Optimal Control
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Lecture - 30
Discrete-time Optimal Control Systems

Welcome friends to the session of our discussion. In the previous discussion we have discussed the continuous time system, LQR problem, LQT problem. So, today we will see whether the variational approach can be applied to the discrete time system. I hope everybody will understand the difference between the continuous and the discrete system. In discrete system we will define a signal at a discrete instant of the time. So, here we will consider the system in the discrete form as well as my performance index also in the discrete form.

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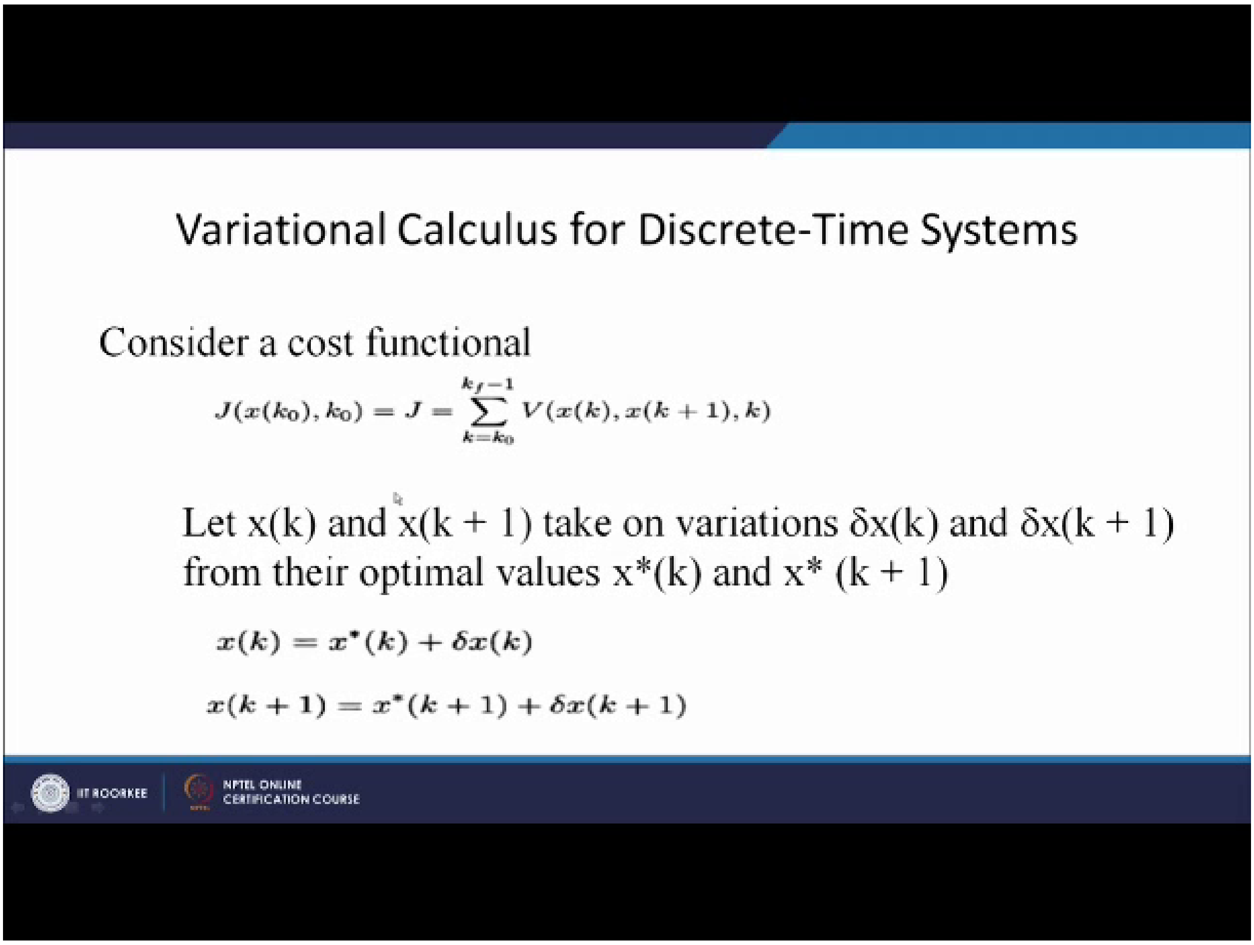

So, here as we have defined the continuous time functional similarly we can define a discrete functional, which is nothing but the summation of a functional V, where V is a function x k x k plus 1 and k.

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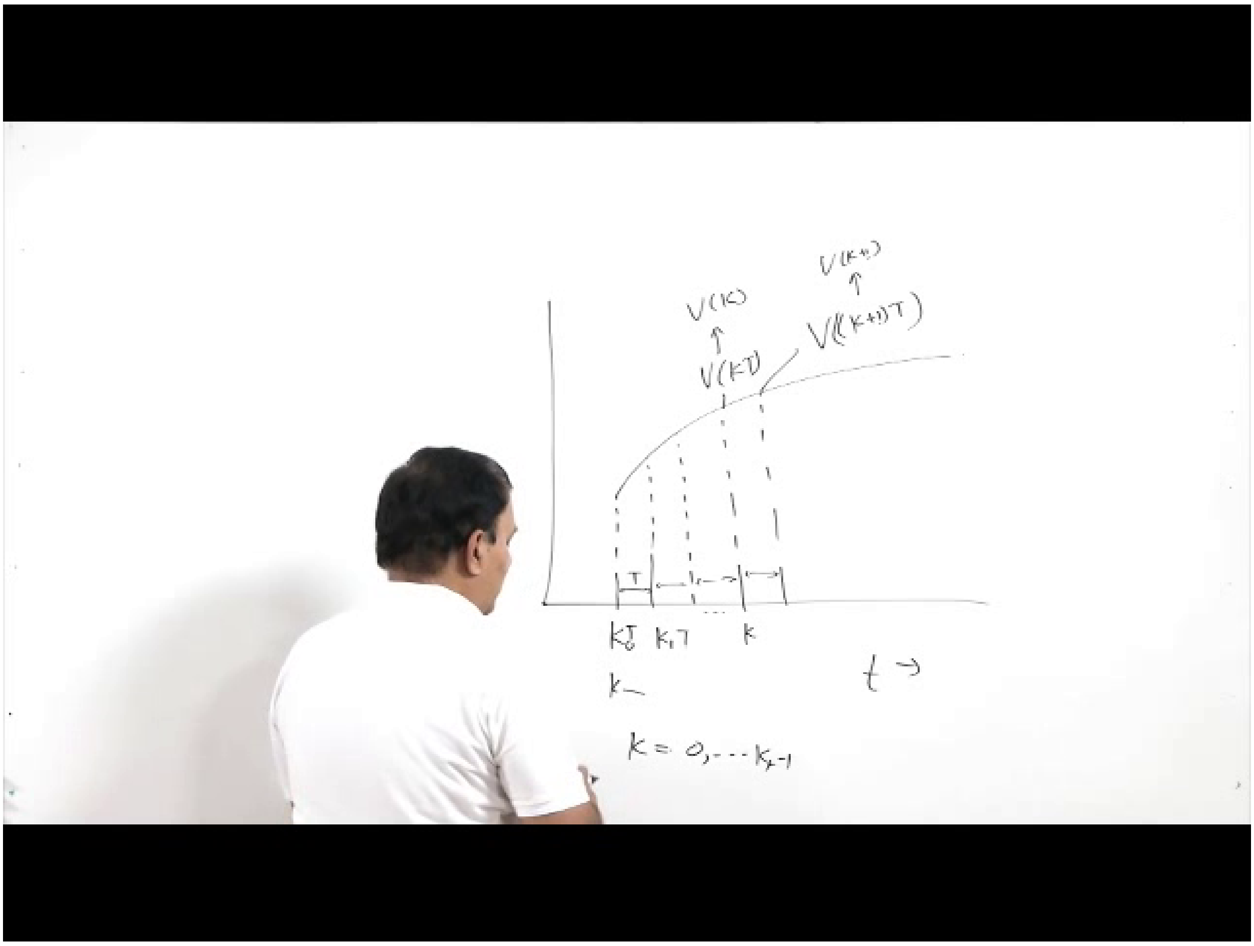

So, what actually we are considering if T is my scale, and if I will say this is V. Suppose any form of the function I will consider and I will have the sampling instant sampling period s t. So, I can say this is my K 0, this is my K1T. So, if I will at the constant sampling I can define the value of the V. So, let us say this is my some point K. So, I can define this as my V k, and more general case if we will take we will write this as V at the instant K T and this point I can write as V K plus 1 T. So, in normal representation we represent this as V k and this simply as V K plus 1.

So, what we are considering here, we are considering a constant sampling period. So, T is constant throughout whole the sampling k represent the instant where we are having the sampling. So, k varies we can say from 0 to let us say k of minus 1. So, in this way we can define our any discrete functional, and as we know what is a functional is nothing but a function of functions. So, what we are trying to do we are considering a functional J as summation of V x k, x k plus 1 k. More general this is V x of k T x of k plus 1 T and k t; so in normal representation we are dropping out the T, and simply representing the function as V of x k x of k plus 1 and k.

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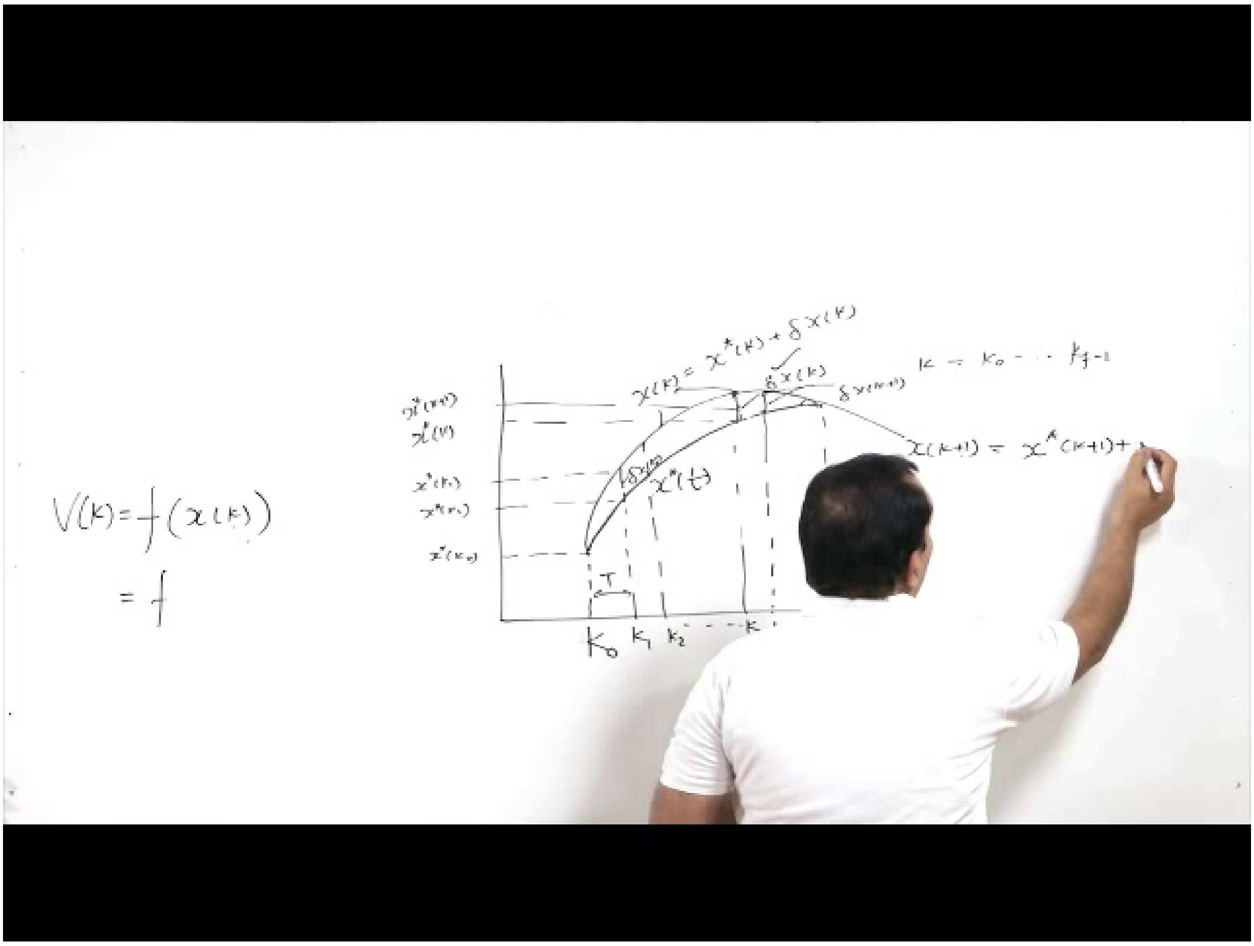

So, my this functional is a nothing but summation of the different scalars, when my k is varying from g k 0 to k f minus 1. So, we consider that. So, V is we can say a function of another function x k. So, if V is a function of x k, and x k let we define this is my x k and particularly let say this is the optimal value. So, V is. So, we are considering let this x k is the optimal value of the x, and we are considering a variation as x k which is nothing but x star of k plus delta x of k what is the k. So, let this instant is k 0. So, this is defined as x of k 0 which is normally defined and here we have considered T is constant, let us say this is K 1 K 2 K and my terminal time is up to K f we are considering each instant.

So, my x is defined at the instant of K 0 x of k 1 this is my if I am defining as my optimal value x star of k 2 a general point, x star of k and each point I will have delta x of k. Like at this point this is delta x k 1 similarly I will have here and let us say this is delta x of k, where k is varying k 0 to k f minus 1. This particular case is shown as my two fixed end point problem, but normally we will we are given with the initial condition and the terminal condition depends what actually is appearing which we will discuss in the later part, but this particular case is shown for a fixed end point problem. So, this means what we are saying, we are considering a optimal x and seeing the variation at this optimal point.

So, we are saying let x k, x k plus 1 take the variation delta x k and delta x k plus 1 from their optimal values of x star k, x star of k plus 1. So, this means if let us say at k th instant my value is x star of k, next instant to this will be k plus 1. So, this value will be x star of k plus 1 at this instant this is the variation, at this sorry this is my delta x of k plus 1. So, this we are assuming at these instant, but these variations are available. So, in journal my x k is defined as x star of k plus delta x k, x k plus 1 as x star of k plus 1 delta x k. So, this means if this is my x t, let this is the continuous trajectory x t at different instant I know the value at K 0, K 1 K 2 K, K K plus 1 k f and at each instant I am defining my x k as x star of k plus this delta x k.

So, this is x k is defined as this point and at this point we are defining as x k plus 1 as x star k plus 1, plus delta x of k plus 1.

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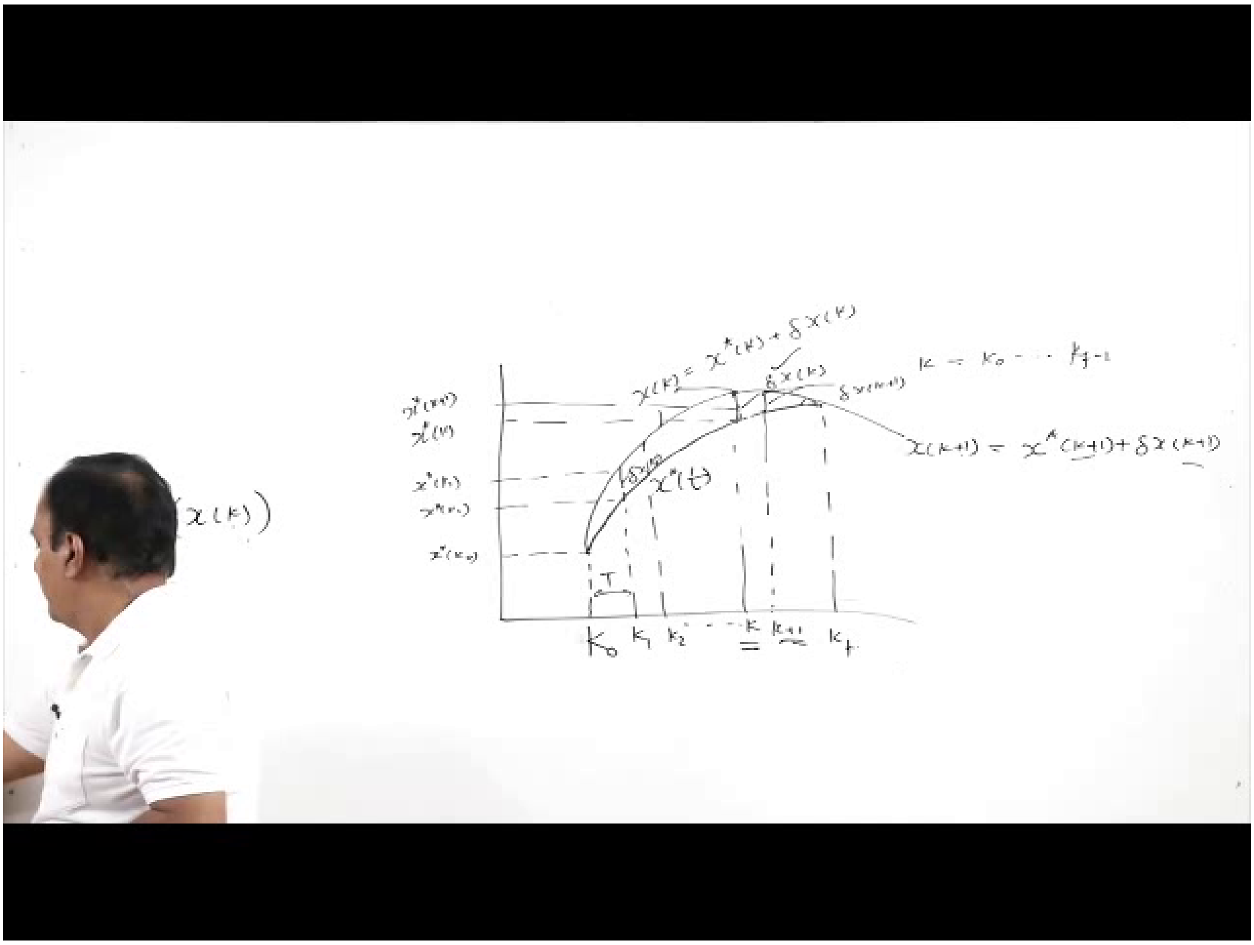

So, this is at the instant k plus 1 and this is at the instant of k. So, I simply define x k as x star k plus 1, delta x k x k plus 1 as x star k plus 1 delta x k plus 1.

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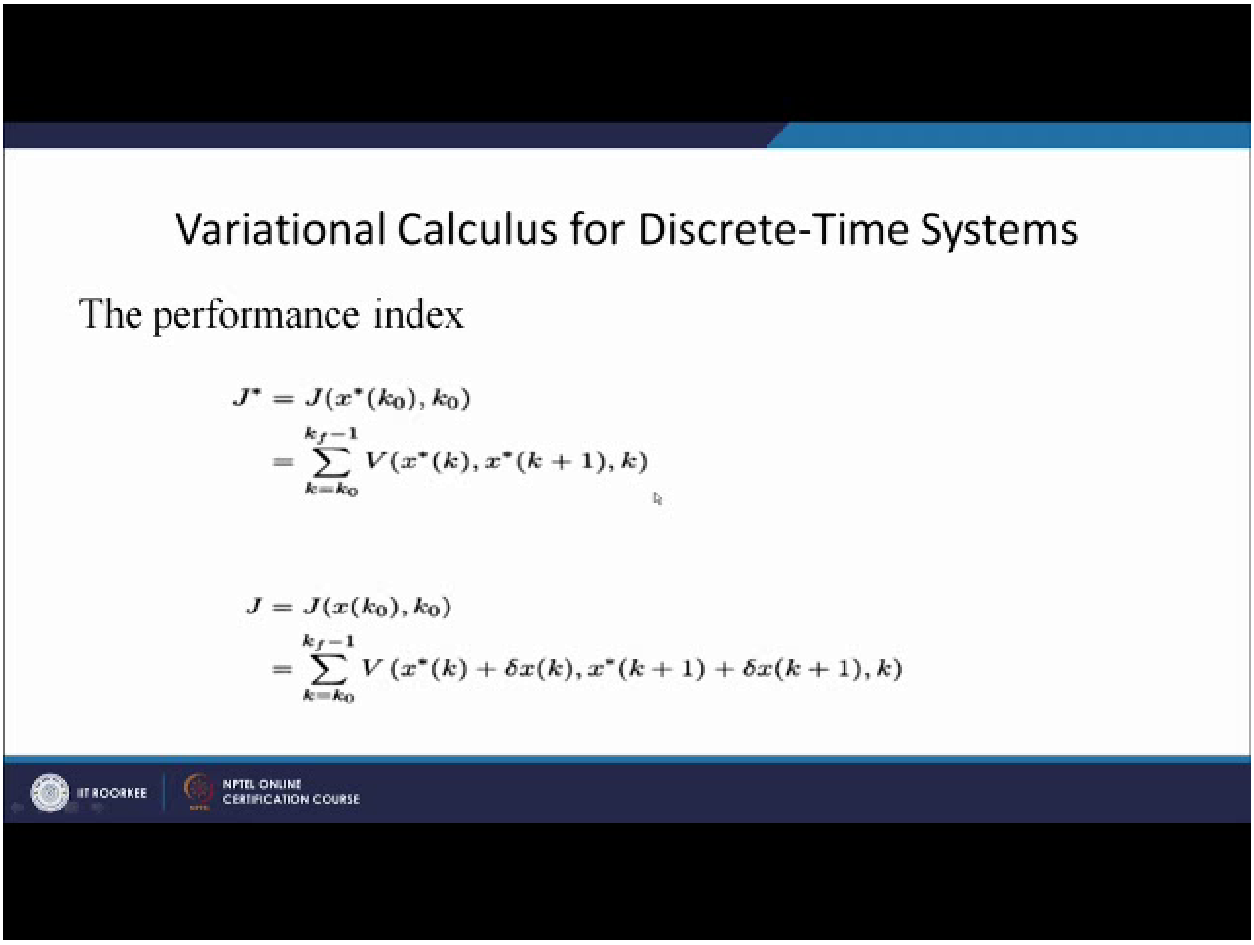

So, with this k x I will have the optimal value of the J as J star given as V of x star of k, x star of k plus 1 k, and at the variation point x star of k plus 1 delta x k plus 1. So, at optimal point which is at x star of k, my J is V x star of k x star of k plus 1 k. And at the variation point my performance index is J defined as V, x star of k plus 1 delta x k x star of k plus 1 delta k x plus 1, x k what is my objective? I am trying to find out the extremal value of the J, and to determine the extremal value of the J my J is defined at the optimal point and some variation point.

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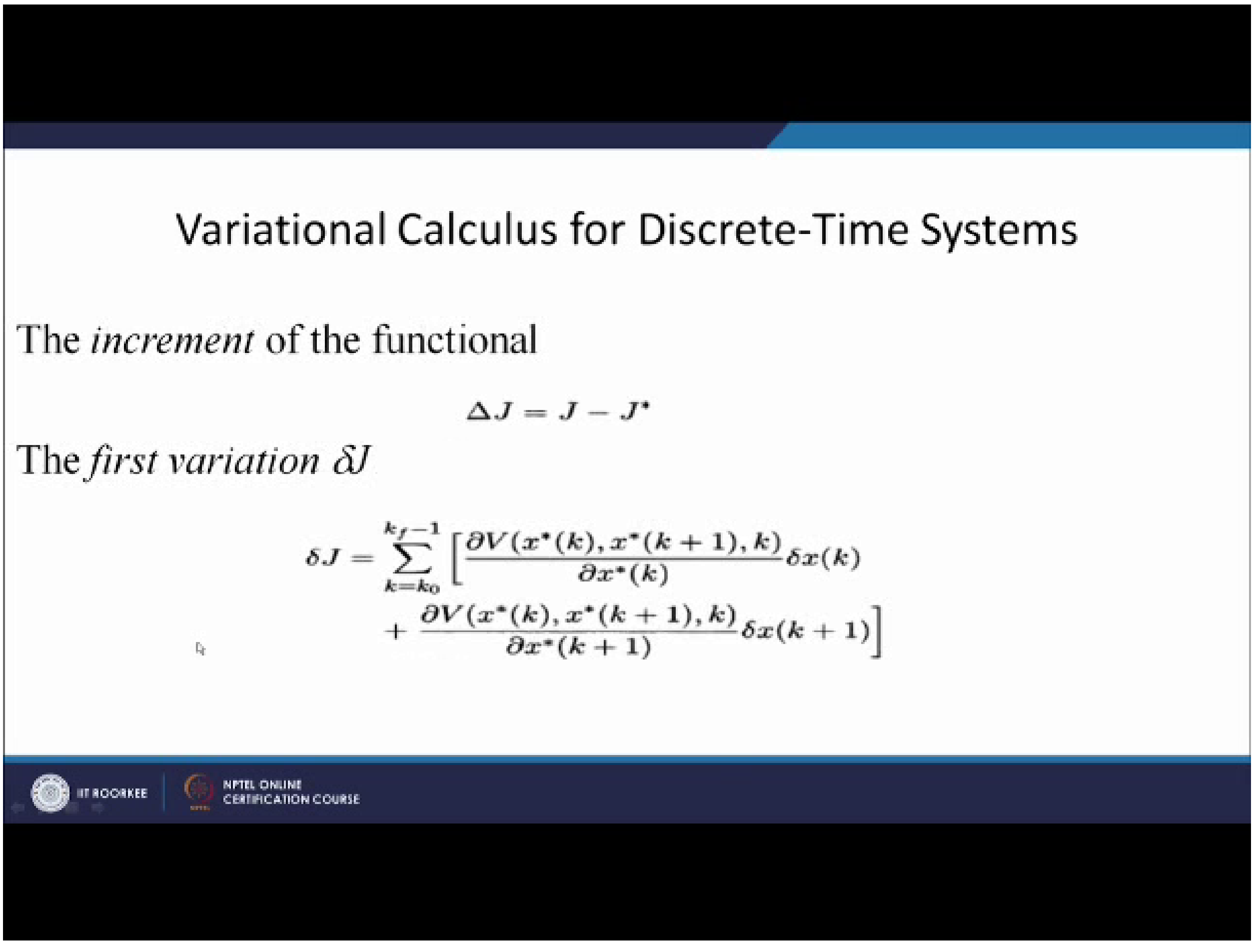

As we know we can define the increment of a functional as J minus J star. In this J minus J star we can determine the first and the second variation by the fundamental theorem of calculus of variation my first variation must be 0 and second variation should be less than 0 for maxima and greater than 0 for minima.

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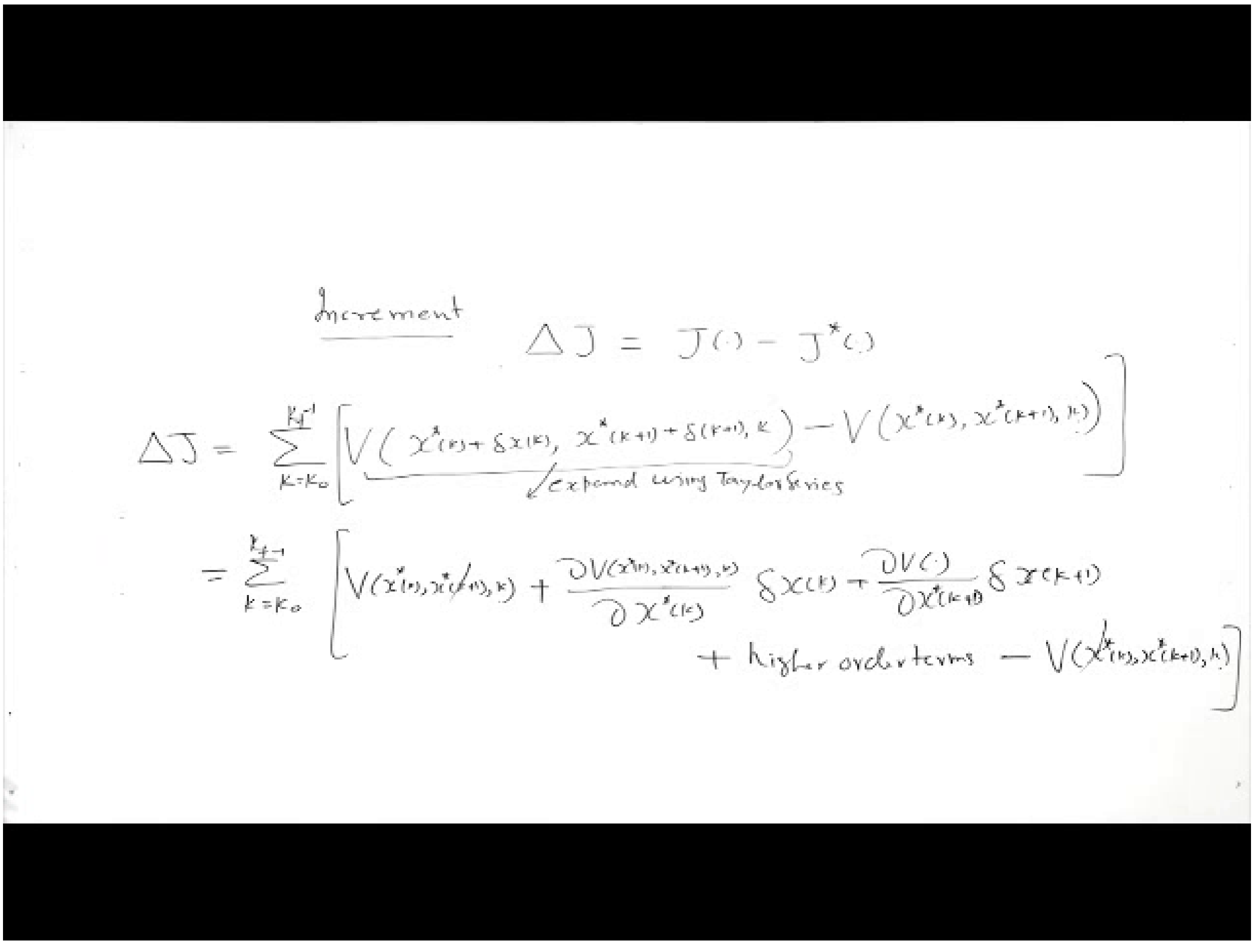

So, for the given value of the J and J star we find out the first increment and increment is defined as delta J this is the J minus J star. So, what I can write for delta J, what is my J? My J is sigma of V x star of k delta x k, K 0 to K f minus 1 we are finding and this is v delta x k x star of k plus 1 delta x of k plus 1 k. So, we can say this is my J minus J star is nothing but V x star of k, k plus 1 k this is also sigma. So, this V minus V at x star, and these we can expand using the Taylor series which is nothing but k 0 to k f minus 1. So, the first term I am expanding using Taylor series.

If I will explain this I will get V my first time will be V x star k, k plus 1, k minus del V I am writing simply like this, where V is a function of k write it complete x star of k, x star of k plus 1, k delta into del x of k sorry this is plus. V plus del V by del x k del V by some simply writing this as dot you can understand this delta x of k plus 1, delta, delta x of k plus 1 plus now second order terms higher order terms minus V x star of k, x star of k plus 1, k. So, this term will be an as such and this term we are expanding using the Taylor series.

So, naturally these two will cancelled out, if I will neglect the high order term I can write my first variation as del V by del x star k, delta x k, del V by del x star k plus 1, delta x of k plus 1. This means I am representing my first variation has del V by del x star delta x k del V by del x star k plus 1 delta x of k plus 1. So, in the first variation I will have the two terms one in the delta x k, other is the delta x k plus 1. Like in a continuous time case we have the x dot term in place of the x k plus 1, and we are converting this x dot in terms of the delta x. So, similarly here we will see can we convert this second term in terms of the delta x of k. So, what actually we have? We have the first variation is del V by del x delta x del V by del x k plus 1 delta k plus 1.

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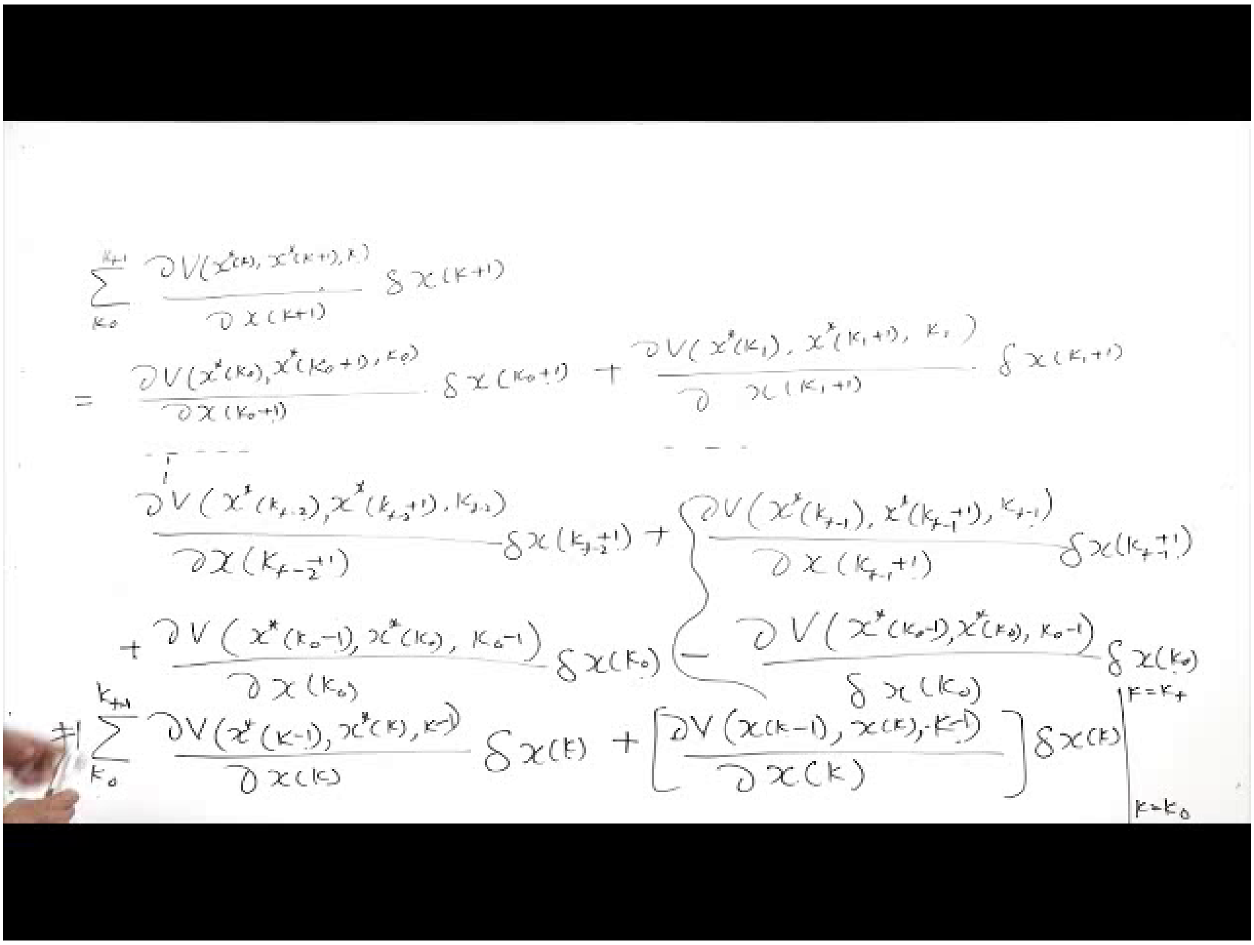

So, just consider the second term delta J is sorry we consider the second term k 0 to k f minus 1 we consider del V x star of k, x star of k plus 1, delta x of k plus 1, delta x k plus 1. So, we are considering the second terms of the first variation as del V x star k x star k plus 1 x k by del x of k plus 1 delta x of k plus 1 we expand this with taking case k 0, k

1, k 2, k f minus 1. So, my first term will be del V, k 0 x star, k 0 plus 1, k 0 differentiated with respect to x of k 0 plus, 1 delta x of k 0 plus 1.

So, I write second term will be del V x star k 0 plus 1, my first term is the k 0 second term will be sorry k 1 x star k 1 plus 1, k 1 by del x of k 1 plus 1, delta x of k 1 plus 1. So, this will be my second term. So, similarly I can have the third fourth up to second last time if I will write, this will be del V k f minus 2 x star, k f minus 2 plus 1, k f minus 2 by del; sorry del x, k f minus 2 plus 1, delta x of k f minus 2 plus 1. So, this is my second last term and what is the last term with k f minus 1 that is del V k f minus 1 x star, k f minus 1 plus 1, k f minus 1 del x of k f minus 1 plus 1 delta x of k f minus 1 plus 1.

So, this give me nothing but the expansion of this summation the second term of the first term variation which we have considered, by k replacing by the k 0 then k 1 second last term k f minus 2last term k f minus 1. So, in this if I will at the two extra terms sorry one term I will add and subtract which is del V x star k 0 minus 1, as per this next will be x star k 0, k 0 minus 1 del x k 0 because we have to take the k 0, k 0 minus 1. So, this is k 0 delta x of k 0 and the same term I will subtract from this del V x star k 0 minus 1, x star k 0, k 0 minus 1 delta x of k 0, delta x of k 0. So, if I will add and subtract these term this term; so I am not going to change my relation.

So, now what we are doing, I am including this first time as my first term of this expression and these two term I am combining together. So, if I will say I am not going to change the number of terms here from k 0 to k f minus 1. So, this whole expression what I have done, I have included this term as the first term of this expression. So, this means I am writing the second terms as sigma k 0 to k f minus 1. So, what I will have? I have only up to this term this is my first term. So, this V, I am changing by x star of this I am writing as x is star k minus 1, k star of k by del x of k my first term is this second third and this is the last term for this expression.

So, if I will take the k 0 my first term will be del V x star of k 0 minus 1 x star of k 0 k 0 minus 1 delta x of k 0 delta k of k. So, this will be my first term this will be my second term this will be my third term and this will be my last term and I am writing this last term as del V x of k, x of k sorry and this is k equal to k 0 and k equal to k f. So, these two combine term can be written as del V by del x of k minus 1 x of k k . So, del V by del x of k at the instant k 0 to k f minus 1. So, now, the second term can be replaced as del V by del x of k, delta x of k del V by del x of k delta x of k at k equal to 0, k equal to k f.

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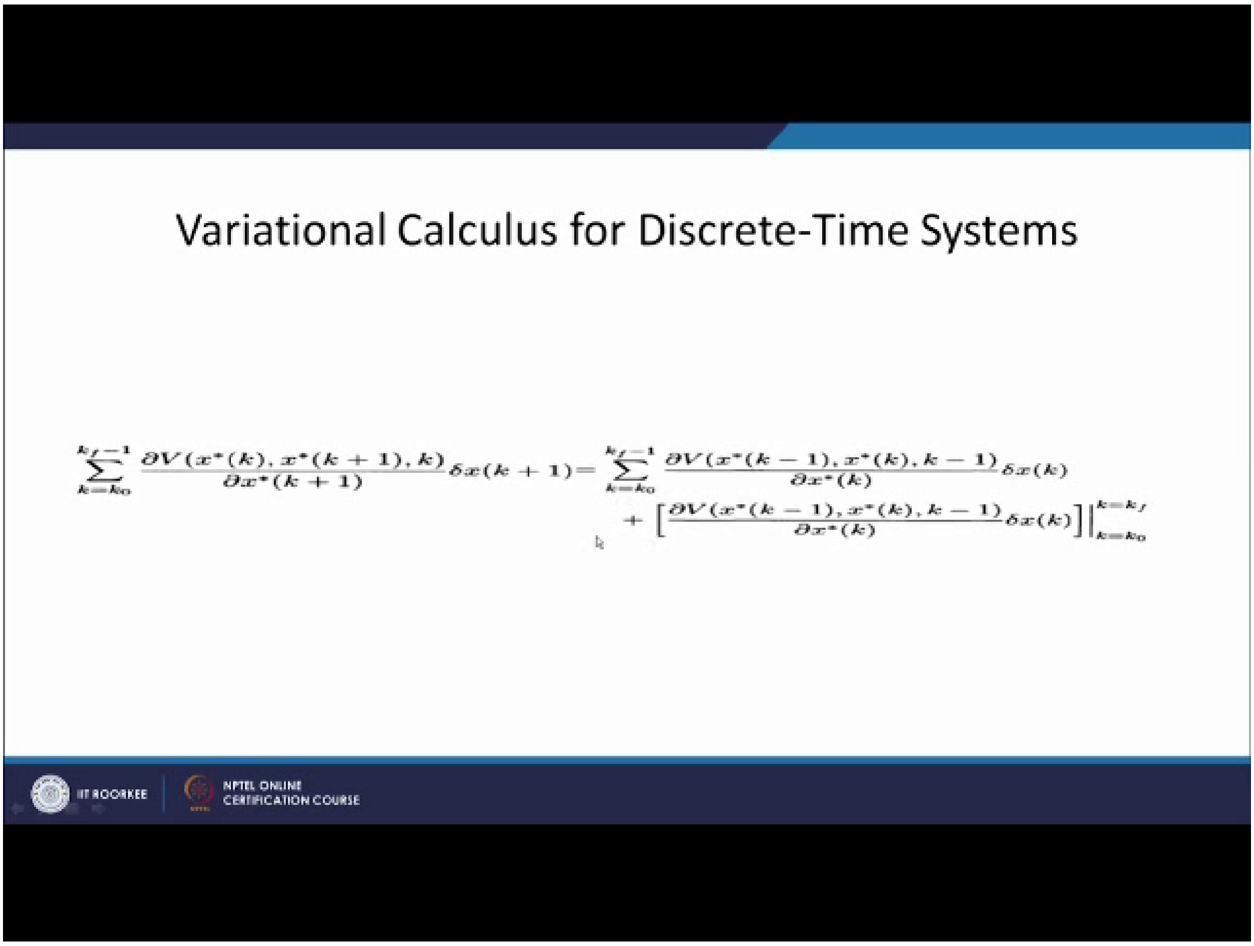

So, these last two term giving me this term and these another terms are giving me summation of this term. So, this del V by del x k we are writing as summation of del V x of k of k minus 1, x star of k, x k minus 1 by delta x of k delta x k. So, this first expression I am getting with these term except the last except these two terms, and these two terms I am combining together to get my second terms of the first variation as sigma of del V x star of k minus 1, x star of k, k minus 1 by delta x k delta x k plus this is my conditional this term is out of the summation, but evaluated at the value k equal to k 0 and k equal to k f.

So, what actually we have done for this second term we have converted this delta x k plus 1 in terms of the delta x k. So, this term we are expanding with the summation of this. So, this whole can be club in the summation with the first term as given at this point. So, this we will have the delta x k, this will have the delta x k, and these two term I can write as my boundary condition given as del V x star k minus 1, x star k, k minus 1 del x star k delta x k from k 0 to k f.

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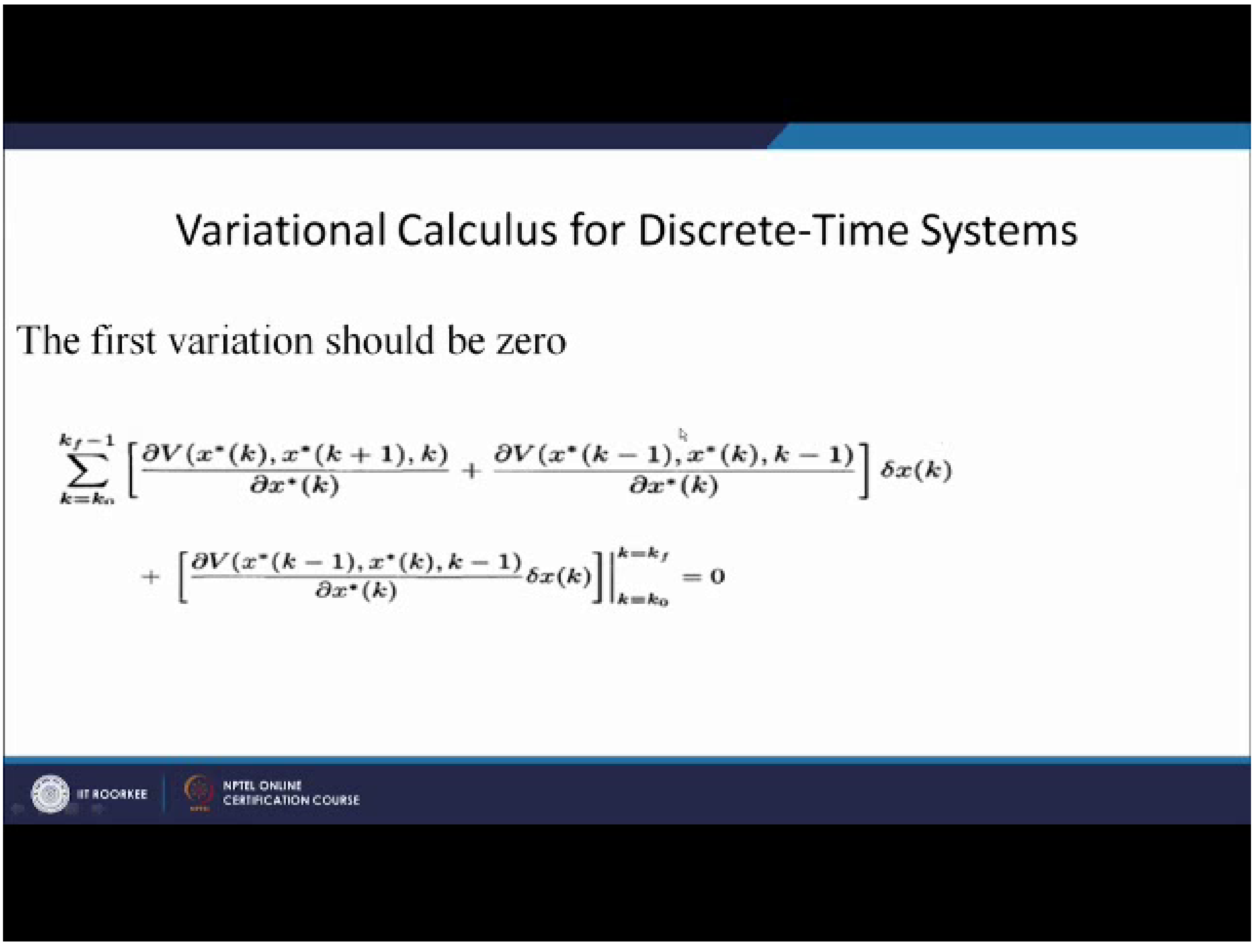

So, the overall I can write by clubbing these two my first variation will be summation of del V by del x k, del V by del x k with delta k, but this V is the function of x star k, x star k plus 1 k and this is the function of x star of k minus 1, x star of k, k minus 1.

So, like we got in our continuous time system del V by del x d by d t. So, because here we have the one step before as I am finding the like in the first term we have the del V at x star of k, this is the one step before x star of k minus 1. So, means analogous to we can consider it like we are taking the differentiation d by d T of del V by del x dot we have taken. So, here we have taken del V by del x k. So, these two term now can be clubbed with delta x k, and this is my last term last two terms which we have taken here this last two term we have clubbed together just to get the value of k equal to 0 to k equal to k f minus 1 these two point I have to evaluate.

So, these give me nothing but my boundary condition and this is nothing but the summation. So, this is my first variation this delta J by clubbing these term this first variation must be equal to 0 this is my fundamental theorem. So, if we find to find out the extreme of a functional its first variation will vanishes at the optimal point. So, I can equate it to 0 as my delta x k is arbitrary. So, delta x k we can take any value; so this whole equation to be 0 the coefficient of the delta x k must independently be 0 and this boundary condition second term will also be equal to 0.

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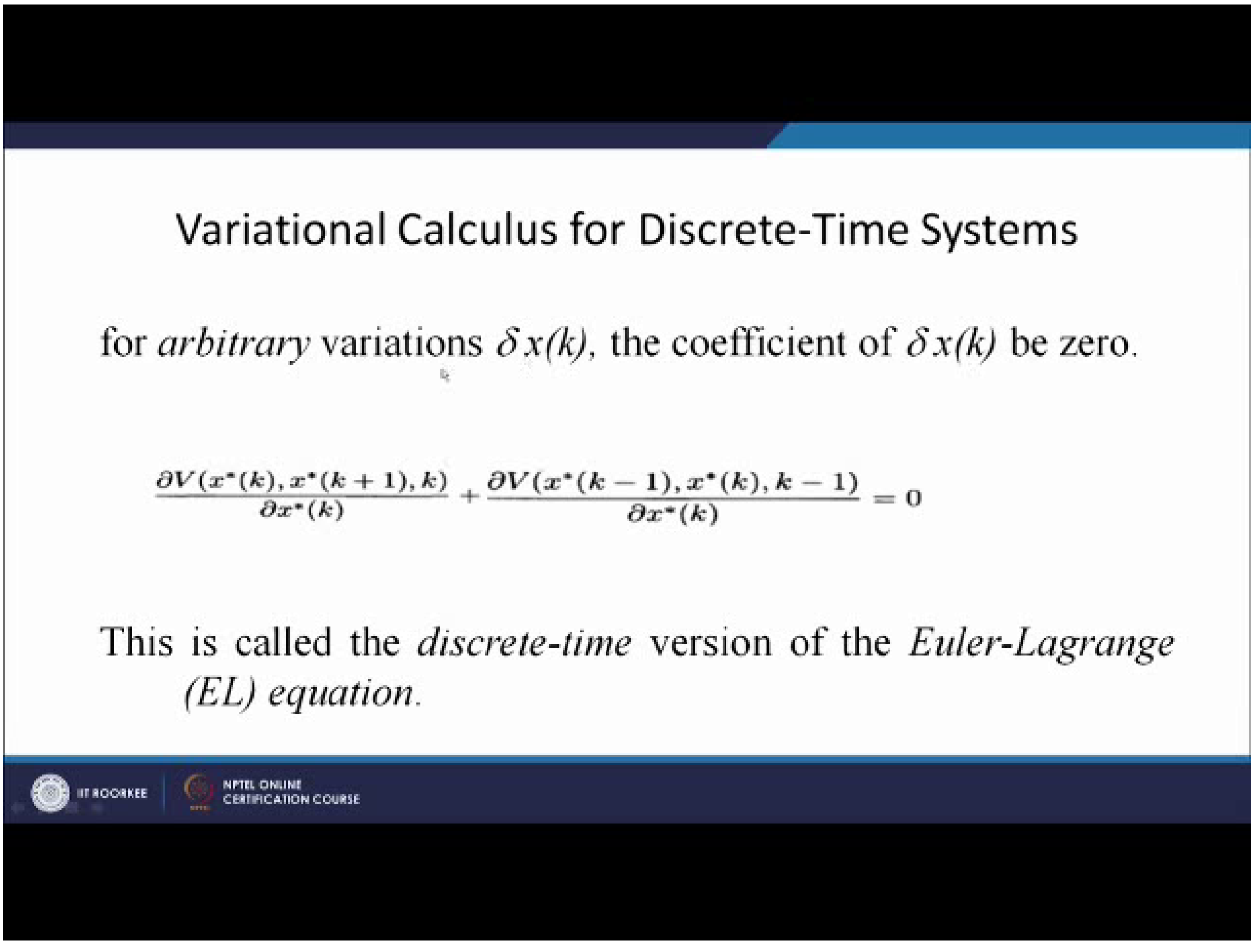

So, we can say the arbitrary variation delta x k coefficient of the delta x k be 0, and this is called my discrete time version of the EL equation.

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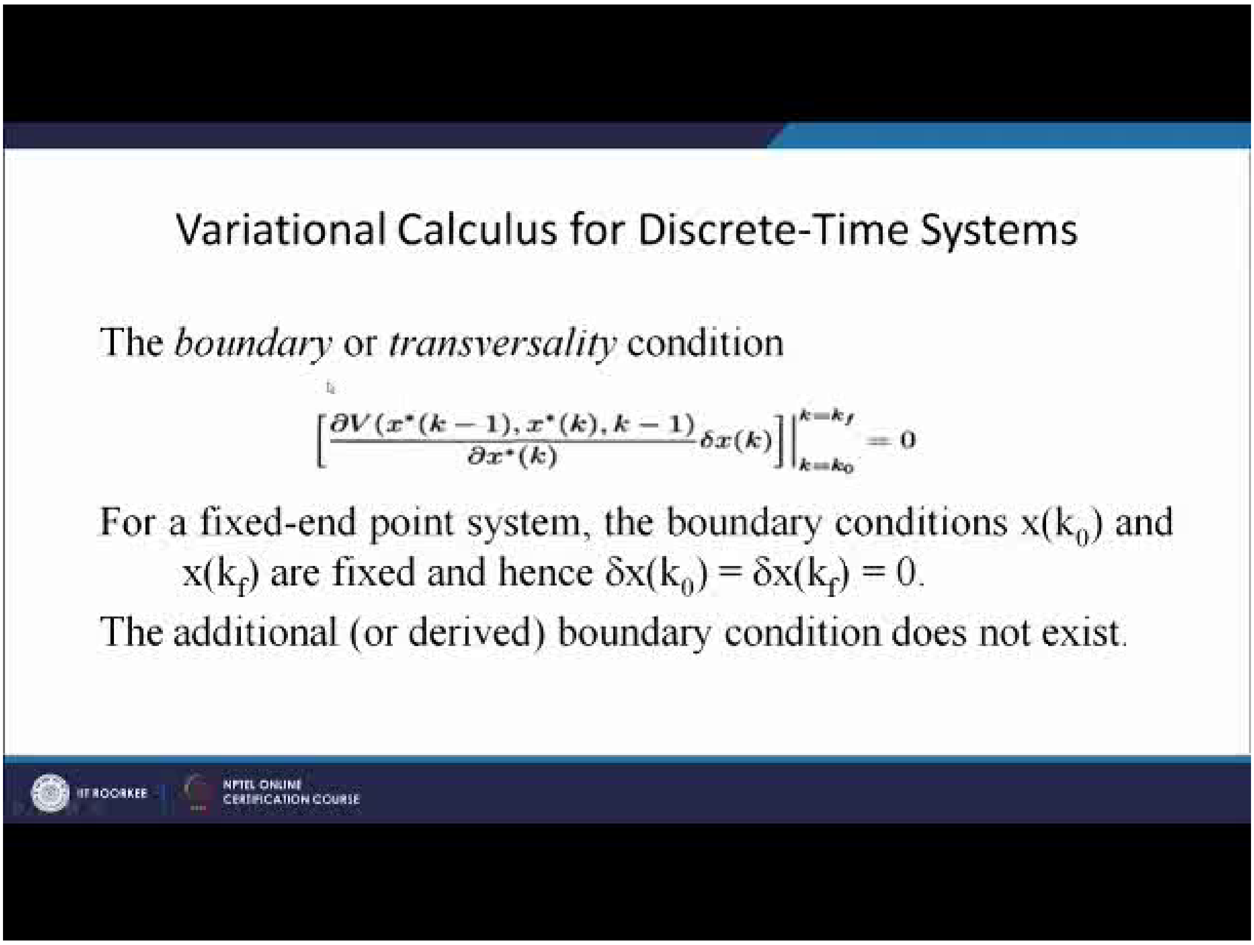

For the boundary condition my second term must be equal to 0, now depending upon my boundary condition.

The different condition can arise if I will the fixed end point condition, then k equal to k 0 and k equal to k f my both the points are fixed. So, in this particular case my delta x k at k equal to k 0 and k equal to k f will be 0. So, this equation will not arise in the system.

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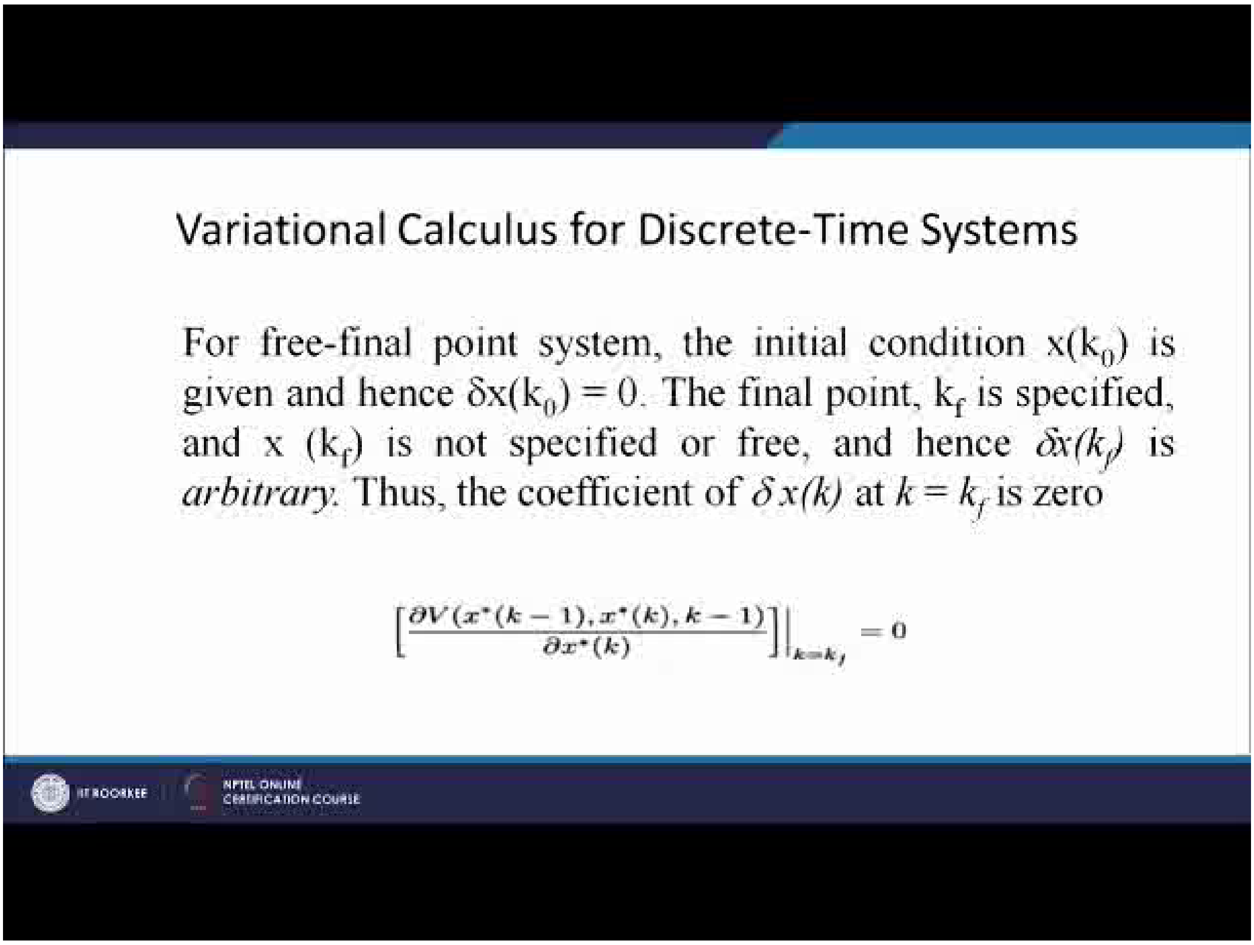

In normal case we will have we are basically given with the delta x 0 is fixed. So, delta x of k 0 will be 0, but normally delta sorry k f is free if k f is free then delta k f will not be 0. So, I can say the my boundary condition with delta x k equal to 0. So, that part will be zero, but my boundary condition will be del V x star k minus 1, x star of k, k minus 1 by delta x k at k equal to k f point then this condition will be 0.

So, similar like we have in a continuous time system, we have solved the extremization of the functional for continuous time case. So, we will land with a EL equation and the boundary condition similarly for the discrete time case we have the analogous EL equation in the discrete form as del V, which is a function of x k, x x k plus 1 k differentiated with respect to delta x k plus del V which is a function of x star k minus 1, x star k, k minus 1 differentiated with respect to delta x k the summation of this will be equal to 0 is my EL equation and at if initial condition is 0 then, and final sorry, initial condition is given and the final a state x k f is free, then this will be my boundary condition. So, I end this session here.

Thank you very much.