Optimal Control

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Lecture - 03
The Basic Variational Problem

Welcome friends to the session of our discussion. In the previous session we are discussing about the basic concept of the calculus of variation. So, we have defined the function; functional increment in a function, increment in a functional. Important in this is the variation of a functional which we will find, if we will, we can define the increment in J as J x t plus delta x t minus J x t.

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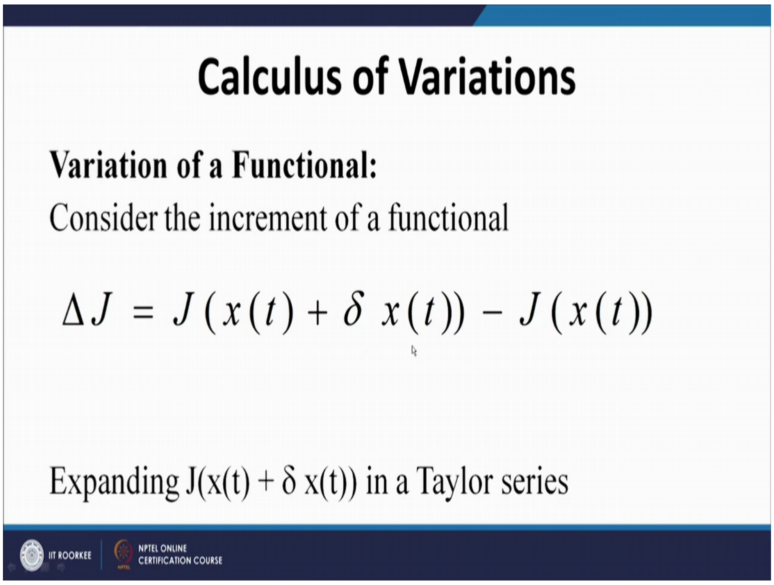

So, this means we are subtracting the value of J at x t point from the value of J which is at the variation of delta x t in the x t we will have.

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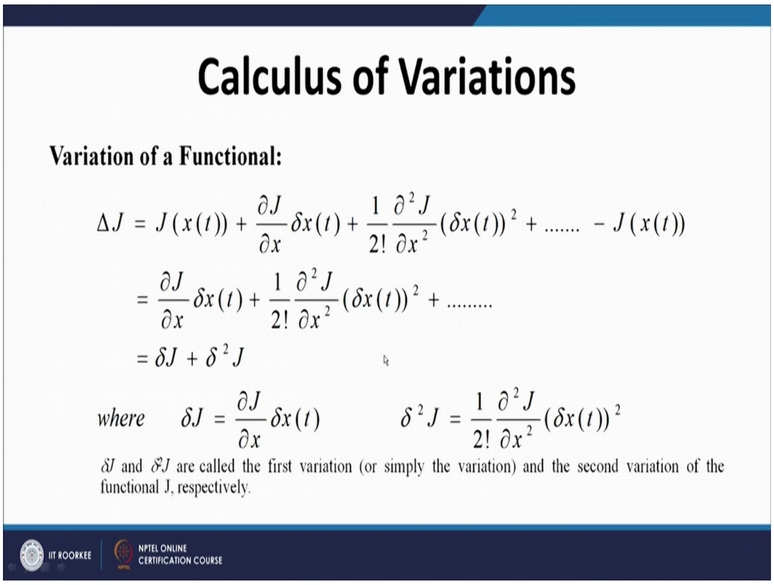

So, to explain this we expand it using the Taylor series and we find that the variation can be expressed as the first variation, second variation and the other higher order terms, where first variation is defined as the linear term which is del J by del x into delta x t. The second variation is 1 by factorial to del 2 J by del x square delta del x t square.

So, this variation will become important once we will find out the optimum values. So, we say, the optimum point my del J should be 0 that will see little bit later on.

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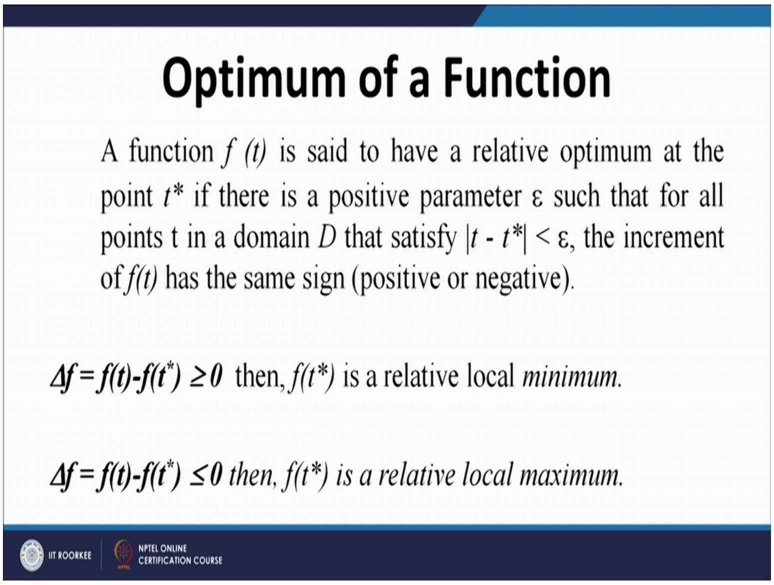

So, today we will start our discussion with the optimum of a function. Means we are trying to find out the optimal value if a function is given. By definition we can define, a function f t is said to have a relative optimum at a given point say t star if there is a positive parameter epsilon such that for all points t in a domain D that satisfy mod of t minus t star less than epsilon the increment of f t has the same sign, either positive or negative.

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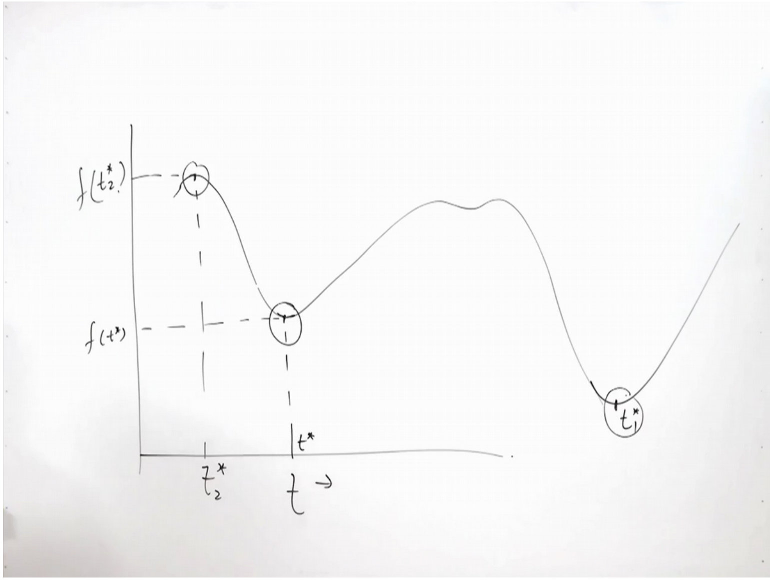

This means if we will have a function f t which can vary in a different ways. So, at a given point we will try to find out what actually will be my optimum point. So, if I will select suppose the lowest point which normally we said to be the optimum. So, in that case what we are saying my t is bounded between t n t star.

So, if I will take this access as a t, so I can say this is my t star point. So, a region we are selecting which is less than what is the region we have taken as the epsilon. So, the increment f t has the same sign. So, this means if I will get the increment of the f t which will be f t minus f t star that always will be positive if I have a local minima. So, this means all the points beyond this either on this side or in this side will have a value greater than f t star. So, if f t star is subtracted from any value here then we said my point is a local minima because local in the sense I have selected this for a very bounded region because if I will go further maybe my another point can appear here, this may be say t 1 star.

So, if I will send this figure and a reason I will select, so if I will explain this region, this region can cover the whole space. In that case I say this is my local minima, but my this point will be the global minima. The same analogy we can apply to define a local maxima and a local minima, but in this case f t minus f t star will be less than 0 because at any optimum point say t 2 star, if t 2 star is this. So, this is f of t 2 star. So, at this point if I will select any point that always give me the negative value for the increment. So, by this week I can define my local minima and in general I will say I will have the optimum point if where the df is vanishes. So, my necessary condition for optimum of a function is that the first differential vanishes means the df will be 0.

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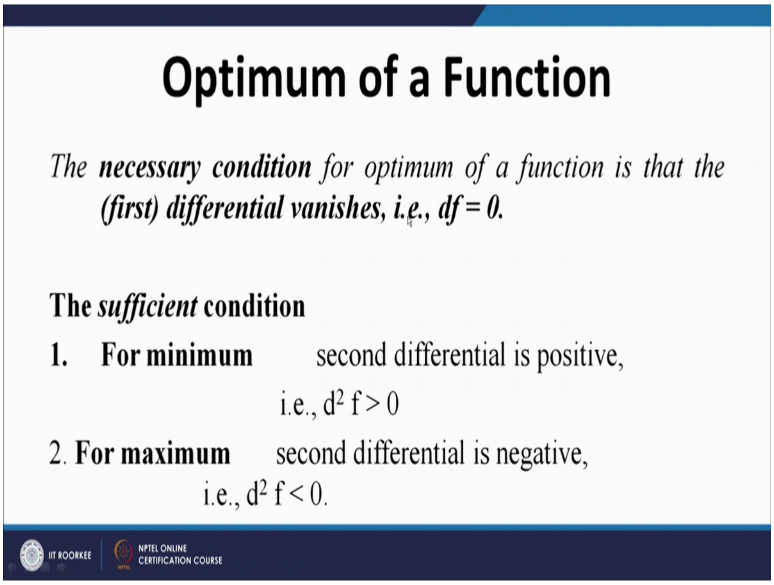

The second differential will be greater than 0 for minima and it should be less than 0 for maxima.

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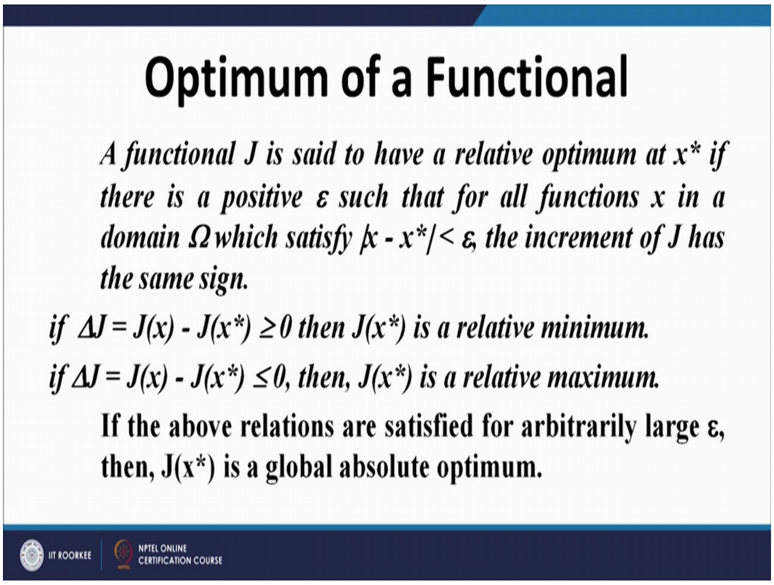

Similarly, we can define the optimum of a functional here functional is a function of. So, J is a function of a function x star. So, what we are saying let I will have the optimum value of the function is a x star. So, in this case there is a positive epsilon such that all function x in a domain which satisfy x minus x star less than epsilon the increment J as the same sign. So, this means if x star is my optimal point then if I will find the increment then J x minus J x star will be greater than 0 then J x of star is nothing, but a relative minimum point. And similarly if del J is less than 0 J x minus J x star is less than 0 then J x star is a relative maximum.

So, if the above relations are satisfied arbitrarily for large sigma then we say J x star is a global absolute optimum, optimum means if it is increment is greater than 0 we say is a minima, increment is less than 0 we say its maximum. And the increment of a functional we have seen that this is given as my first variation, second variation and the higher variation.

So, naturally if my first variation will set to 0 then the sign of del J is governed by the my second variation and at optimum point as my slope will goes 0. So, when the first variation will become 0 that will be my optimum point and the sign will be governed by the second variation.

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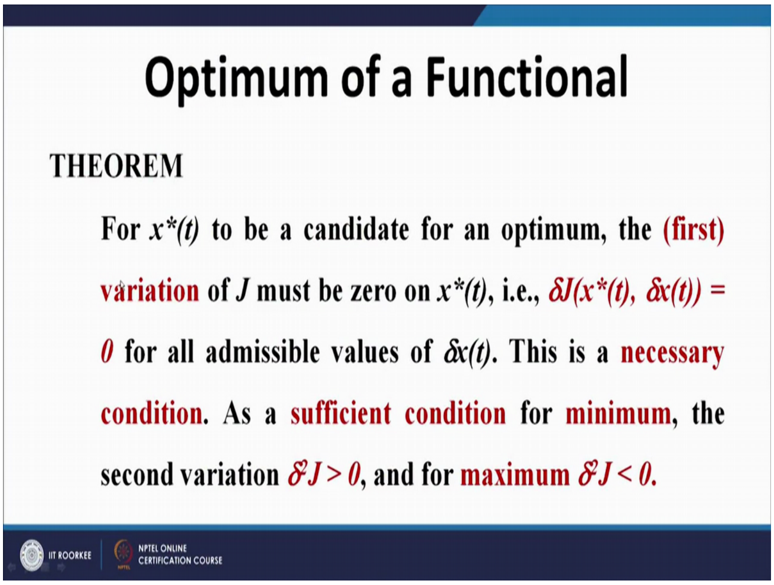

So, on this basis we will have, the fundamental theorem of calculus of variation which state that: For x star to be a candidate for an optimum, the first variation of J must be zero, this means we are saying J is equal to 0 if x star is an optimum point. So, this will be my necessary condition and has a sufficient condition for minimum, del to J must be greater than 0 and del to J must be less than 0 for maximum that we can see again from my increment point if my first variation is 0 then the sign of del J will be governed by the second variation. So, del J should be positive for minima, therefore, del to J should be positive. Del J should be negative for maxima, so del to J should be negative. So, that is my fundamental theorem of calculus of variation.

At optimum point my del J will be 0 which satisfy by necessary condition and del to J is greater than 0 if I have a minimum point and del to J is less than 0 if I will have the maximum point.

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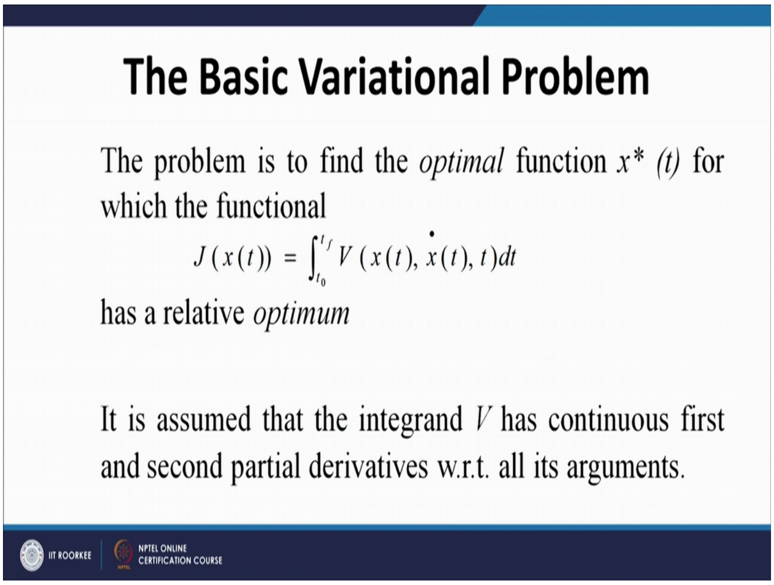

So, based on this we apply our knowledge of function functional increment and variations to determine what is the optimal value of a function x star which will give me the optimum value for J. So, my objective is the problem is to find the optimal function x star for which the functional J will have a relative optimum point.

So, what I have to do? Basically I have to find the increment, from increment we will find out the first and the second variation, we set the first variation to 0 for my necessary condition and this will lead to me a differential equation which we have to solve subjected to the boundary condition and then we will check whether my second variation is positive or negative, if it is positive this will be maxima if it is negative then this will be minima.

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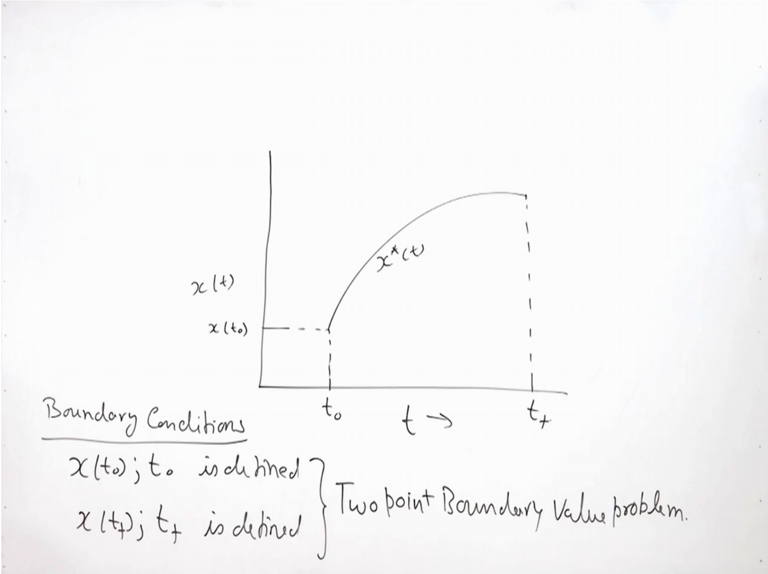

So, for the given J we assume let my x star, I already have say x star. So, let this represent my optimal trajectory x star of t. t 0 is my starting point, at t 0 let I will have the value as x t 0, this is my t f point. Now once we are solving this optimal problem we must be defined with the boundary condition what may be the boundary condition means? What is the value of my function x t at t 0 and t f, normally from control point of if we will see x t 0 and t 0 is defined, defined means this values are given from where we are starting and what is the my initial conditions.

But final condition depends upon the nature of the problem. The first problem we are taking in which x t f and t f is defined; this means we also know what is my t f and what is my x of t f. So, this is known as the two point boundary value problem. So, we are given with the initial point, we are given with the final point - t 0 t f both are given, x t 0 and x t f are also given, so my optimal trajectory starting from the t 0 and terminating at the t f.

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So, at final points I know where is my trajectory. Let us consider a variation of delta x t. So, I can say this is my x t which is not at the optimal point. So, we are considering let there exist a optimal trajectory x star t starting from the t 0 and terminating to the t f point we are considering a variation of delta x t in x star of t, my terminal points are bounded. So, my variation is also starting from the t 0 point and terminating to the t f point.

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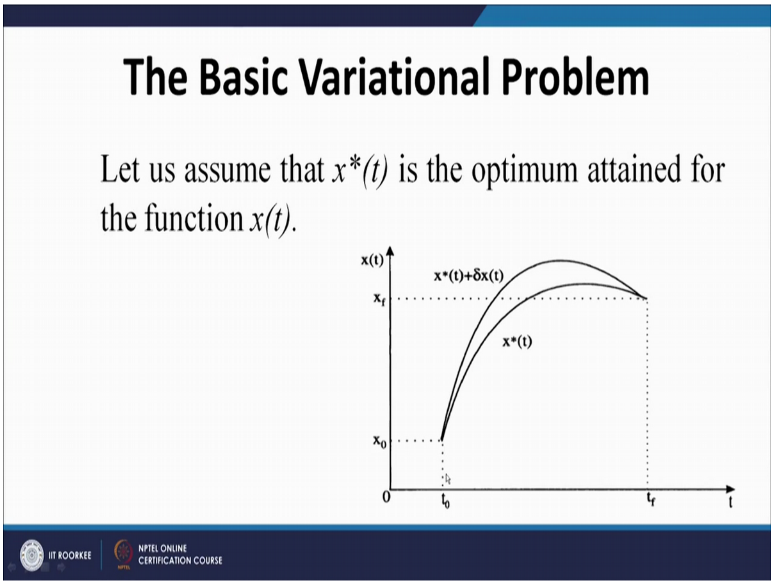

So, this is my condition I am given with the t 0 and x t 0 t f and x f which is the value at the t f point this is my optimal trajectory and this is the variation in the optimal trajectory. So, for a given value of the x, I can write J at the x star point and J at the x point.

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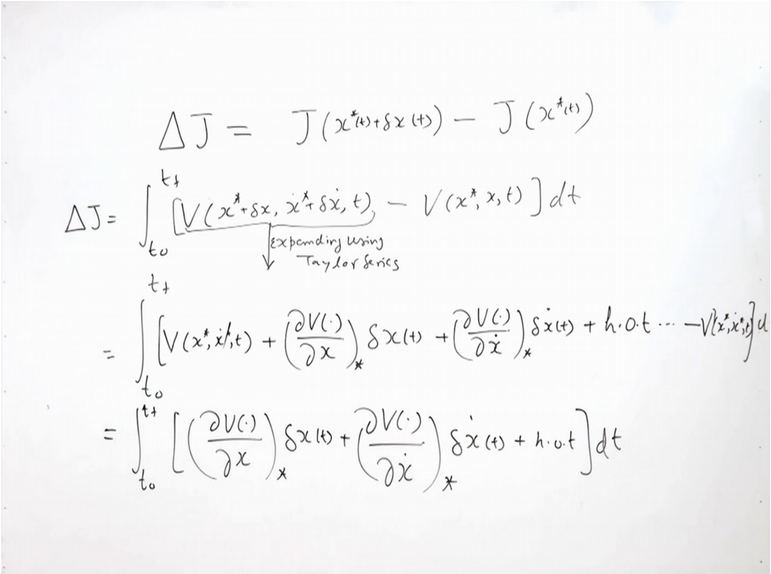

So, as my to start my problem my objective is to find out the increment in the J and by definition increment on the J is defined as J at J x star plus delta x t point minus J at x star of t point. This will give me the increment in J, so J x star t plus delta x t and as we have considered V to be a function of x dot and t.

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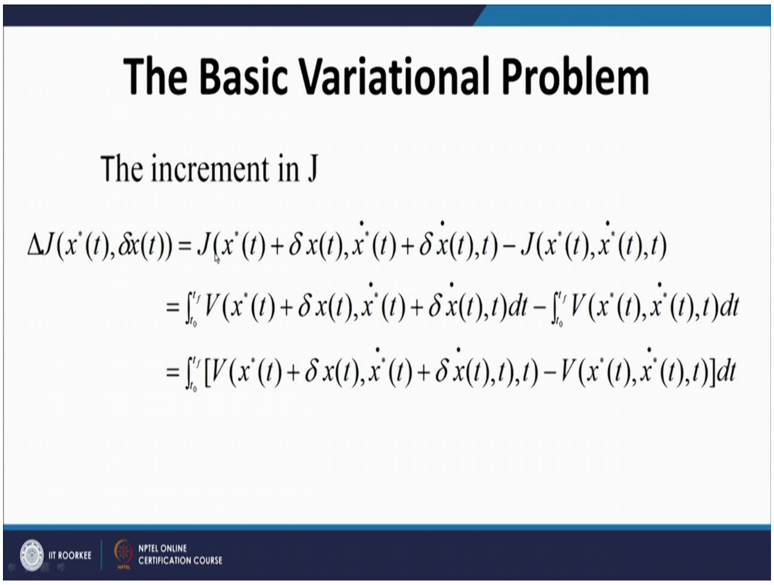

If I am giving a variation in x t, so there is a variation in the x dot t also. So, that is why we are saying x t plus delta x t x dot of t plus delta x dot of t t minus J of x star t x dot of t and t. So, that is we have considered a general from of the functional which is the function of x star t as well as the function of the x dot of t. And we know J is V of x star t x dot of star t and t. So, this is integral V at x plus delta x point minus V at x star of point, these 2 integral we can because there limit is similar t 0 t f we can combine together and we can write V x plus delta x x star plus delta sorry; x dot star plus delta x dot star sorry delta x dot t minus V x star of t x dot star of t and t, and these my objective is to find out what actually will be my increment.

So, now, this first term I can explain using the Taylor Series. So, we can write delta J as integral t 0 to t f V x star x dot star t. So, I am dropping out the t by writing here because t is repeated. So, I am simply writing as V x star x dot star sorry; this will be x star plus delta x, x dot star plus delta x dot and t minus V x star x and t d t. We are expanding the first term only. So, only the first term we are expanding using Taylor Series. So, what actually we will get - t 0 to t f. So, my first term will be V x star x dot star t, my second is the first variation del V by del x at my optimal point into delta x delta x of t plus del V by del x dot delta x dot of t plus higher order term and my last term is minus V x star x dot star t into d t. So, in this I can cancel out this term. So, I am left with t 0 to t f del V by del x at optimal point I can write delta x t plus del V by del x dot delta x dot of t plus higher order terms into d t. V already have been cancelled out. So, we are simply defining the del V by del x, del V by del x dot delta x dot 1 by factorial to my higher order term which is my the second order term plus the another higher order terms.

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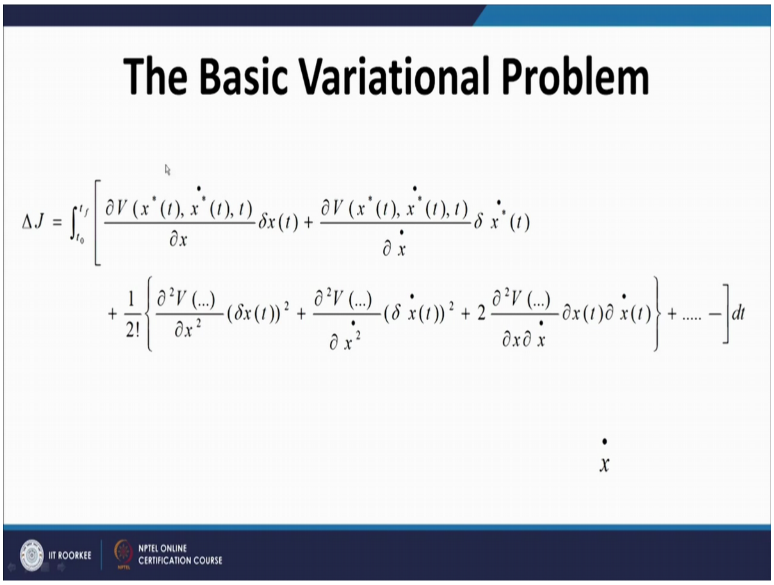

So, del J I can write in this form. The first 2 term are basically representing my first variation and this is representing my second variation.

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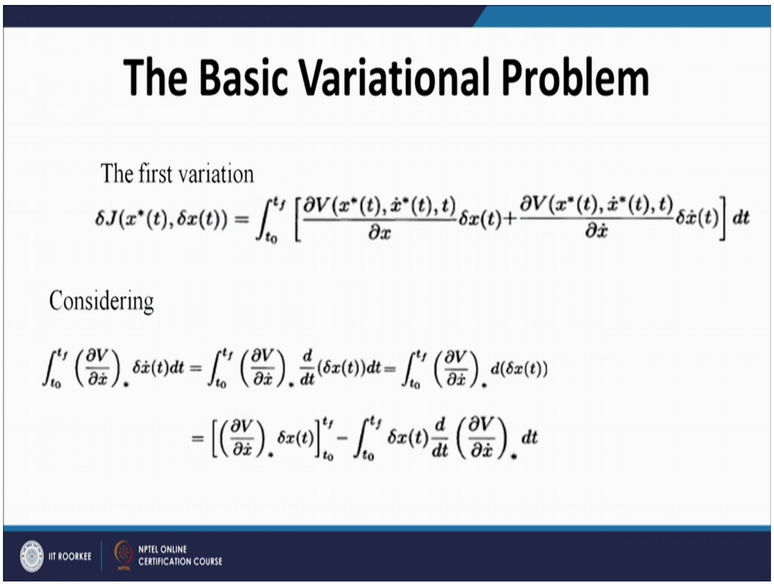

So, I can write my first variation is delta J equal to del V by del x delta x plus del V by del x dot delta x dot. So, the first variation will have the 2 terms - one in delta x t form, other is as the coefficient of delta x dot of t. So, in the next we will try to convert the second term which is in the delta x dot of t form in delta x t form and this we can do using the property of the integration and integrating it by parts.

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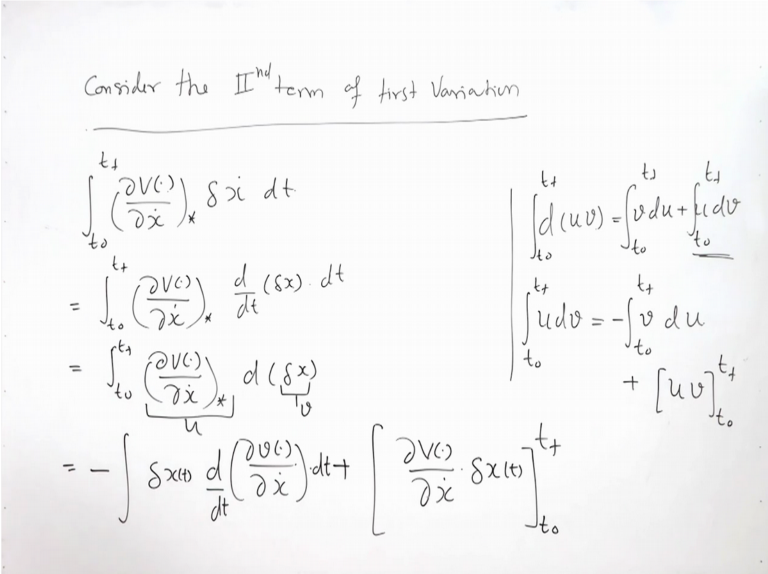

So, we are considering consider the second term of first variation and what is my second term? That is integral t 0 to t f del v, I am simply writing del V dot by del x dot at optimal point into delta x dot d t. So, we are considering only this term del V by del x dot delta x dot and this we are integrating by parts. So, we are you using say if I will write the d u v. So, this is nothing but v d u plus u d v. If I will integrate this with the given limit let us say t 0 to t f, t 0 to t f, t 0 to t f and write this term. So, I can write integral t 0 to t f u d v as minus integral t 0 to t f, v d u plus. So, if I will integrate this, what actually I am getting? I am getting simply u v, t 0 to t f means u and v we evaluated at t 0 and t f point. So, this if I will write as t 0 to t f v dot by del x dot and this I will write d by d t into delta x.

So, delta x what I am writing d by d t o delta x. So, this is giving me the delta x dot and this is multiplied with the d t. So, naturally if d t d t will be cancelled out. So, this is t 0 to t f del V by del x dot and this I will right the delta x. So, what I say? Let us say this is my u d of v and delta x is my v. So, what I can write from here? So, this is my u d v, u of d v is minus integral of v, v is nothing but my delta x, d u means d of del V dot by delta x dot. So, this is the v d u plus u d v and u is my del V dot by del x dot into delta x of t because this delta x is a function of time. This means this is also my delta x t because I have drop out the t, I am taking this t inside varying from t 0 to t f. So, this is my case.

So, if I will write this is d by d t and I multiply d t here. So, this term I can simply represent as del V by del x dot delta x t t 0 to t f which is my last term here and minus integral t 0 to t f delta x t d by d t del V by del x dot into d t. So, this second term I can expand by these 2 terms. And if I will substitute this second term in this form what actually I will get. So, you can see here this is del V by del x delta x, this is d by d t del V by del x dot delta x t.

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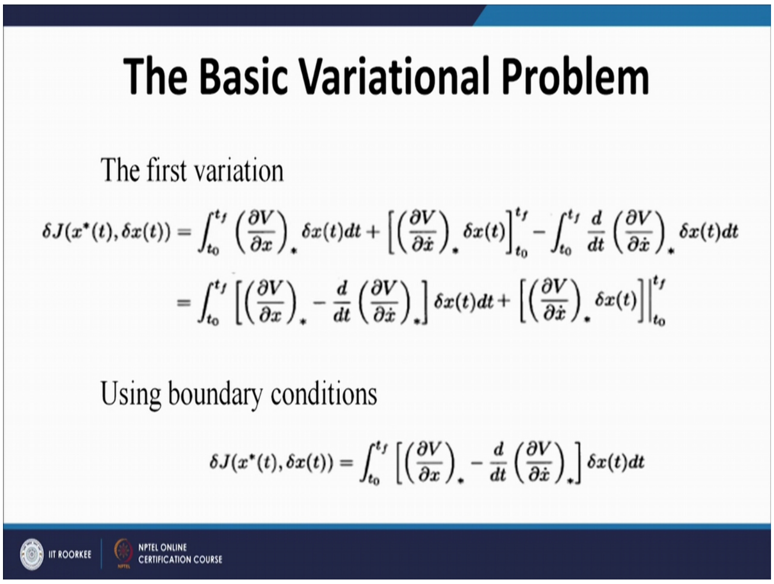

So, I can simply write this as del V by del x delta x d t this is my first term, this is boundary point minus d by. So, this whole represent my second term.

So, this 2 integral I will club together to write as del V by del x minus d by d t del V by del x dot multiplied with delta x t. Now this delta x t is a common factor plus del V by del x dot delta x t. Now see what the boundary conditions we have considered. So, in this, if we will see; so the 2 term here one is integral term, other is the term with the limits t 0 to t f. So, what is, if I will place the value of t 0 and t f here? So, this is this variation is at the t 0 point and delta x t f is the variation at the t f point. So, that is why this boundary condition. So, I can say the last term in this expression is giving me the boundary conditions while the first term giving me a differential equation in d by d t form.

So, this is a non-linear equation which I am getting and another term and getting as a boundary condition.

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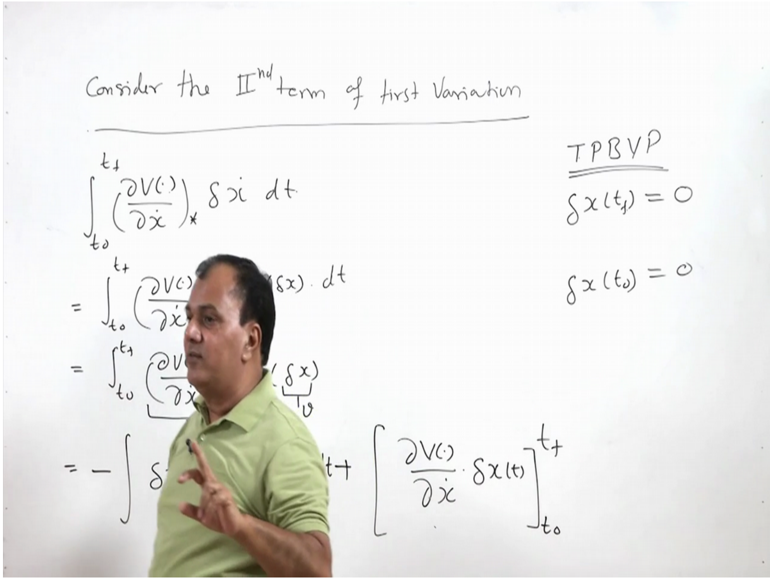

In our, this two point boundary value problem my initial and the final point is. So, two point boundary value problem TPBVP, two point boundary value problem if you are considering. So, this means I have the one term delta x of t f and another term is delta x of t 0 and as we have considered the case at t 0 my this value is fixed so there is no delta x of t 0 this will be 0 and a t f again my this point is fixed. So, this will be delta x of t f. So, this means I can set this value to 0, this value to 0, but only in the case of the two point boundary value problem. Later we will see the free end point problem in which we will make, we will keep the initial point fixed, but we will make the final time as well as the final state to be free.

So, in this case if we will consider the two point boundary value problem. So, my first variation is simply equal to t 0 to t f del V by del x minus d by d t del V by del x dot delta x t d t and what is my fundamental theorem to get the optimum value? My first variation must be equal to 0 at the optimum point.

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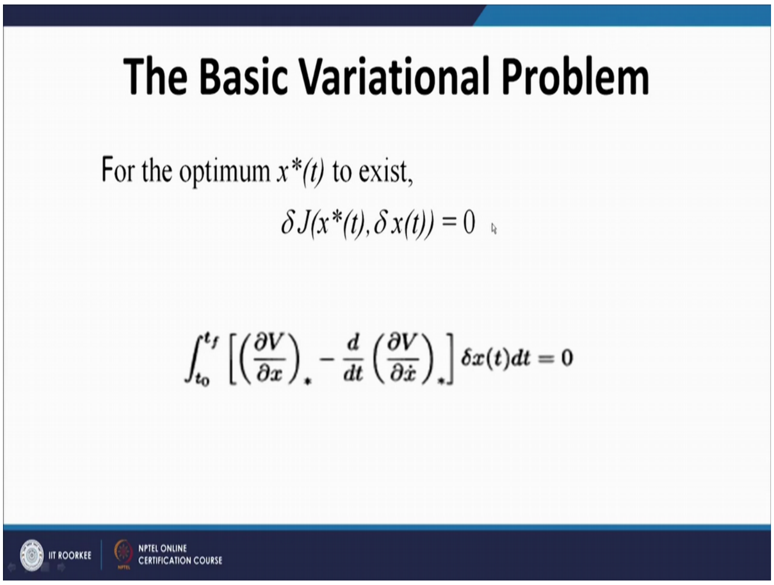

So, for optimum x is start t 2 adjust my delta J must be equal to 0, this means my t 0 to t f del V by del x minus d by d t del V by del x dot delta x t must be equal to 0. But this term is with the integral, now if you will see this whole rule of the 2 major parts one is in this bracket del V by del x minus d by d t del V by del x dot and the delta x t.

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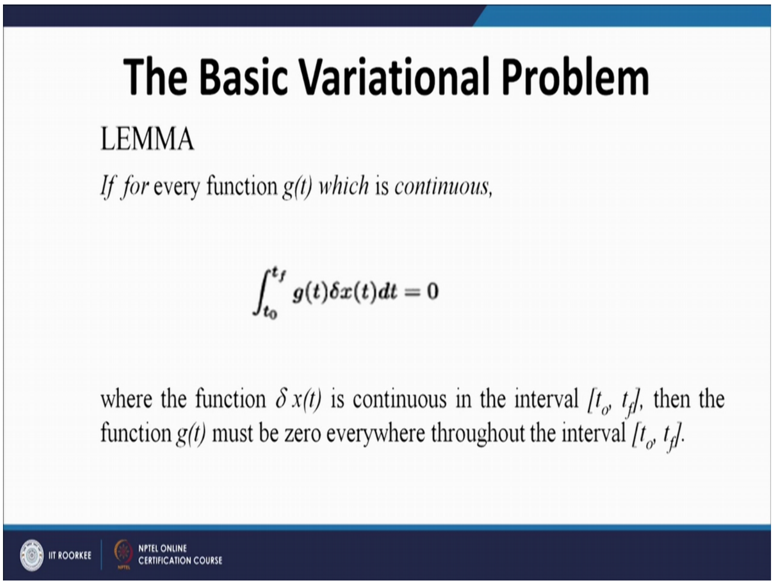

So, we use the fundamental lemma which is given as t 0 to t f g t delta x t d t equal to 0. Now in this case my delta x t is the variation which is arbitrary which normally cannot be 0. So, if delta x t cannot be 0 this means my g t value should be 0, this is called my fundamental lima for every function g t which is continuous given as t 0 to t f, g t delta x t d t equal to 0 where the function delta x t is continuous in integral then the g t must be 0 everywhere throughout the integral. So, throughout the interval t 0 to t f if delta x t is continuous.

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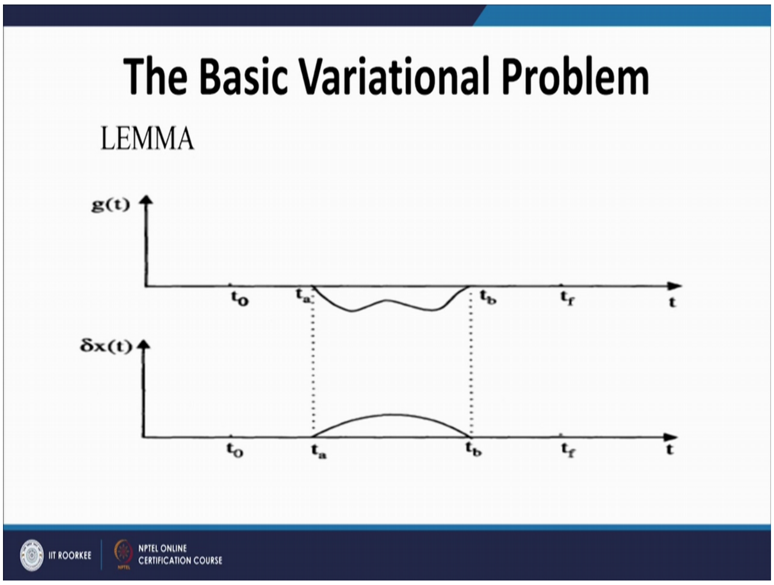

This we can prove your let in some interval delta g t is not 0 if delta g t is not 0, x t is changing d t is changing. So, during this interval what is my assumption? The product of g t and delta x p d t must be 0 that is not true. So, this is contradicting my assumption let g t delta x t is 0. So, this means if delta x t is arbitrary my g t will be 0 and in this case I can say this is my delta x t and if my g t is this. So, for this integral to be 0 deltas x t is arbitrary and continuous; my d t will be 0 this means my del V by d x minus d by d t del V by del x dot will be equal to 0.

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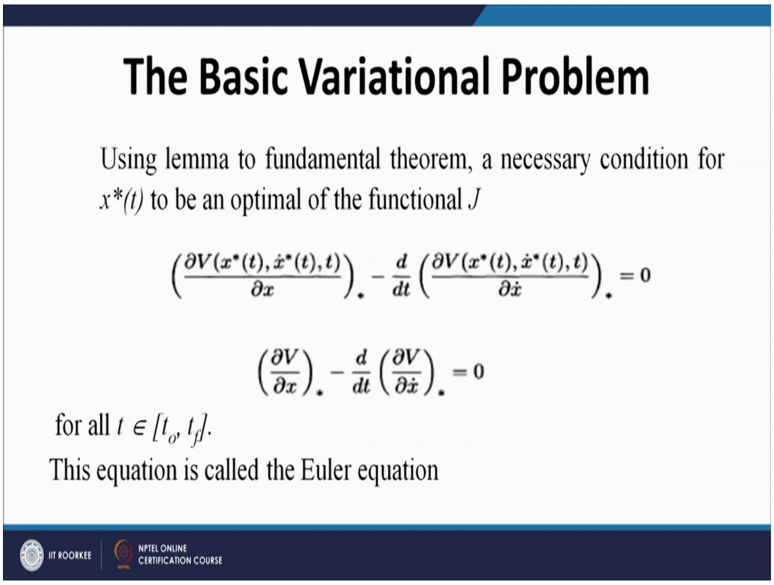

In simple form we can directly right has del V by del x at optimal point minus d by d t del V by del x dot at optimal point that must be equal to 0 for the complete interval from t 0 to t f and this is called my Euler equation.

So, I stop my discussion for this session at this point and more detail of this basic variational problem we will discuss in the next class. And we will also take up an example to see the application of this basic variational problem.

Thank you very much.