the lambda k plus del S which is nothing, but giving me the terminal cost delta x of k equal to 0 where S x k f k f if I will evaluate this point is nothing, but half of x prime k f f of k f x of k f. So, in Lagrangian form I got this three EL equations and the final boundary condition given as this. So, this optimal control problem can directly be solved using the Lagrangian approach or further we can simplify this using the Hamiltonian.

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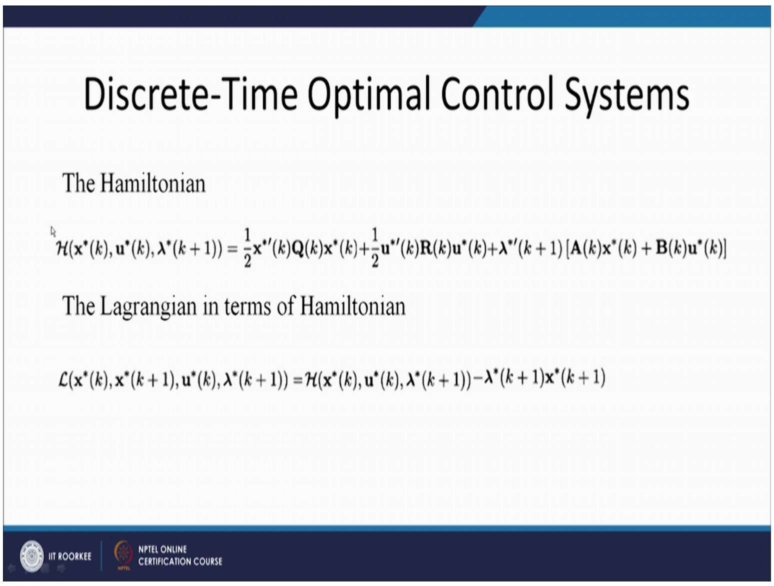

So, we define our Hamiltonian H what is H? Is compared to L plus lambda prime here we will take in place of the g f where we are considering x dot equal to f. So, only we will be consider f here in place of the x do sorry f minus x dot is g. So, for this case my g is f minus x. So, this is the difference between Lagrangian and the Hamiltonian.

So, here we take V dot plus lambda prime f and what is my V n lambda prime if we will see. So, from my performance index my first two term half of x prime x k plus half of u prime R u this is my V plus lambda prime k plus 1 and this is my f A k x k plus B k u k. So, this define my Hamiltonian and in terms of Lagrangian if I will write this. So, what was my Lagrangian here? We have consider the Lagrangian is half of x prime q x half of u prime R u now lambda prime k plus 1 A k x k plus B k u k up to this, this is my H minus lambda prime k plus 1 x of k plus 1 is subtracted. So, I am writing my L in terms of the H minus lambda prime k plus 1 x star of k plus 1.

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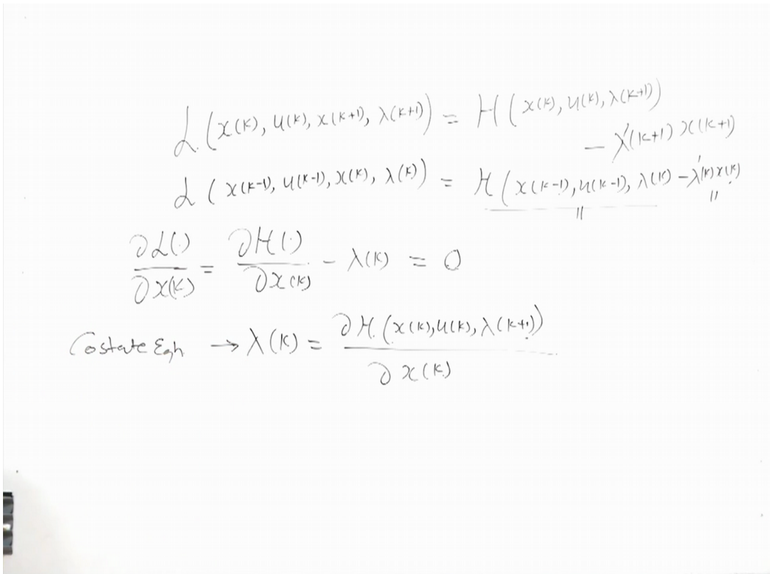

So, this is giving me Lagrangian, now if this Lagrangian in terms of the Hamiltonian I will use with these equations. So, I can get all the EL equation in terms of my Hamiltonian. So, I am writing my L, L is a function of x k, u k, x k plus 1 lambda k plus 1 and this is equal to my H which is also the function of x k, u k, and lambda x plus 1. So, H is not a function of x of k plus 1 this is the function of x k, u k, lambda k plus 1 and minus we have lambda prime k pus 1 x of k plus 1. So, we will have this value because lambda x of k plus 1 is separated out from the H as we can see from this no x k plus 1 term appear in the Hamiltonian. So, now, this L in terms of the H we can use in these equations.

So, my first equation is del L b y del star of k. So, del L, what you will get by this first term? We are L we are taking this with x of k if we will differentiate this with respect to x of k I have only term with the H. So, this I can write simply as del H by del x of k while this term will give me the 0. My second term is del L x star k minus 1. So, if I will write L as x of k minus 1 u of k minus 1 this will be x of k lambda k and similarly here will be the H x of k minus 1 u of k minus 1 lambda k minus lambda prime k x k and this I have to differentiate with respect to x k. So, naturally only x k term will appear. So, with this second term we are getting only as lambda k. So, this term I have to differentiate with respect to x k. So, this will give me 0 and this give me only lambda k. So, this is minus lambda k and this will be 0. So, what I will get? Lambda k as del H which will be the function of my x k u k lambda k plus 1 by delta x k.

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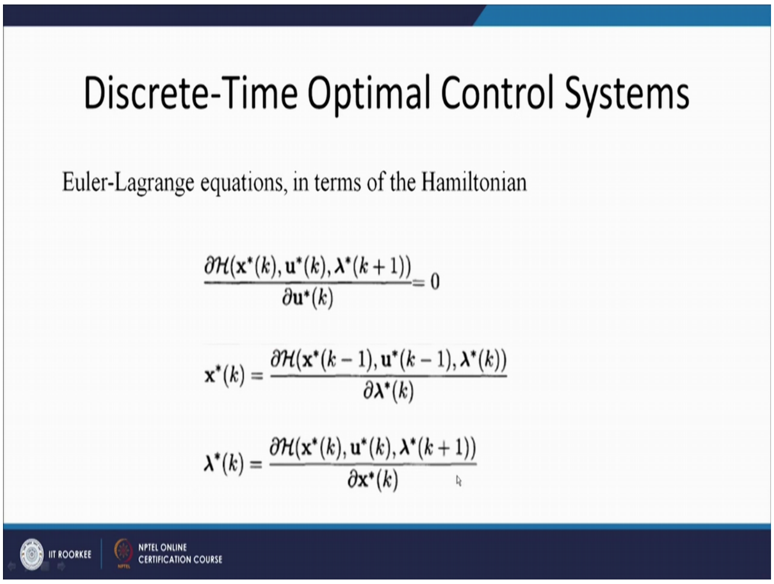


And this is nothing but my this equation I am getting lambda k equal to del H function of x k u k lambda k. So, this nothing, but giving me the co state equation. So, this will be my co state equation.

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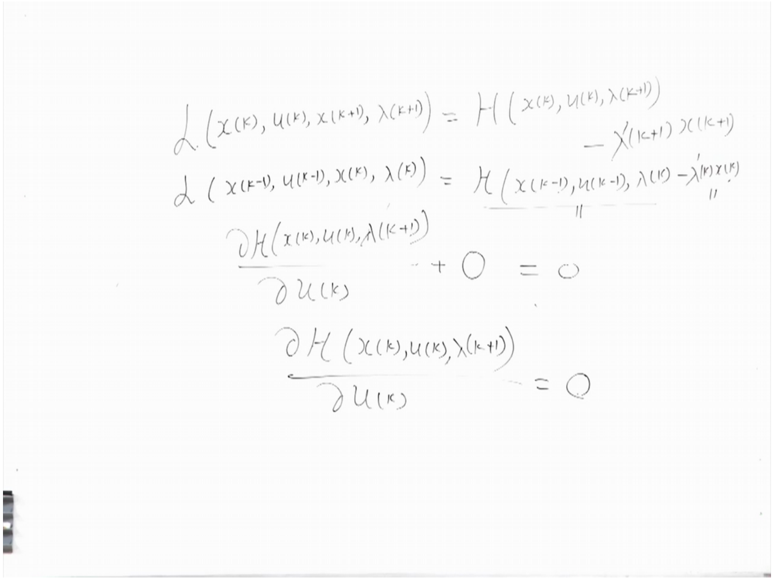

Next we have to take del L by del of that is u k this means the second equation we are taking del L by del u k, second EL equation I am placing this Lagrangian in terms of the Hamiltonian. So, with respect to k, k plus 1 u k I have to differentiate this. So, this will give me. So, fast I am differentiating the first equation which is L of x k u k x k plus 1 lambda k plus 1.

H is the function of u. So, naturally I will get del H x of k u of k lambda k plus 1 by delta u of k. My next term then I have to differentiate this L with respect to u k and here if you will see H is not a function of u k, this is also not a function of u k. So, this term will be give me 0. So, if I will write this equation. So, this is nothing but del H by del u k plus 0 equal to 0. So, this means this must be equal to 0, so my this equation will be nothing, but del H with x k u k lambda k plus 1 by delta u of k that will be 0. So, by this I am getting my this equation del H by del u with x k u k lambda k plus 1 equal to 0. Then I have my third equation as del L by lambda k del L k minus 1 by lambda k. So, this means I have to differentiate these two with respect to lambda k.

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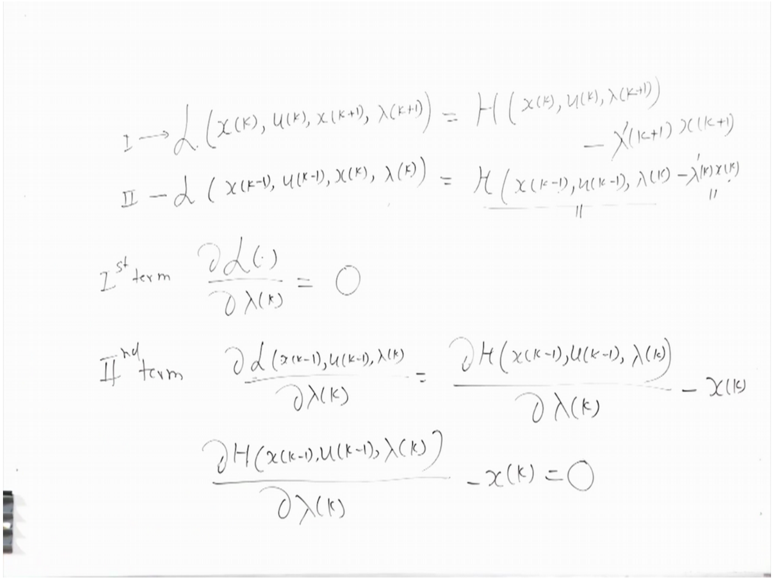


So, from the first term this is del L this I am saying first let us say this I am saying second del L by del lambda k what we will get? So, we having del L by. So, this term I have to differentiate with respect to lambda k and this whole term is independent of the lambda at a k sorry; this will give me 0. While the second term what actually I will get? x of k plus 1 u of k plus 1, so this H is a function of lambda k as well as my second term is also the function of lambda k. The second term will give me before the second term I am taking, this is x of k minus 1, u of k minus 1 lambda k by del of lambda k again. So, if I will differentiate this what I will get del H which is x of k minus 1 u of k minus 1, but function of lambda k.

So, this means H I can differentiate with respect to lambda k and this lambda prime k x k if I will differentiate this with respect to lambda k this will give me nothing, but x of k and by adding these two term, so this is the first term which is written here, this is the second term by addition equal to 0 is giving me the EL equation. So, this give me nothing, but del H x of k minus 1, u of k minus 1, lambda of k differentiated with respect to lambda of k minus x of k equal to 0. So, x of k equal to this. So, I will get my from the third equation I will get my equation is x star k delta H x star k minus 1 u star k minus 1 lambda k by lambda k. So, I got, so this is my control in terms of the Hamiltonian del H by del u equal to 0 x star k del H by del lambda k x star k plus 1 u star k plus 1 in this we have to mind this. This is my state equation and lambda star k plus 1 as x star k u star k lambda k plus 1 this is my co state equation.

So, this, that is why we have taken as the lambda k plus 1 to get this equation in lambda k. So, what actually we are doing in the next now this x star k I am getting say this two terms, are in terms of the x k u k lambda k plus 1, this is also in terms of the x k u k lambda k plus 1 except this middle term which is x star k minus 1 u star k minus 1 lambda star k minus 1 lambda sorry differentiated with respect to lambda k. So, this we can write as.

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$$x^*(k+1) = \frac{\partial H(x(k), u(k), \lambda(k+1))}{\partial \lambda(k+1)}$$

$$H(x(k), u(k), \lambda(k+1)) = \frac{1}{2} x'(k)Q(k)x(k) + \frac{1}{2} u'(k)R(k)u(k) + \lambda'(k+1)\left[A(k)x(k) + B(k)u(k)\right]$$

So, I can write this equation as x star k plus 1. So, that my this term can be converted into the H of k. So, this is delta H now this will be the function of x k, u k, here lambda k plus 1. So, [FL] my H is similar in all the three equation control delta x of because we have made the k plus 1 sorry this is delta with respect to I am differentiating lambda k plus 1. So, what is the advantage of having this? Because now I can define my H x k, u k, lambda k plus 1, how we are defining this? This is my V which is nothing, but half of x prime k Q k x k plus half of u prime k, R k u k plus lambda prime k plus 1, A k x k plus B k u k. So, now, directly now I can have the differential of this H with respect to x k with respect to lambda prime k plus 1 and with respect to u k. So, my all equations with the given Hamiltonian as H I can directly have del H by del u equal to 0.

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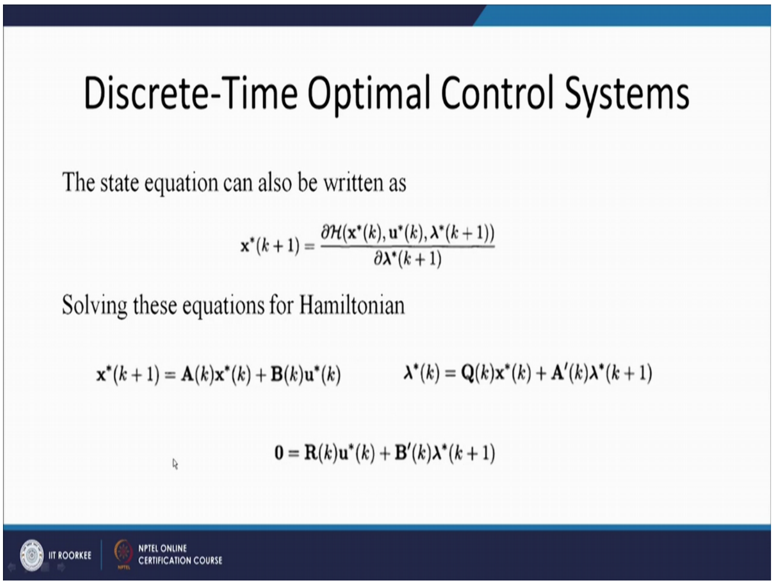


I can have del H by del x equal to 0 to get my co state and del H by del lambda prime k plus 1 because this is defined as the lambda k plus 1 direct differential with this. So, that is why initially we have taken in place of the lambda k we have taken the lambda k plus 1. So, my H k if I can define the Hamiltonian directly I can write my equations.

So, I stop my discussion here for this session and further how we will utilize these equations to develop the difference Riccati equation that we will discuss in the next class.

Thank you very much.