Optimal Control

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Lecture - 27
LQR with a Specified Degree of Stability

Welcome class for this session of our discussion. In the previous lecture we have discussed about the Riccati equation in form of the matrix Riccati equation and the algebraic Riccati equation. We also seen that how we can solve a Riccati equation for the time varying, and the time invariant case as well as we have also shown that a regulator problem. If we will implement the optimal controller or optimal control or u equal to minus k x then the closed loop system is a stable system. We have also seen the frequency response of a closed loop regulator system which is which give us a sufficient be stability margin, here gain margin is between half to infinity and phase margin is minimum 60 degree.

So, today we will see the few alternate or say the some other forms of the Riccati equation, in which the first one we are discussing that is LQR with the specified degree of stability. Say in a stable system as we know we are placing the pole into the left half of our x plane a normal ones we are solving the Riccati equation we ensure that our system is a stable system, but if you want to place the pole beyond a certain level.

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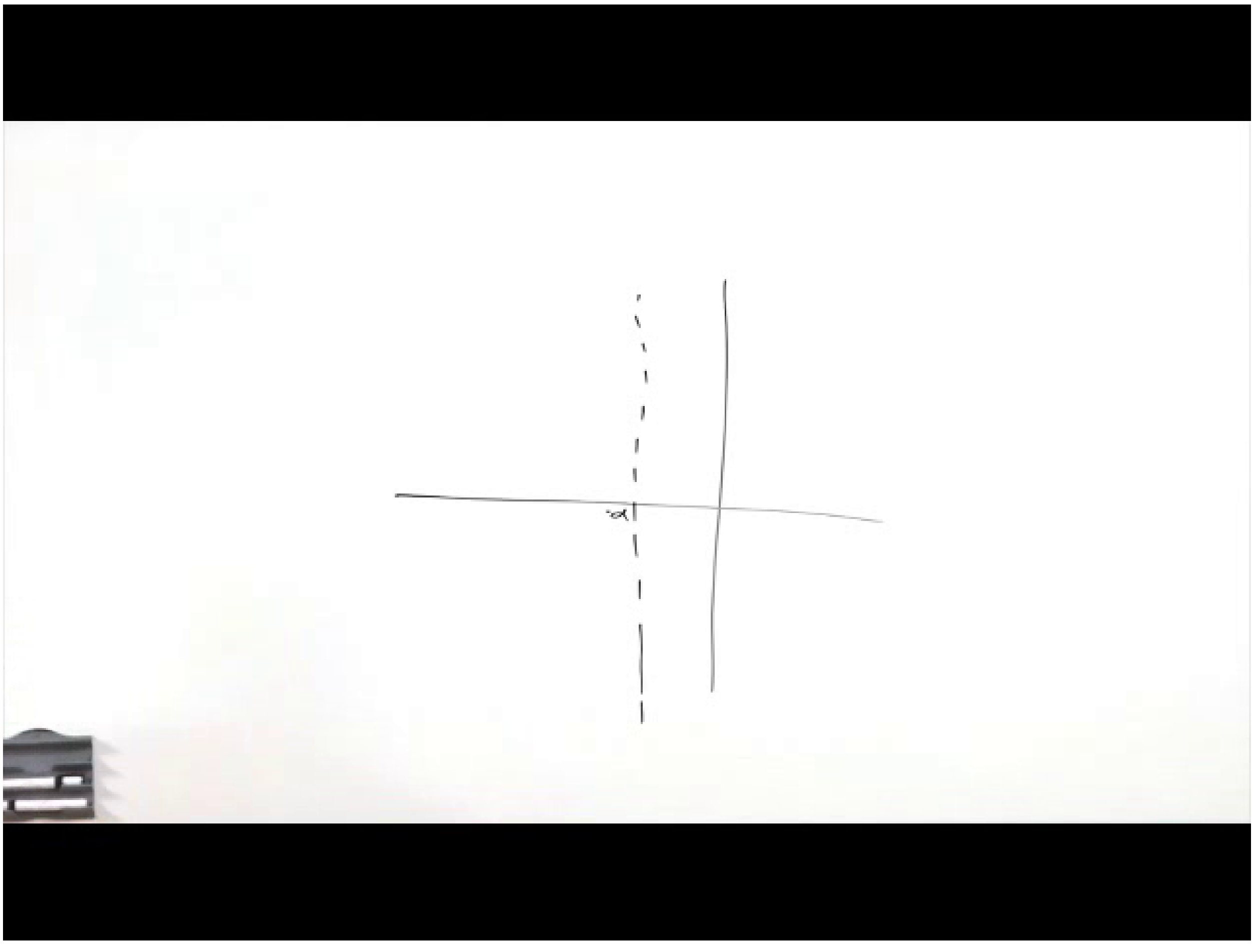

Let us say this is my alpha, I want to ensure that in optimal placement my all the poles remain left to the line at minus alpha. So, in that case we can utilize the concept of our a specified degree of stability, in which we consider a system x dot equal to A x plus B u.

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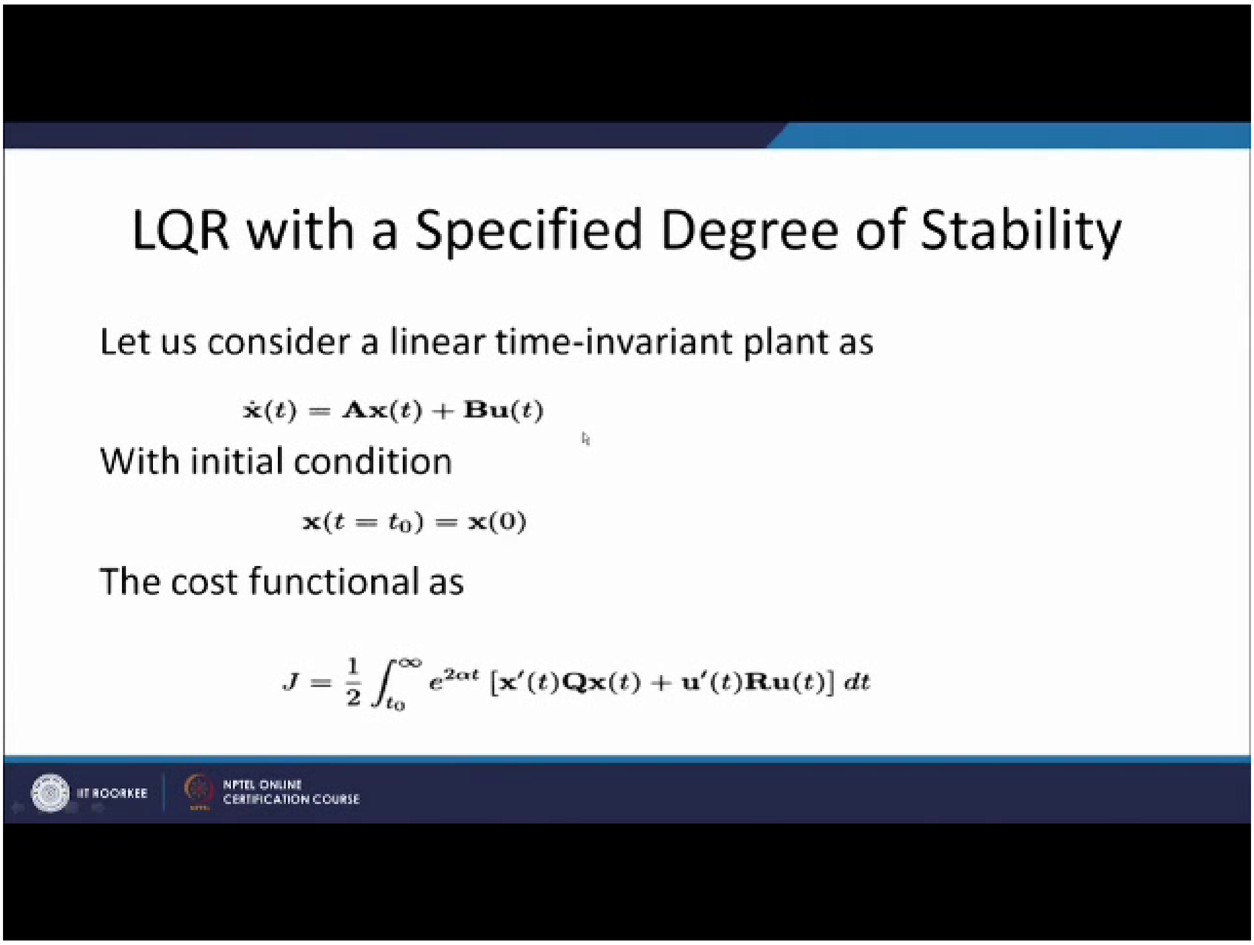

So, we are cons taking this example only for a linear time invariant system. So, A and B we are considering to be the constant. Initial condition is considered as x t 0 equal to x 0 some initial condition is given and to ensure that my pole will lie left to the line of minus alpha, we modify our performance index by multiplying the x prime Q x and u prime r u with a term e to the power 2 alpha t.

So, we will see how with this modified performance index we can ensure the placement of the pole left to the line minus alpha. So, we have considered the system x dot t is f e a x plus b u.

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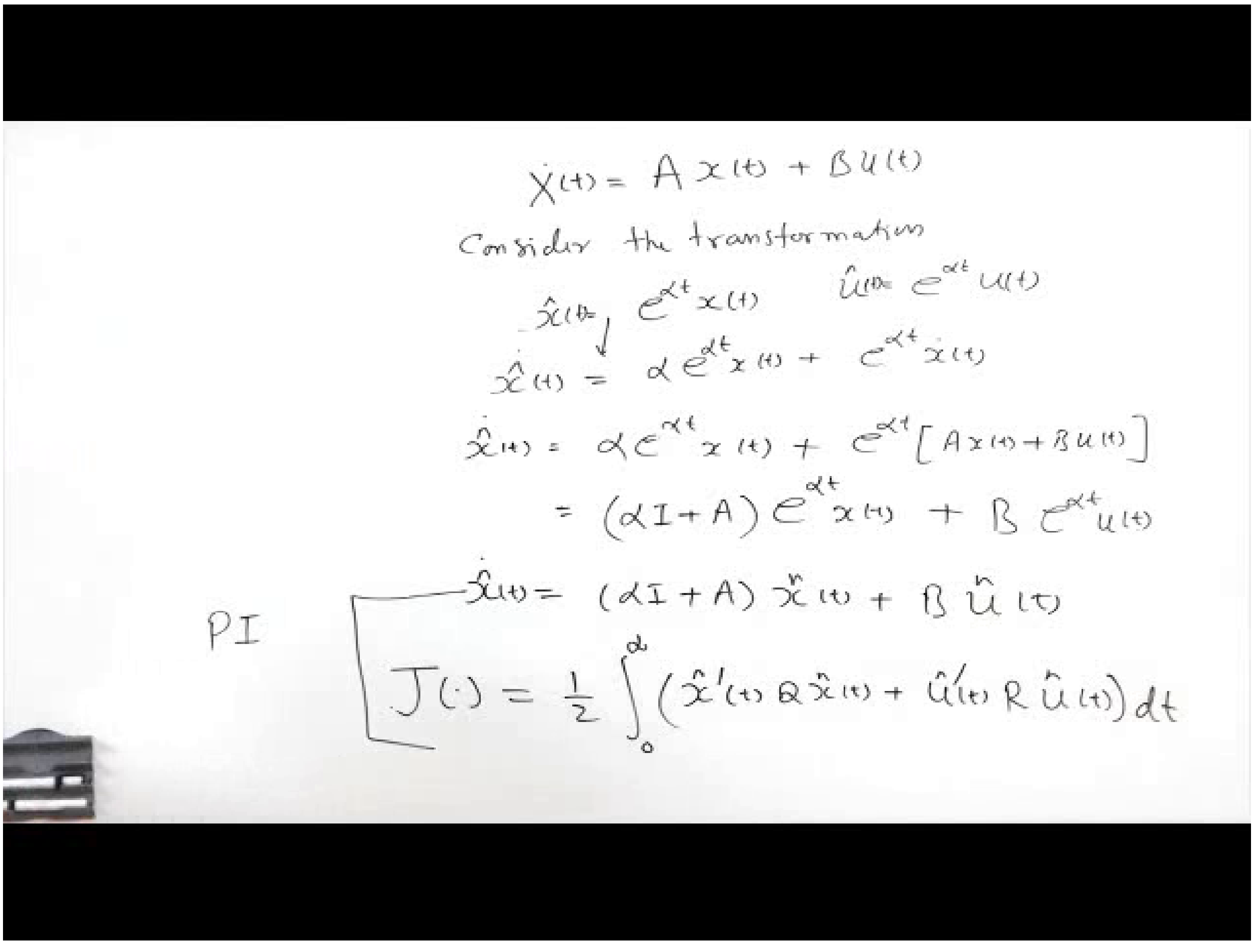

So, my system is X dot t linear time invariance. So, A we take as a constant A x t plus B u t. Now if we will see this is not the exact form which we have solve before in that case we consider our j as only x prime q x u prime r u and e to the power 2 alpha t is not there. So, if we are adding this term this means my PI is modified. So, the Riccati equation which we have used before we cannot use it directly, power approach here is we will convert the given system along with the performance index into the A standard form of my system representation and the PI.

So, that directly I can use the concept of my Riccati equation. To get the transform we consider the transformation, we consider x hat as and u hat is e to the power alpha t u. So, we considered this transformation. So, from the first one if you will take the derivative of this, we have x hat dot t sorry this is t this is x t, this is u t and this also u t. So, if I will take this. So, what I will get alpha e to the power alpha t, x t plus e to the power alpha t x hat dot t. So, by chain rule I am differentiating this. So, differentiate this equation. So, we got x dot hat alpha e to the power alpha t x t e to the power alpha t x dot of t.

So, I write this as I keep this term same alpha e to the power alpha t x t and in place of x dot t I can write as A of x t plus B of u t; so if I will explain this and club the x t terms together. So, I get alpha I plus A e to the power alpha t x t plus B, because alpha is a positive constant alpha is a scalar. So, e to the power alpha t will also be a scalar. So, I can write this e to the power alpha t is e to the power alpha t u t. And we have considered this as x t is e to the power alpha t and u t as e to the power alpha t u t.

So, the transform system I can write is x hat dot of t is alpha I plus A, x hat of t plus B u hat of t. So, in transform system my matrix A is modified as alpha I plus A and if we will see what is my performance index. So, my PI is integral 0 to infinity, because we are considering a infinite horizon, time invariant regulator system and if we you will see my product here is e to the power 2 alpha t. So, I will take 1 alpha t with x prime and another alpha t with x t. So, if I will multiply this, I can modify my performance index is x hat prime t Q x hat of t plus u hat of t R hat of t. So, this is prime, this is prime and this dt.

We are except if my e to the power alpha t and u hat is e to the power alpha t u t. So, e to the power 2 alpha t we have taken here. So, we can consider 1 alpha t with x prime, 1 alpha t with x t, 1 alpha t with u prime and 1 alpha t with u t. So, I will place this value. So, I will get the same PI as I am getting before. Now, if I will consider this system and this PI this is a my standard form, but the only difference here is in place of the x we will have the x hat and u hat.

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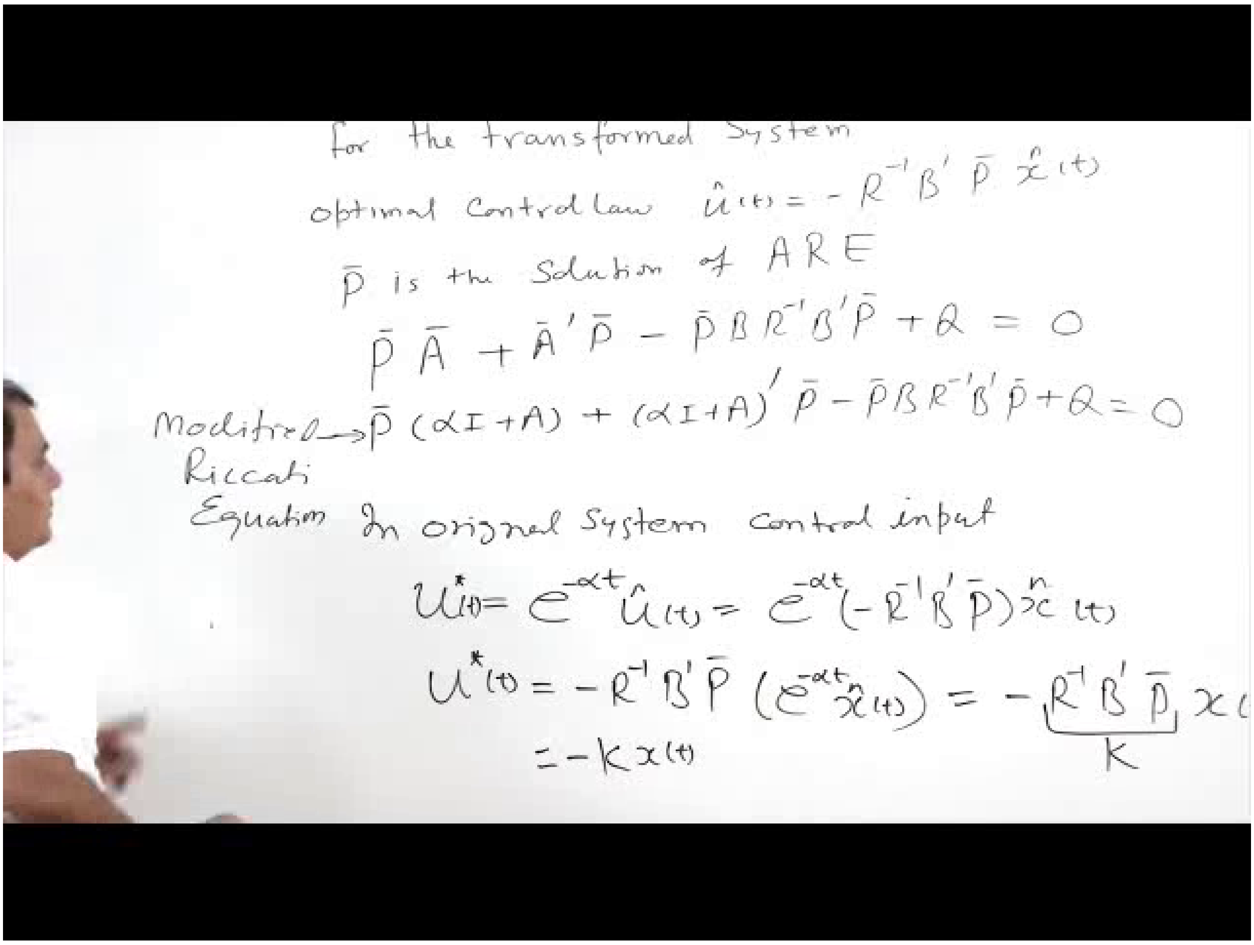

So, for this problem: so for the transform system of my optimal control law is u hat of t and as my infinite horizon regulator problem I know solution for this system is minus R universe B prime because my B matrix is unchanged let us say P bar x hat of t.

So, for this system my optimal control law is u hat as R inverse B prime P bar x hat of t and P bar will be the solution of my Riccati equation, where P bar is the solution of my algebraic Riccati equation and what will be the algebraic Riccati equation? If I will consider my this whole matrix to be, let us say A bar. So, my ARE will be P A let us say P bar A bar plus A bar prime P, A transpose of P minus P B R inverse B transpose P plus Q equal to 0. So, this is my algebraic Riccati equation for my transform system and in this A bar is alpha I plus a. So, P bar is nothing but the solution of my alpha I plus a alpha I plus A prime P bar its will also be the bar minus P B R inverse B prime P plus Q equal to 0.

So, what we have to do for my transform system I will solve this Riccati equation. So, this is giving me the value of my u hat, u hat is R inverse B prime P bar x bar of t, but in this my objective is to find out the optimal control law for the original system. In original system my control input is u star and which is what we have considered u hat e to the power alpha t u hat. So, this is e to the power minus alpha t, u hat of t because u hat of t is we have considered e to the power alpha t, u t. So, this is my optimal control and u hat of t is P bar n x set of t as e to the power minus alpha t is a scalar. So, I can write my u star hat as minus R inverse B prime P and e to the power minus alpha t x hat of t; and if all minus alpha t x hat of t is nothing but sorry this is nothing but my minus R inverse B prime P bar x of t.

So, this means in original system I can directly feedback my state multiplying with this as a K. So, u star t is nothing but minus k x t. So, I can directly implement my a state feedback system to the original system except the P bar which will be the now solution of my modified Riccati equation. So, I can say this is my nothing but modified Riccati equation. So, original system I can directly find out the K by solving my modified Riccati equation which will ensure that my all the poles are away from the minus alpha line parallel to the j omega x is.

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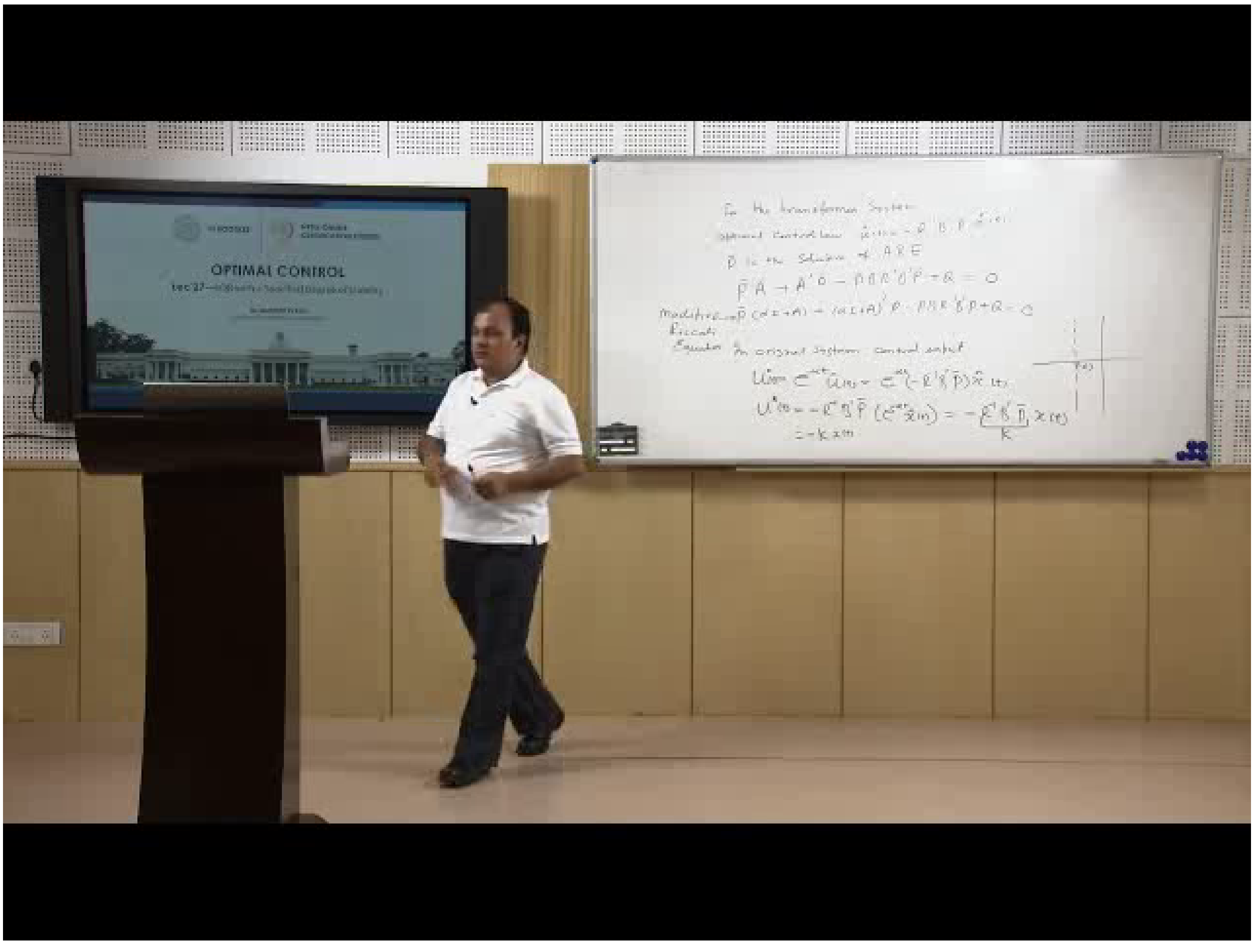

So, this solution will ensure that my all the poles are beyond this minus alpha line. So, just to summarize this what we are saying, if you want to find out the optimal control of for the given system x dot t as A x t plus B u t which will minimize the performance index j e to the power 2 alpha t x prime Q x plus u prime R u.

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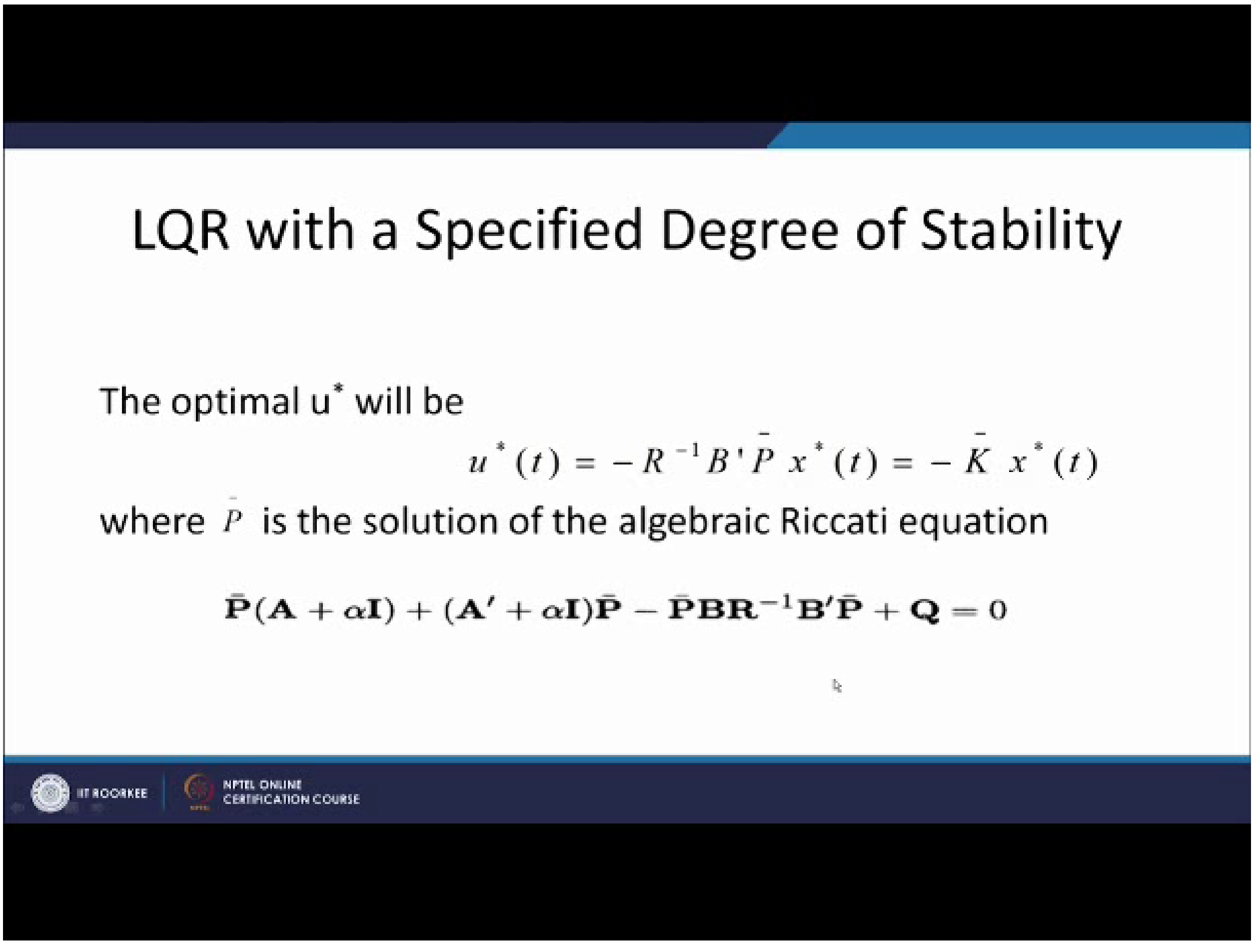

So, my u will be for original system minus R inverse B prime P bar x bar of t which is nothing but k bar x t where P is the nothing but the solution of my modified Riccati equation. And this solution will ensure that my whole pose lie left to the line minus alpha. So, we will see through an example how this will modify the pole location in the original system.

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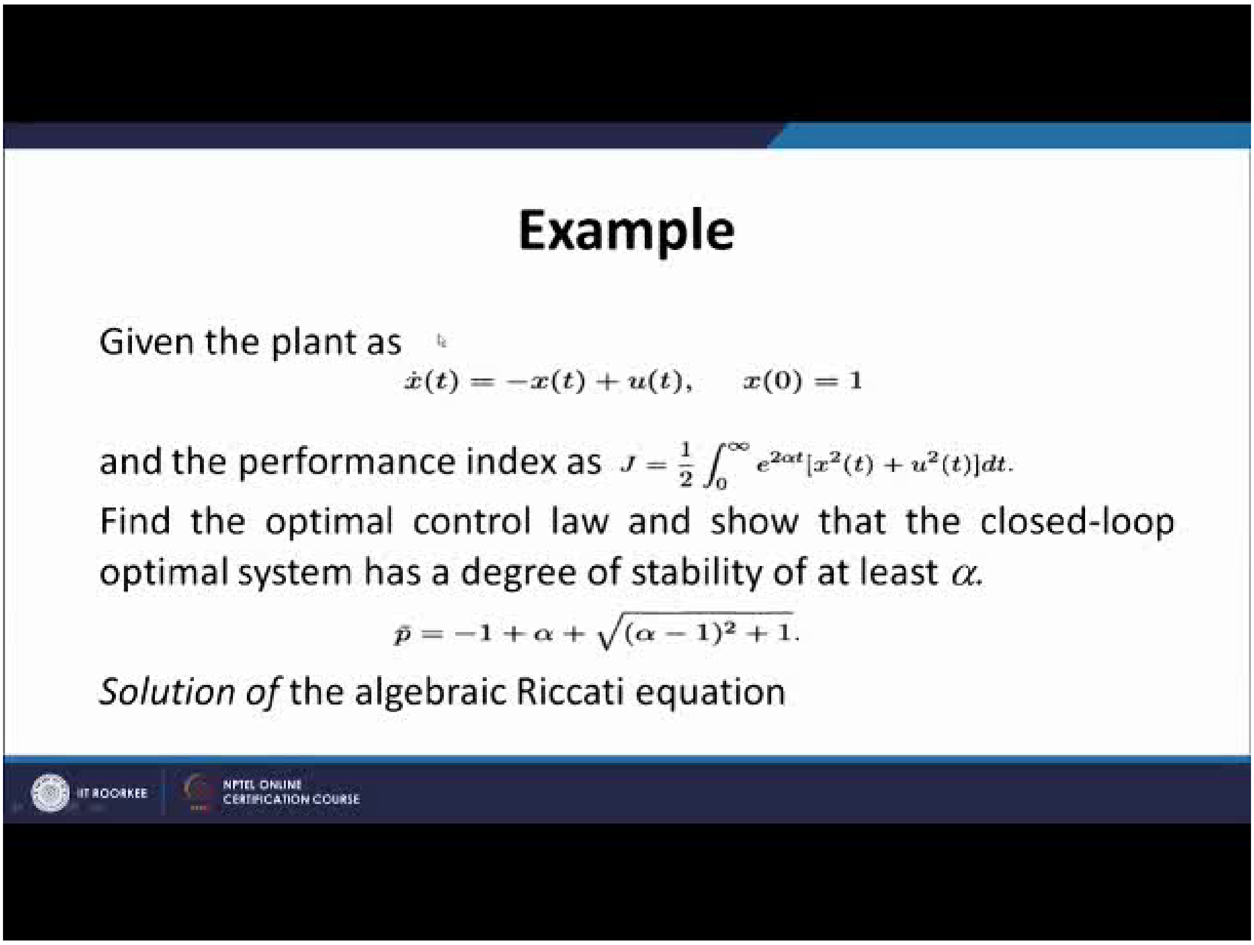

So, let us take a very simple example we take a first order system x dot t equal to minus alpha t plus u t with x 0 equal to 1 and we try to minimize the performance index given as j equal to 1 by 2, 0 to infinity e to the power 2 alpha t x square t plus u square t dt. Objective is to find the optimal control law and show that the closed loop optimal system has a degree of stability at least alpha; this means my closed loop system will have its eigenvalue into left to the alpha line.

So, means my degree of the stability is alpha. So, minimum degree of stability by this we can achieve as alpha.

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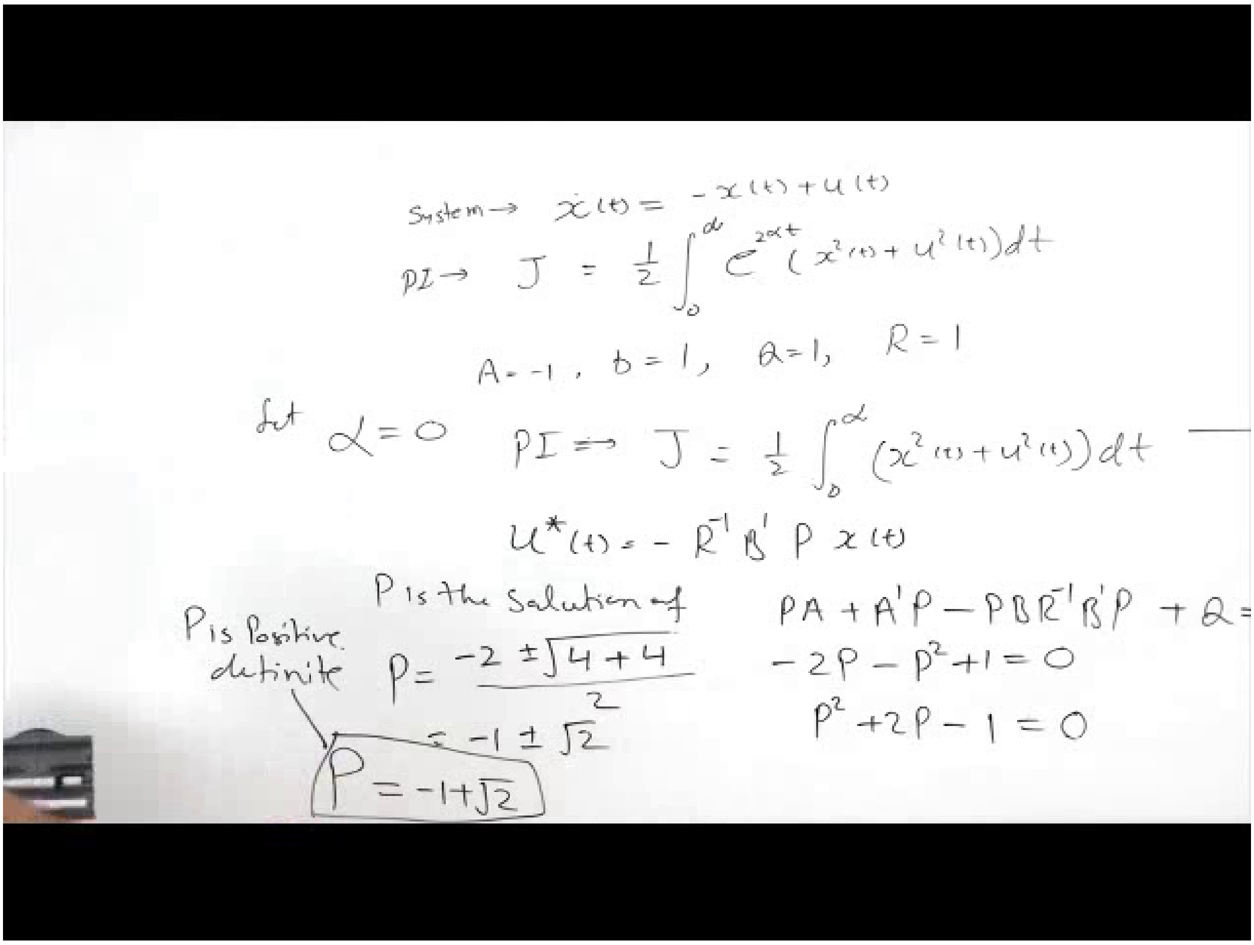

So, my system is x dot of t equal to minus x t plus u t and performance index this is my system this J is x square t plus u square t dt. So, in this case my A is minus 1 b equal to 1 and q equal to 1 R equal to 1. So, let us take if we do not use the A specified degree of stability concept means we take let alpha equal to 0. If alpha equal to 0 so, my PI is simply 1 by 2 0 to infinity and this term will become unity. So, this is nothing but x square t plus u square t dt, so for this system if Ii will find out what is my control law.

So, my optimal control is nothing but R inverse B prime P x t and P is the solution of nothing but my PA plus A prime P minus PB R inverse B prime P plus Q equal to 0. So, this will give me A is 1 a prime sorry A is minus 1. So, minus 2 PB R inverse B prime this is unity minus P square Q is one plus 1 equal to 0. So, I have equation P square plus 2P minus 1 equal to 0 and for this my P will be minus 2 plus minus B square minus 4 a b a to. So, this is nothing but minus 1 plus minus under root 2.

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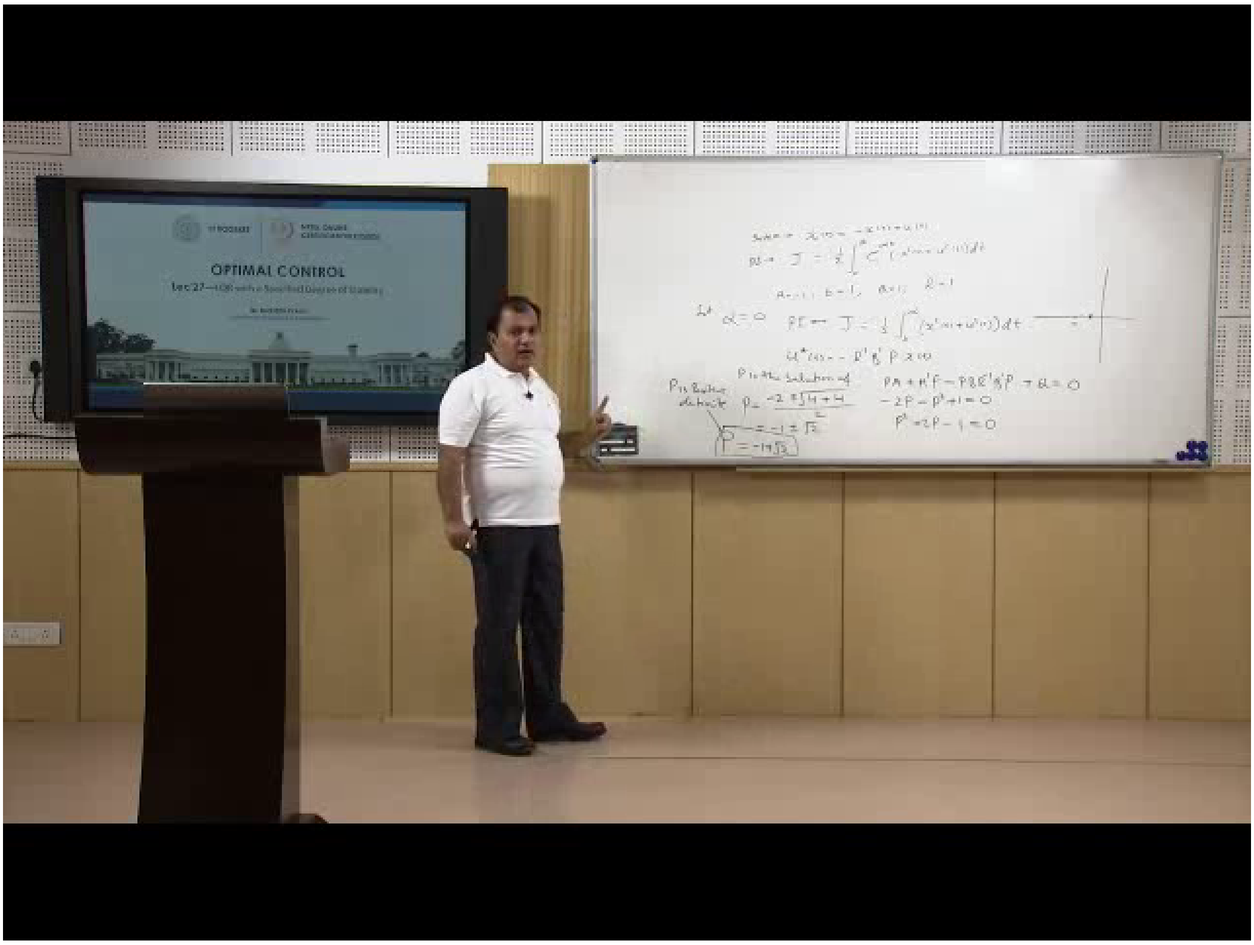

So, with alpha equal to 0 I sure that my system is stable. So, in this case what the P value we will take our P should be a positive definite. So, my P will be minus 1 plus under root 2, P is positive definite.

So, we take only the positive value. So, I will select my P as minus 1 plus under root 2. So, my pole will lie if this is minus 1 somewhere point 4 somewhere here when I will optimally locate with alpha as 0. So, I am considering this alpha to be 0. So, my polar located into left of the f plain, but value less than minus 1, but if you I will consider the alpha what will be the case in that case.

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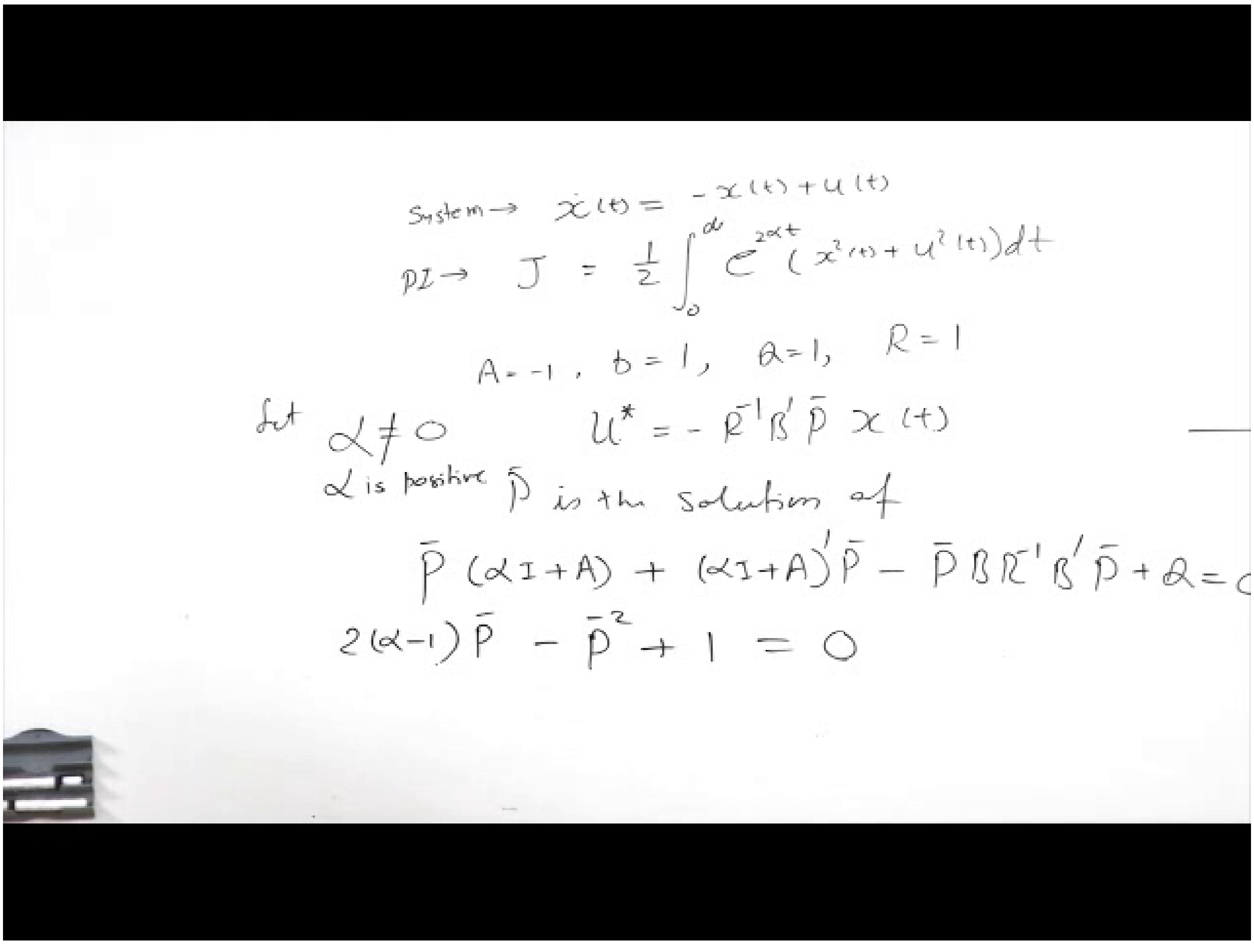

If alpha not equal to 0 and alpha is positive. So, my optimal control law is if alpha not equal to 0 and alpha is positive. So, my this is minus R inverse B prime P bar x t and now P bar is the solution of my Riccati equation which is now P alpha I plus A plus alpha I plus A prime P minus sorry P bar we are considering minus PB R universe B transpose P plus Q equal to 0.

So, A is I is 1 A is minus 1. So, if I will write this, this is nothing but. So, 2 alpha minus 1 P, this two will give me alpha minus 1 P minus P bar, P bar is square plus Q is 1 equal to 0 and if I will find the root of this. So, this will give me the solution as P equal to minus 1 plus alpha if square root of alpha minus 1 is square plus 1.

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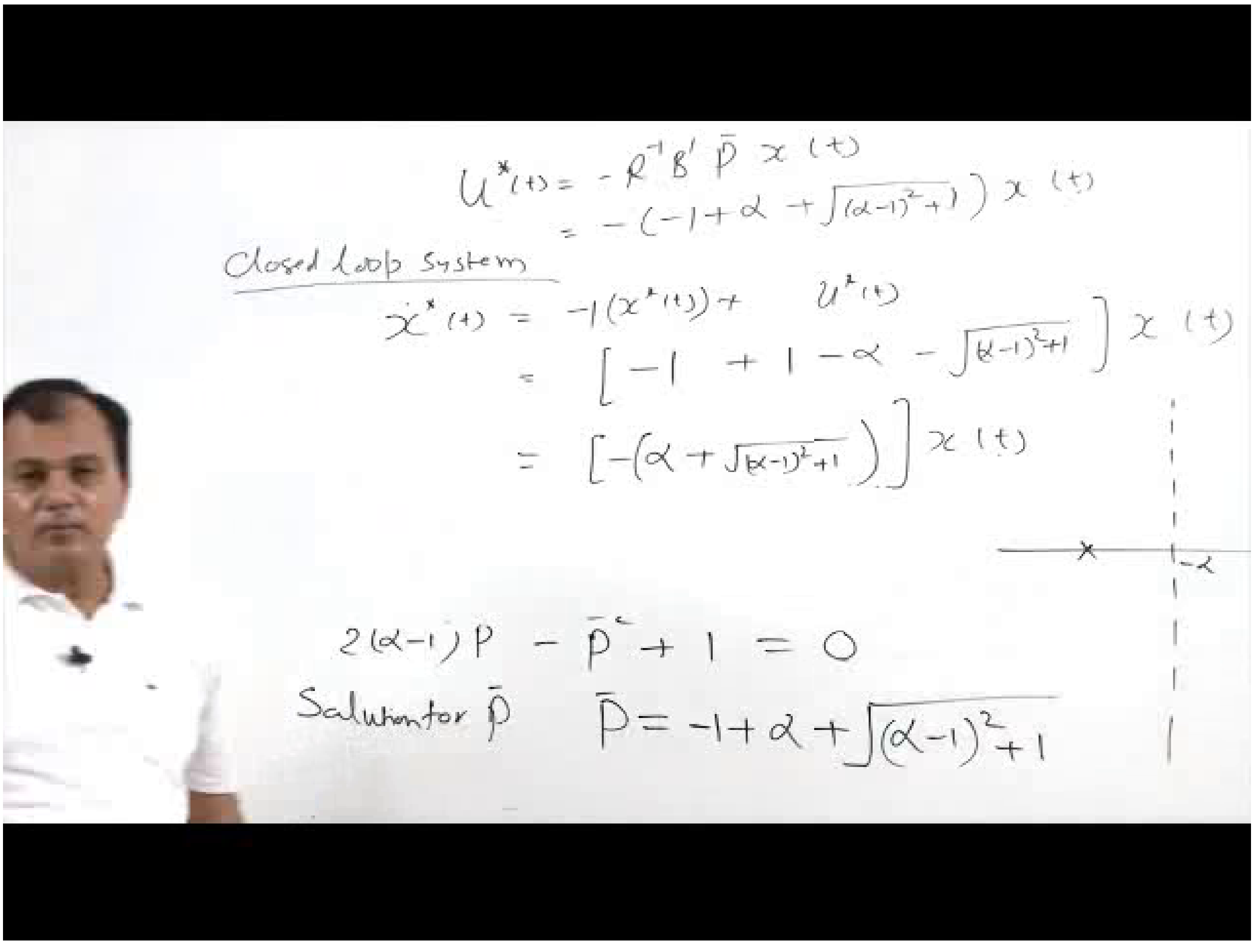

So, the solution of this equation is solution for P. So, my P bar is has given minus 1 plus alpha plus a square root of alpha minus 1 a square plus one. So, this is my P bar. So, my optimal control. So, my U star t will be minus R inverse B prime P bar x t and P bar I can take this R inverse is 1 B prime is 1. So, this is coming out to be minus of minus 1 plus alpha, alpha minus 1 a square plus 1 into x t.

If this is so, then my closed loop system will be x dot t as A x t plus B u star of t and u star of t is. So, this is A is 1. So, we will have A is minus 1 B equal to 1 u star of t. So, in this case if we are placing this u. So, minus 1 plus 1 if I will multiply this 1 minus alpha minus under root alpha minus 1 a square plus 1 into x of t. So, if we will simplified is this is nothing but minus alpha minus alpha plus under root alpha minus 1 a square plus one into x of t. So, for closed loop system, this is giving me the root. So, the eigenvalue of my system and this eigenvalue I can see is away from the minus alpha line.

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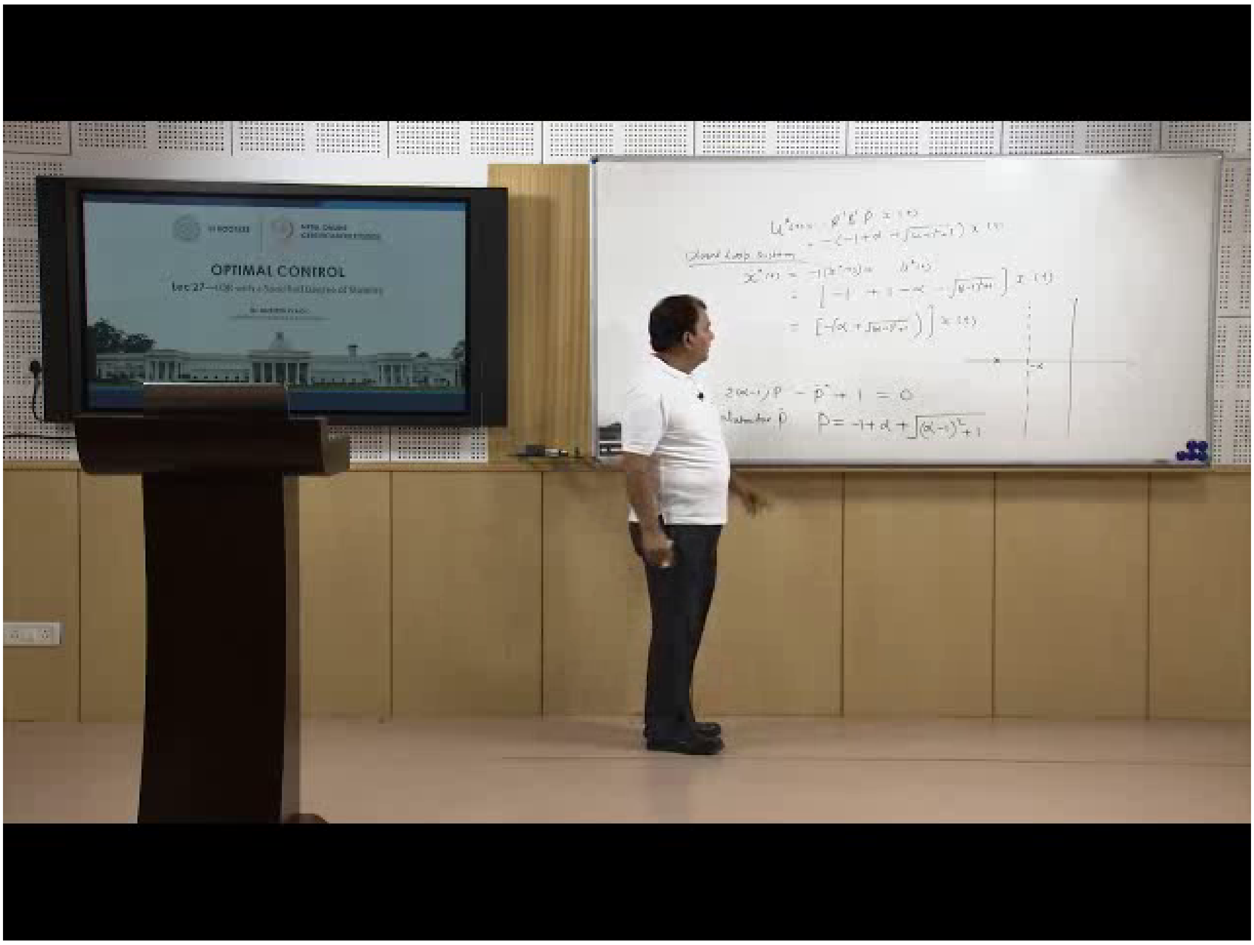

So, if I will consider this as my minus alpha. So, naturally my eigenvalue will be located somewhere here because this is minus alpha minus 1 more time is added. So, I can ensure the astability means my all eigenvalues will be left to the minus alpha line. So, this is the advantage of utilizing the a specified degree of a stability in a closed loop system, in which we can ensure that my eigenvalues always will be placed left to the minus alpha line or in certain case I can ensure a ensure the a stability of my given system.

So, this is the one form of this another form is called the inverse matrix Riccati equation, which we will discuss into the next class. So, I stop this class here.

Thank you very much.