Optimal Control
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Lecture - 29
Linear Quadratic Tracking System

Welcome friends in today's session. So, today we will discuss about the linear quadratic tracking system. Till now we have discussed in the previous lecture a linear quadratic regulator problem. In regulator problem we try to bring our all the states to the origin. So, there is no change into the reference input reference input is considered to be the zero, is system is subjected to the disturbance and starting from an initial condition then system we try to write the system to the origin means at its operating point. In tracking system we will try to follow whatever be my reference input. So, this means the R will not be zero, and system is subjected to track how my R will change.

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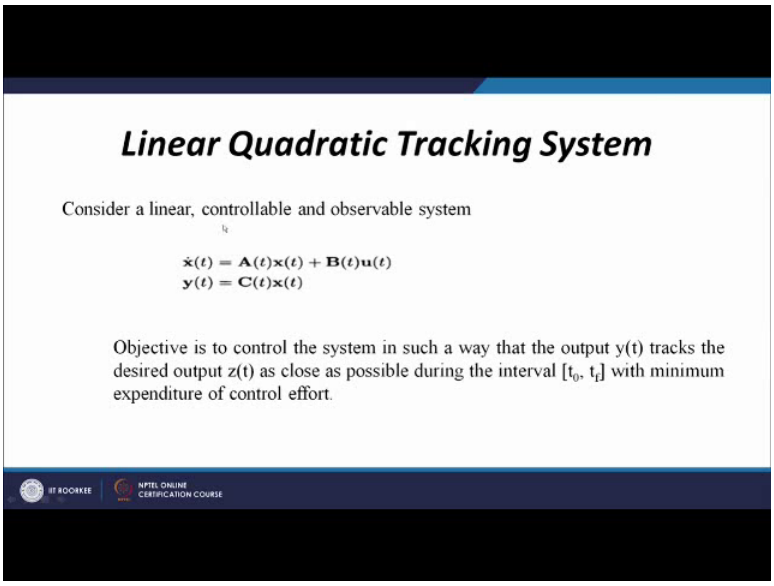

So, we can consider a observable system, this is the primary condition to design a quadratic regulator. So, my system should be controllable and observable.

So, we consider a system is x dot t equal to A x t plus B u t y equal to C x t. So, pair A B is controllable and the pair A C is observable. Objective is to control the system such a way that my output y t will track the desired output z t. So, and it is as close possible in a define interval of time with minimum expenditure of the control efforts. So, we can formulate a quadratic problem is a given plant x dot equal to A x plus B u y equal to C x A B is controllable A C is observable, we are trying to find the optimal control law which will track the desired output z t in a define interval of time with minimum expenditure of the control effort by minimizing a given performance index.

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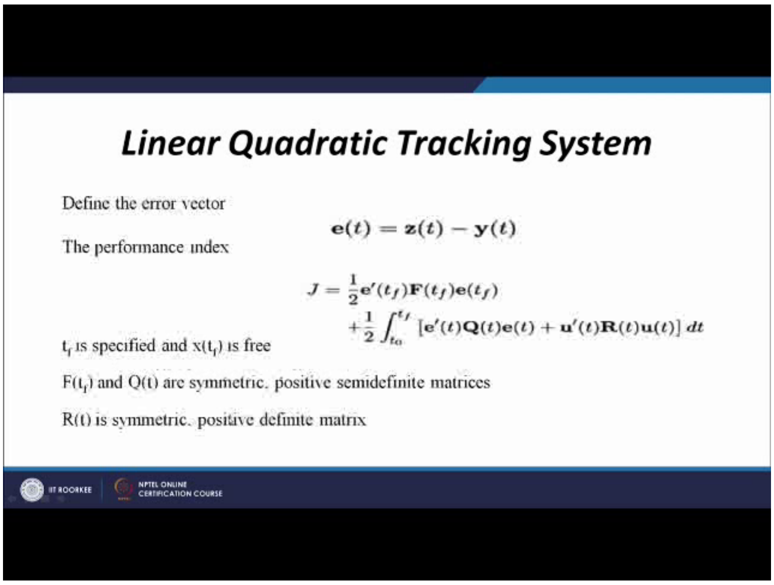

Now, next question here is what should be my performance index. So, as we said my desired is z t and actual output is y t. So, I can define my error is z t minus y t. So, I can define a performance index as half of e prime t f, F of t f e of t f which define my terminal cost because at terminal I want to make this error as small as possible, plus half of my quality performance index will be here e prime Q e plus u prime R u. Where e we are defining is the error. So, what we are trying to do? At terminal we are trying to minimize the error and also from a time interval from t 0 to t f we are trying to minimize the error as well as we are trying to minimize the control effort.

So, that my output y t is as close as possible with the z t if my error is minimum. So, these y my actual output will try to track my desired output. Here in defining this problem my F Q R will have the same format as we have 4y LQR problem. So, f t f and the Q t f are the symmetric positive semi definite matrix while R is a symmetric positive definite matrix. So, my objective of the problem is for a given plant I have to find the optimal control law which will minimize the performance index J. So, ultimately we are trying to minimize the error e and the control effort u with the terminal error is e prime t f which is. So, my t f is a specified in this case and x t f is free.

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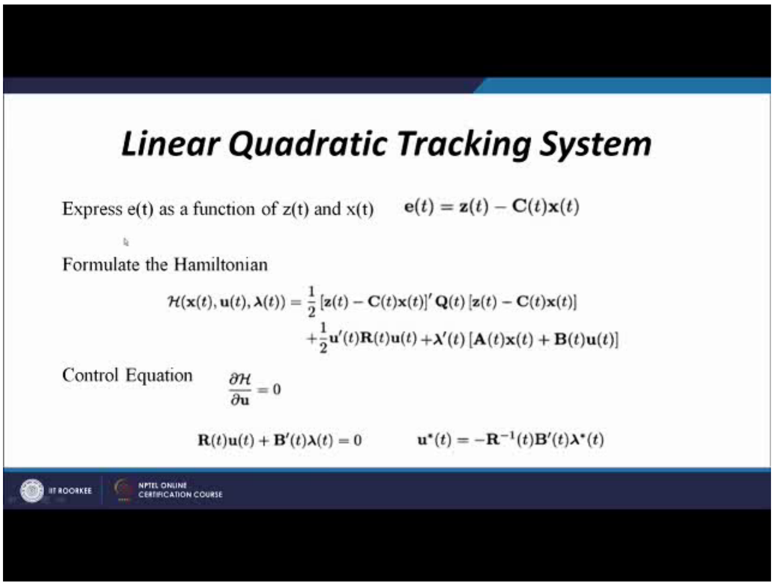

So, for this system as we are defining the e has z minus y and y is nothing, but my c x. So, I can define e as z minus C t x t. So, this problem of optimization we solve using the Hamiltonian approach and what is my Hamiltonian as we know.

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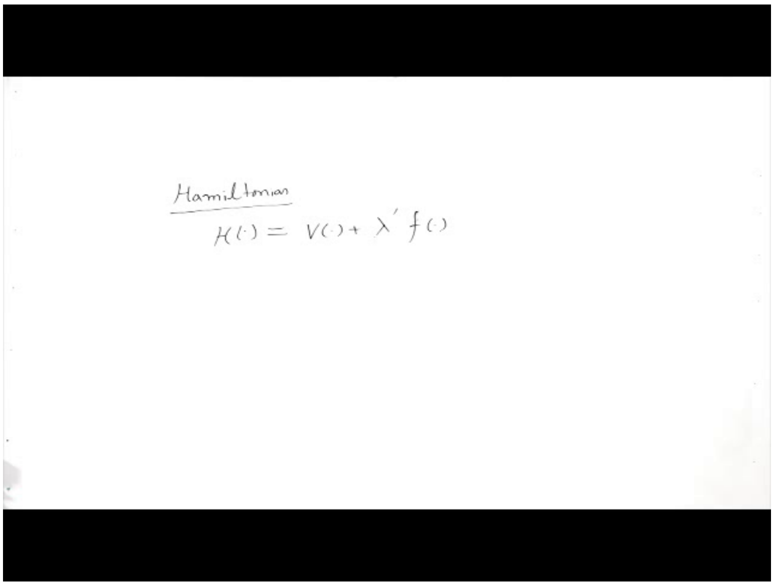

So, define my H as V plus lambda prime f what is my V? So, V is e prime Q e plus u prime R u this is my V and f is A x plus B u. So, I am defining my Hamiltonian is e prime Q e and e is replaced by z minus c t. So, this is z t minus c t x t prime, Q t z t minus C t x t plus half of u prime R u s h plus lambda prime A t x t plus B t u t. So, this give me the Hamiltonian and with the help of the Hamiltonian I can write what is my control equation what is my state equation what is my co state equation.

So, in this case I will take my del h by del u equal to 0, if I will differentiate this h with respect to u. So, I can see in the first part half of e prime Q e does not depends on the u. So, my u will appear only in the half of u prime R u and differentiation of this with respect to u give me the R u and lambda prime A x t you will again independent of u and lambda prime B u if I will differentiate this matrix equation I get the B prime lambda. So, I get the same equation as am getting in the LQR my R t is sorry my u is minus R inverse B prime lambda. So, nothing, but I am writing this equation in terms of the u. So, this is R inverse B prime lambda t. So, are exp sorry expression for the u is similar as I have in LQR problem. So, this is my control equation next we will write down the state equation a state equation is x dot t equal to del H by del lambda.

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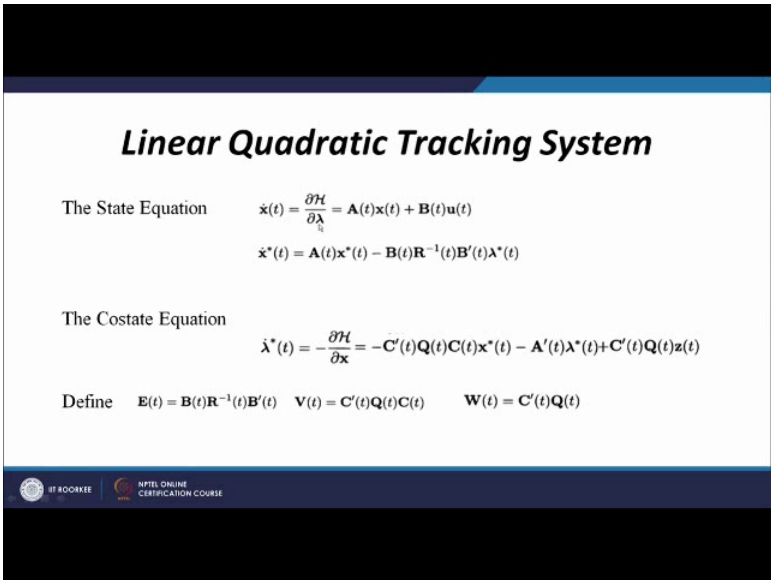

This means del H by del lambda I am differentiating this h with respect to lambda and I can see in this whole the last term is dependent on the lambda. So, if I will differentiate this I will get nothing, but A x plus B u. So, what actually we are getting x dot t equal to A t x t plus B t u t and u we already have drive there B R inverse B prime lambda is nothing, but my u. So, I can say my this state will be my this state equation is nothing, but giving me the optimal state if I am using the optimal value of the u. Next we will write the co state equation which is lambda dot t as minus del H by del x means I am differentiating this h with respect to x with negative sin. So, we are the x will involve z t is independent. So, my x involve with the C t x t and here C t x t. So, this will be give me the a squared term. So, what I am getting C prime Q c x t and A prime lambda t plus C prime Q z.

So, C prime Q z I will get from if I will expand this. So, my one term will be in the x prime x which is giving me the C prime Q c with negative sin if I am expanding this my another term will be in terms of the x z which is giving me C prime Q z and if I will multiply this A prime sorry lambda prime A x t differentiate this. So, this giving me nothing, but the A prime lambda, the co state equation is C prime Q C x, A prime lambda t plus C prime Q z.

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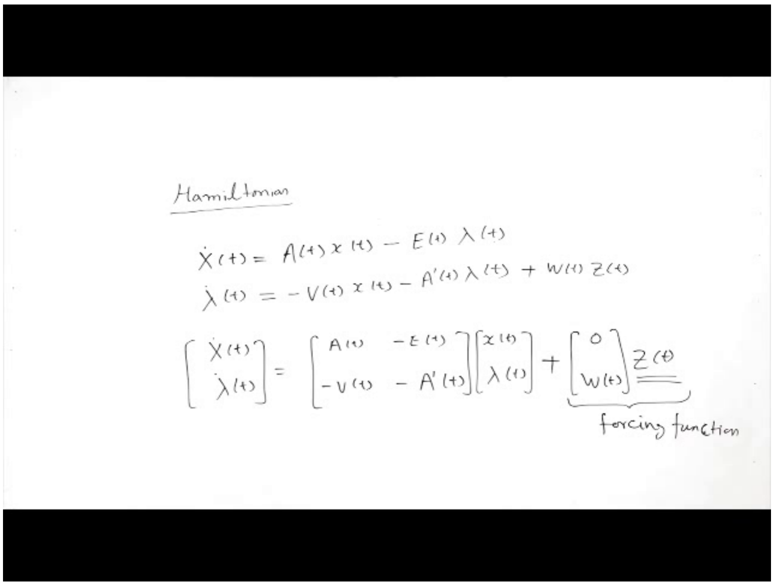

So, I can see in the co state equation my z term is appearing. So, here we consider E as B prime R sorry B R inverse B prime, this means B R inverse B prime I am writing as a E. So, my A state equation I can write in the form of x, X dot t as A t x t minus E t lambda t. So, this minus B R inverse B prime this I am writing as the E.

So, this is my x dot a x minus e lambda this I am writing this and co state equation lambda dot t I am writing as minus C prime Q c, and C prime Q c we are writing is V t. So, I am writing this as minus V t x t another term I am assuming here as C prime t Q s w this I am write as minus A prime lambda, A prime t lambda t plus w t, z t where W is we have considered as C prime Q. So, E is B R inverse B prime, V is C prime Q c and W is C prime Q. So, I am writing a state equation is A x minus e lambda and lambda dot s minus v x minus A prime lambda plus w z.

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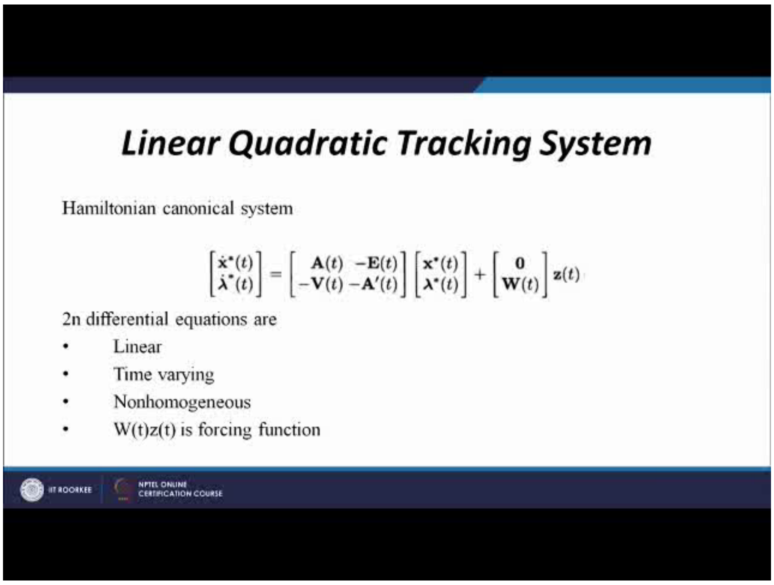

So, if I will write a state and the co state equation in terms of the Hamiltonian canonical form. So, naturally my system will become X dot t is A t minus E t minus V t minus A prime t and this will depend on your x and lambda; plus the another term which is appearing here is w t Z t. So, if we will compare this. So, I am getting this extra term and naturally my this extra term will be nothing, but the forcing function because z we have taken as my reference or the desired input. So, Z t will nothing, but will be my desired input and what we will have? For x we will have the n number of the equation lambda dot again giving me the n number of the differential equations. So, I have the total number of the two and differential equations, and these differential equations are linear in nature time varying non homogeneous and this last term is nothing, but representing my forcing function.

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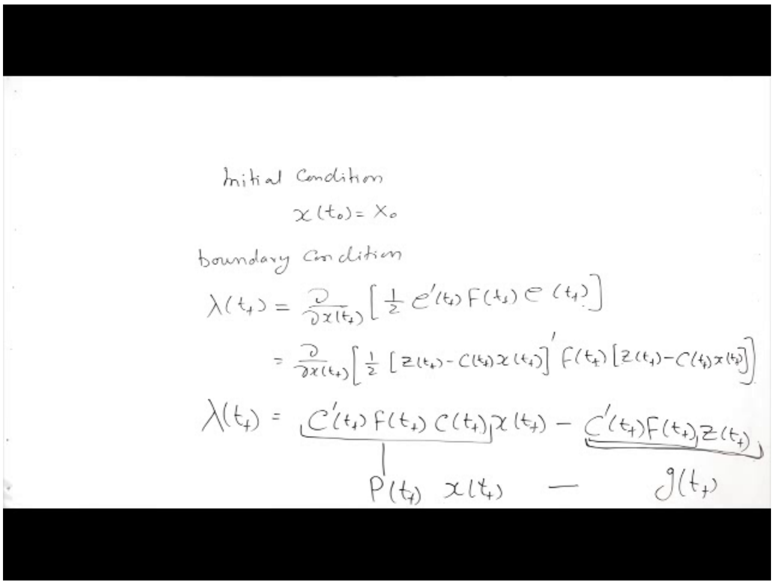

So, with these equations my I am given with the initial condition is x t 0 let us say x 0 and to solve this I also have the another boundary condition must be known which can find out from the terminal condition. So, my boundary condition will be lambda t f s with my terminal cost is half of e prime f of t f, e of t f and e is nothing, but my z minus C x C x of t f prime F of t f and this is z t f minus C t f x of t f. So, I have to differentiate this whole with respect to x of t f if we will do. So, so we will get C prime f of t f, C of t f x of t f minus C prime t f, F of t f Z of t f. So, this if we are again expanding this. So, my z t f term will be independent of x of t f. So, this will not appear C x prime x prime C prime F c.

So, C prime F c a square of x this is giving me the x t f and once I am expanding the term with c x with Z of t f C x with Z of t f I will get C prime t f F of t f Z of t f . So, now, if we will see this is giving me nothing, but the relation between the co state vector which is the lambda and x, but the additional term here we are appearing is C prime f of z. So, by this intuition what we can say we can consider by x as P x of t f minus z t. So, this means I have this is representing a matrix at the t f point.

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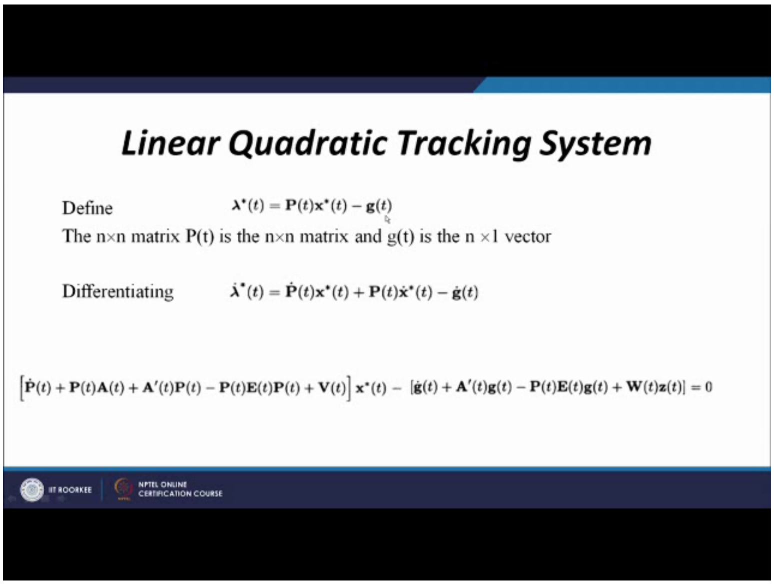

And if you will see this is representing another matrix at the t f point. So, what we are saying? We are considering and if multiplied with the sorry if multiplied with this Z t f. So, this whole term will be a vector. So, we are considering this as my P t f as per this and this I am considering the whole at g of t f.

So, this P of t f x of t f minus this g, so this we are considering as my transformation from lambda to x.

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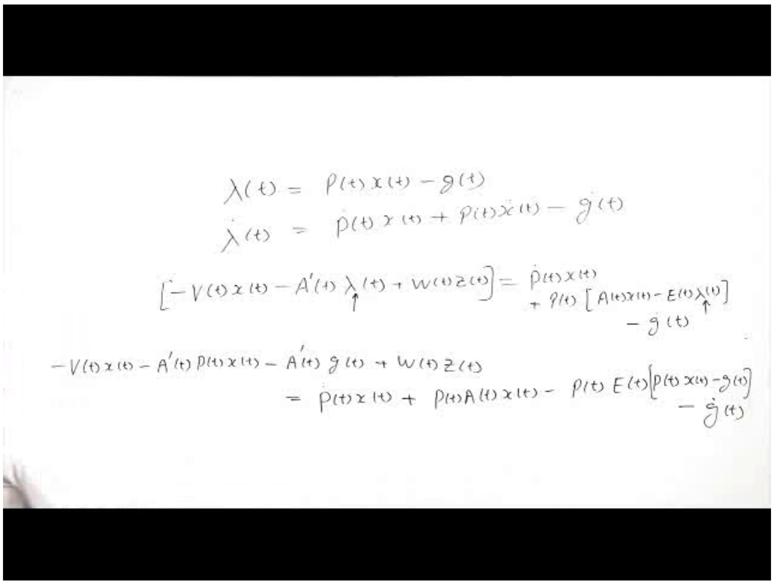

So, lambda I can consider I can define this transformation as lambda t equal to P t x t minus g t. So, we have considered lambda t as P t x t minus this g of t. If I will differentiate this lambda dot t, P and x both are time varying. So, this is the P dot t, x t plus P t x dot t minus g dot of t we get this expression and see from my Hamiltonian system I substitute the value of lambda dot t as minus V t x t minus A prime lambda t. So, lambda dot t I am replacing by minus V t. So, this I am replacing as minus V t x t minus A prime t lambda t. So, this is nothing, but my lambda V t x t minus A prime lambda t plus w t z t plus w t z t.

So, this is my left hand side while in the right hand side what we have. P dot t x t I write it same plus P t in place of x dot t by A t x t minus E t lambda t that part 0 minus g dot of t. So, we keep in this equation lambda dot t as P dot x t plus P x dot t minus g t we are writing lambda dot and x dot and if we will see lambda t as P t x t minus g t. So, in this equation I will replace this lambda and this lambda I substitute as P t x t minus g t. So, my complete equation here will be minus V t x t minus A prime t. Now P t x t P t x t minus A prime t, g t plus w t z t. So, this is my left hand side P dot t sorry I will return to this later on. So, in this equation this lambda t is replaced by P t x t minus g t.

So, this is my left hand side right hand side is P dot of t, x t plus P t a t x t minus P t E t and in place of lambda t we are writing P t x t minus g t minus g dot of t. So, in this equation we will have the coefficients in terms of the x t and the one coefficients in terms of the g. So, if I will arrange this equation I can write this P dot t plus P t a t, P dot plus P t a t minus P t e t P t, minus P t e t P t with x of t by expanding this plus V t x t this we are getting from this side minus g dot of t plus A prime t g t minus P t E t g t plus W t z t equal to 0. So, by arranging this whole equation we will get the overall equation is one with the coefficient of the x t and other g t minus A prime t g t minus P t e t z t plus W t z t equal to 0.

So, P t to satisfy the matrix differential Riccati equation, so if this overall equation is equal to 0, x t is my arbitrary. So, it can have any value. So, if x t is arbitrary. So, the coefficient of the x t will be individually 0, and the another term that this whole term must also be equal to 0.

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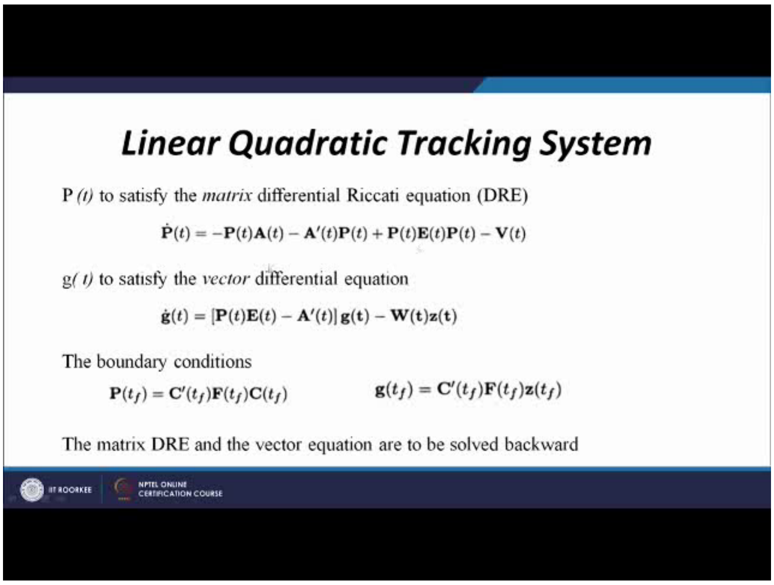

So, this means if I will write this as a 0. So, my P t as minus P t A t, A prime t P t plus P t e t P t minus V t. So, this equation will satisfy nothing, but has the coefficient of the x t equal to 0. So,

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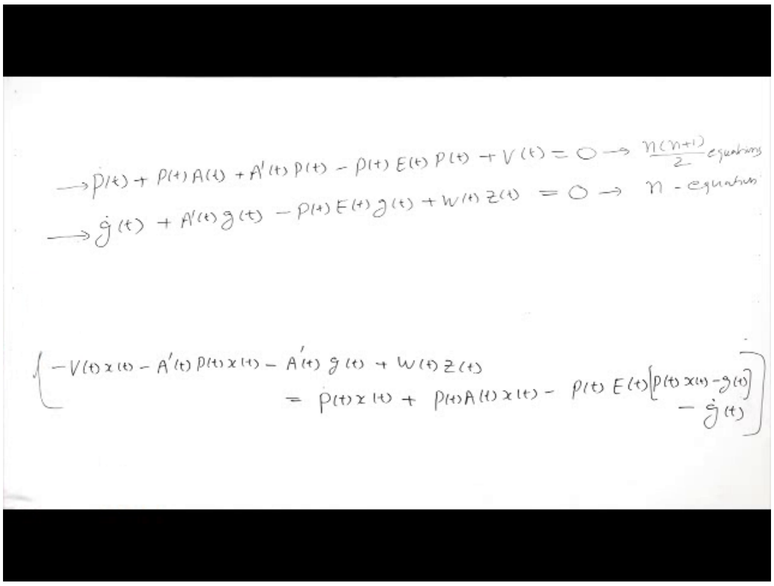

So, my coefficient of the x t is P dot t plus P t a t plus A prime t, P t minus P t E t plus V t. So, what we are saying? My this whole coefficient this must be 0 as my x t is arbitrary similarly g dot of t plus A prime g this is my plus A prime g, minus P e g minus P E g t plus w t z t plus w t z t this must be equal to zero.

So, we got these two equations to be 0, my this term will be 0 this term will be 0 and if I will arrange these equations. So, I get the first equation is P dot equal to this, and this is nothing, but my matrix differential Riccati equation for quadratic regulator system, and g t to satisfy my another vector equation which is P E minus A prime g t minus W t z t. So, my first equation is the matrix differential Riccati equation and the second equation is nothing, but a vector differential equation means I have the n number of g vectors and how many equations are here because P is a symmetric positive definite matrix.

So, this is n into n plus 1 by 2. So, my total number of the equations, this will give me n into n plus 1 by 2 equations and this give me n equations. So, this n into n plus 1 by 2 plus n equations I have to solve and what was my conditions? As we have seen their my P t f is C prime t f F of t f C of t f and g of t f is C prime t f F of t f z of t f z is my reference input which is known to me f is known to me C prime t f is known to me. So, the terminal condition g t f this whole vector is known to me similarly P t f c is known f is known. So, this whole terminal condition is known. So, n into n plus 1 by 2 plus n equation we have to solve with terminal condition as P t f given for the first equation g t f given for the second equation and naturally if the t f point is given. So, we have to solve this equation in backward.

So, the backward solution for this equation we can have, and by solving this we can find out the P and g now see what was my u?

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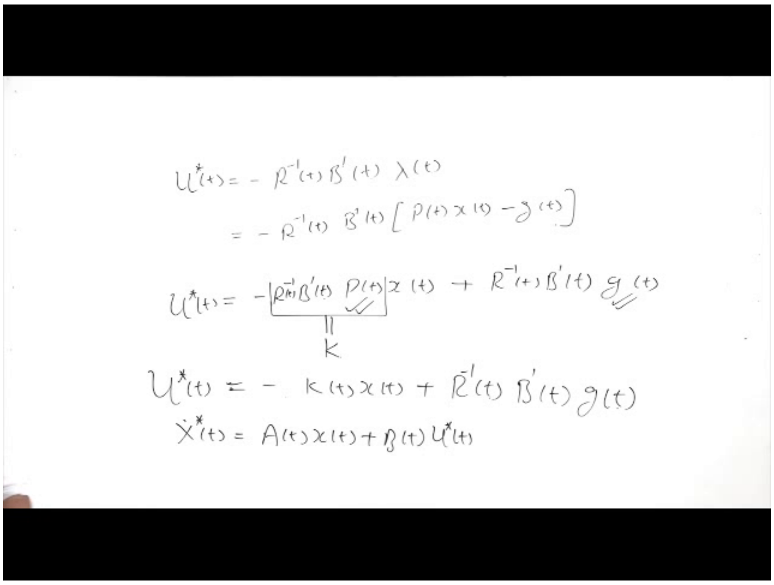

So, we have taken the u t has minus R inverse B prime lambda t, x t and lambda t what we have taken? This is my sorry we do not have x t in this. So, my u t is R inverse t B prime lambda and lambda t is nothing, but my P t x t minus g t. So, u will become minus R inverse B prime P t, x t plus R inverse B prime g t. So, to obtain the optimal value of the u, I must know what is my P and what is my g, this P and g we are calculating by solving the matrix differential Riccati equation with terminal condition is P t f equal to C prime t f F of t f, C of t f and g we are finding by the from the solution of the g dot t.

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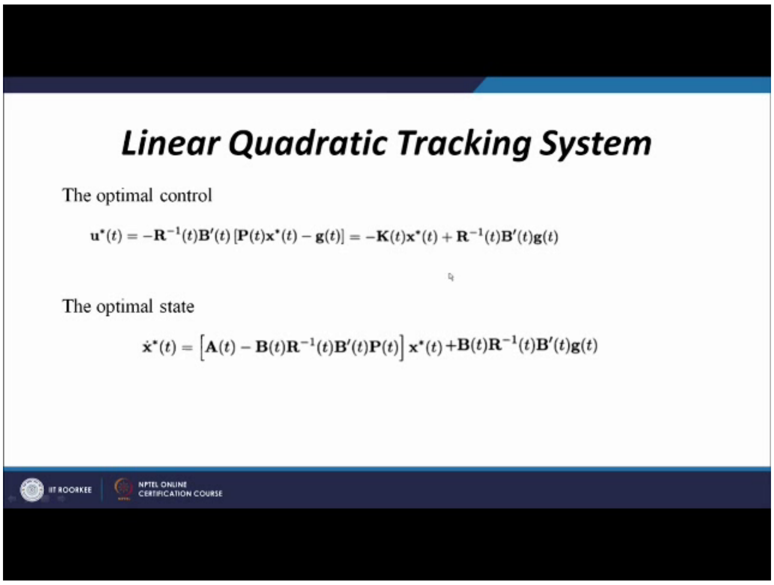

With terminal condition is C prime t f f of t f z of t f. So, these non-linear differential equation can be solved using any numerical techniques. So, once we know the value of the P and g, we know the value of my u which is nothing but my R inverse B prime P t x t minus g t or which we are writing as minus K t x t minus R inverse B prime g t. So, nothing, but we are writing this as my K. So, this is minus K t x t, R inverse R we have already know I know my B matrix g I can determine from this. So, this means I am finding my optimal u as given by this expression and if I will place this optimal u n a t minus sorry A x plus B u u is replace by this. So, this is my optimal state and this whole will give me a give me my close loop system.

So, in this wave I can obtain the optimal control law u t as minus K t x t plus R inverse B prime g t, u if I will replace by this n equation x dot t is A t x t plus B t u t. So, this will give me this optimal u give me the optimally state if u I will replace it. So, I will get this u in terms of the x. So, with this a state feedback I can get my closed loop system which is given as the x dot t equal to A minus B R inverse B prime P x of t ma plus B R inverse B prime g of t. So, once I solve this differential Riccati equation and this vector equation I got the value of the P and g, if P and g is known to me I can directly find what will be my optimal control and the optimally states.

So, I stop this lecture here in the next lecture we will discuss about the discrete time systems.

Thank you very much.