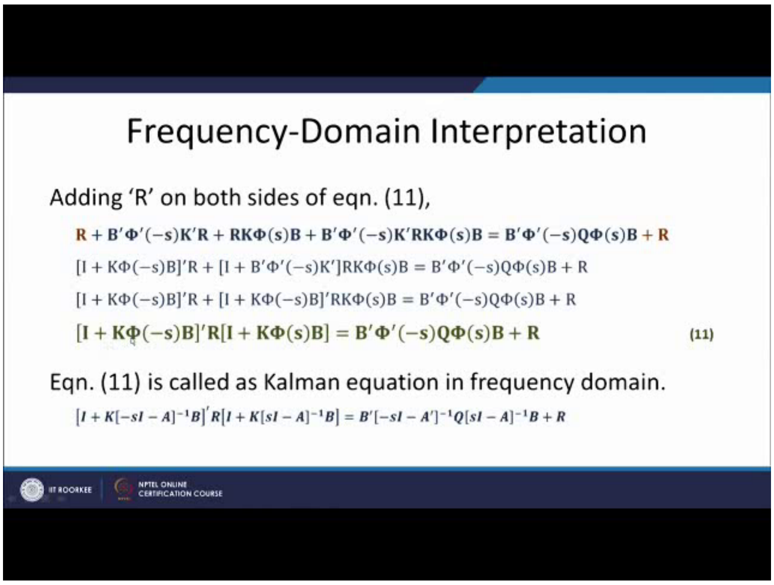
Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture - 26
Frequency Domain Interpretation of LQR (Linear Time Invariant System)
(Continued)

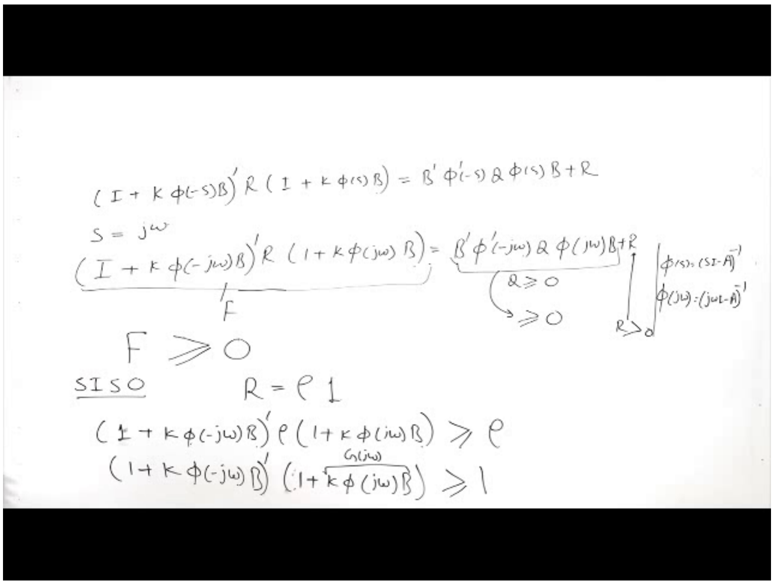
So welcome class, in this lecture we will continue our discussion where we left in the previous class. We are discussing about the Frequency Domain Interpretation of the LQR System and this we are doing for the; Linear Time Invariant case.

(Refer Slide Time: 00:53)



In the previous class, we have seen that we can find matrix sorry Kalman's equation in the frequency domain is I plus K phi minus s B; whole transpose multiplied R plus I plus K phi s B; equal to B prime phi prime minus s, Q phi s B plus R; here phi s is nothing, but my state transition matrix which is given as s I minus A inverse and K phi minus s is nothing, but my minus s I minus A inverse.

(Refer Slide Time: 01:28)



So, this equation if we will write; so this is my Kalman's equation in frequency domain; in frequency domain, we can replace S by j omega. So, I can write this equation as I plus K phi; minus j omega B, K phi j omega B, B prime phi Q phi; j omega B plus R. So, here what is phi S, so if my phi S we are taking as s I minus A inverse. So, phi j omega means j omega I minus A inverse. So, I have replaced this so what I will have; if my S is replaced by the j omega. So, my left hand side and the right hand side are nothing, but the complex quantity and this will have the magnitude and the phase.

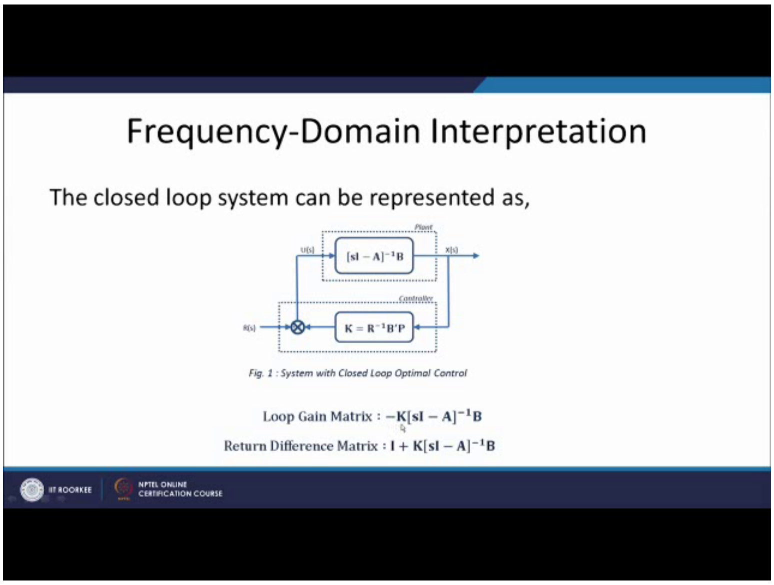
In terms of the magnitude and the phase, we can represent our left hand side and the right hand side. Now, see first the; what is the right hand side, so we know Q is a positive semi definite matrix. Here we have the phi prime and the phi B prime and the B, these always are the squares; so, this means they will also be the positive definite. So, this means the whole quantity is greater than or equal to 0, my R is always greater than 0 because this we have selected as a positive definite matrix.

In this equation, my right hand side B prime, phi prime, minus j omega, Q, phi j omega B my; this term is greater than or equal to 0, R is greater than 0. So, if I say my left hand side is represented by F, so my F will be naturally greater than; at the most I can say greater than or equal to 0. Now, we take example of a single input, single output for a SISO case, I can represent my R as some positive number multiplied with a identity matrix.

So, R is rho into 1 where R should be positive definite, so rho should be a positive number because this R I am taking as to be a positive number or identity matrix. So, I can say my; for single input, single output this will be; I will become 1 plus K phi, minus j omega, B prime, R; I can take as a rho into 1, 1 plus K phi j omega B is greater than equal to rho and rho if I will write; so, I can simply right as 1 plus K phi minus j omega B; whole prime rho, rho I can take it there or this is multiplied with 1 plus K phi j omega B greater than equal to 1.

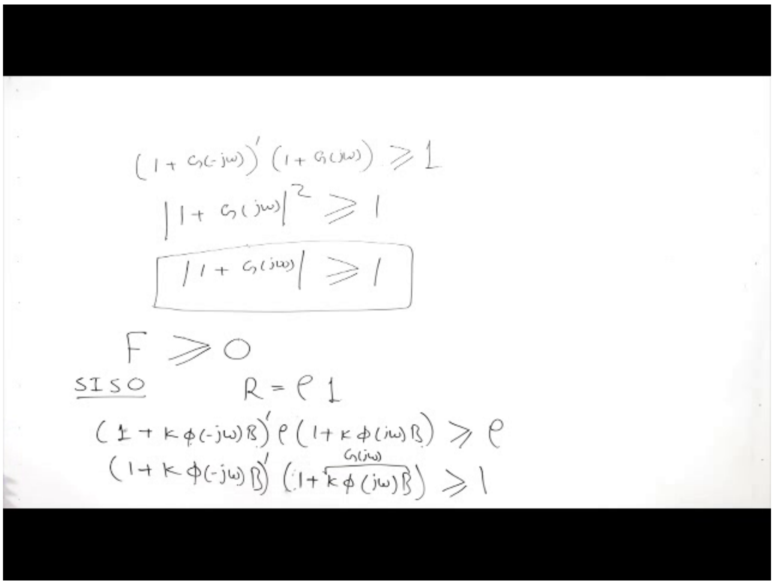
Now, if we will see; what is the K phi j omega B? If you recall, we have the loop gain as K; s I minus A inverse B; where s I minus A inverse is nothing, but my phi S.

(Refer Slide Time: 08:13)



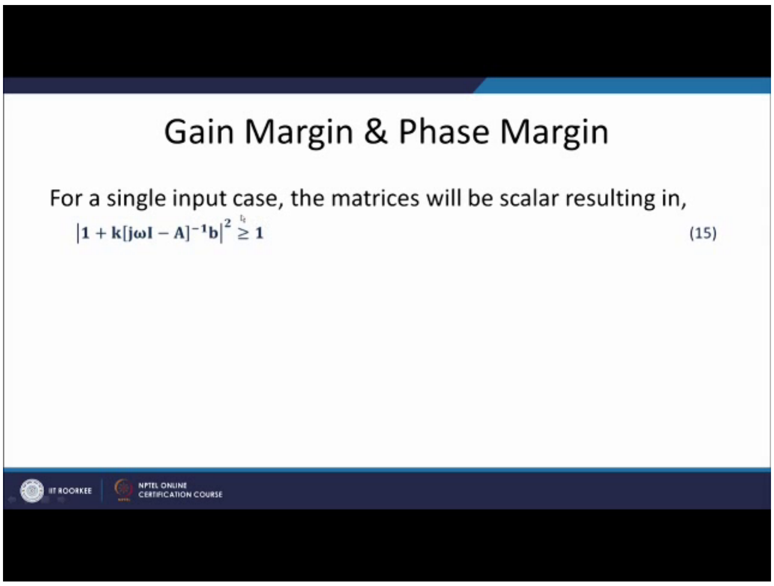
So, this term nothing but representing my G j omega, so what I can write.

(Refer Slide Time: 08:43)



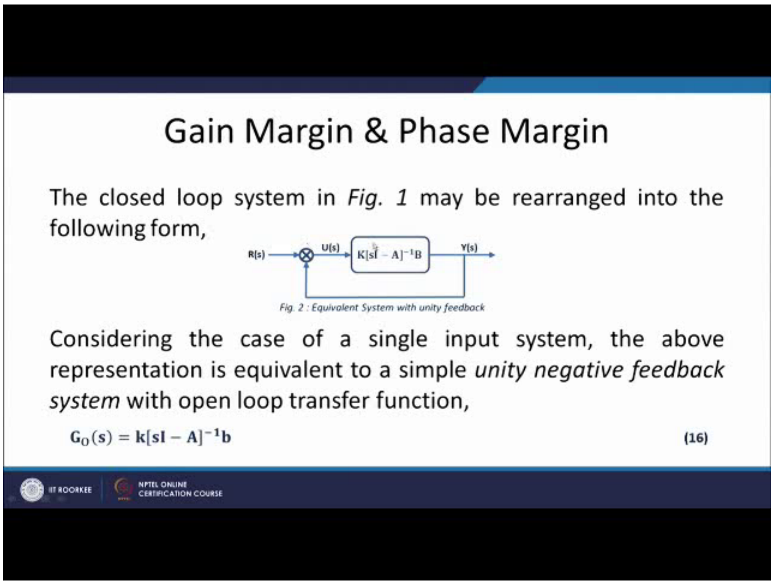
So, I can write this as 1 plus G minus j omega, prime multiplied with 1 plus G j omega and this is greater than equal to 1 or in terms of magnitude, I can write 1 plus G j omega square is greater than or equal to 1 or I can write 1 plus G j omega magnitude is greater than or equal to 1. So, I got this condition and which I have 1 plus G j omega greater than or equal to 1. So, G j omega is what my 1 plus K j omega minus A inverse B is square is greater than 1 because I am taking a single input, single output case.

(Refer Slide Time: 10:00)



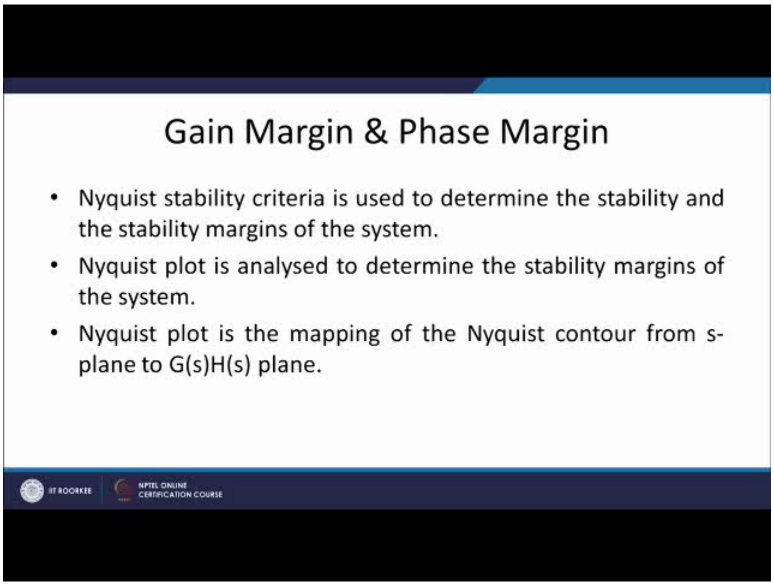
So, B this B I am simply representing is the; small b.

(Refer Slide Time: 10:19)

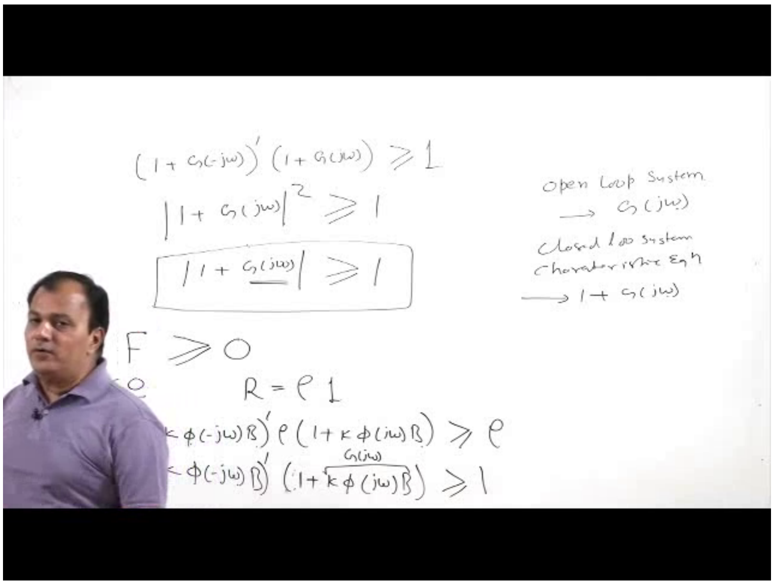


So, my closed loop system if I will include the loop gain this is nothing, but K s I minus A inverse B is nothing, but giving me the my complete closed loop system, where G 0 s; I am representing is K s I minus A inverse b.

(Refer Slide Time: 10:39)

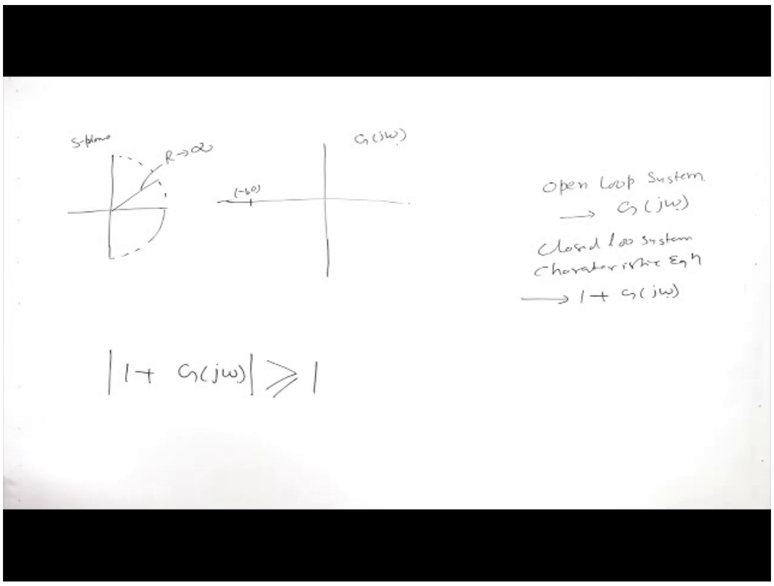


(Refer Slide Time: 10:50)



So, for this system if G j omega if, I will draw the Nyquist plot; so my open loop system is; I am representing as G j omega and closed loop system; characteristic equation, I am taking as 1 plus G j omega. So, I can check the stability by the Nyquist plot for the system of G j omega as a open loop and the closed loop system stability analysis, I can get it by the Nyquist plot. So, Nyquist stability criteria can be used to determine the stability and the stability margin of the system. Nyquist plot is analyzed to determine the stability margin of the system and what is the Nyquist plot; this is nothing, but the mapping of the Nyquist contour to the G j omega plane.

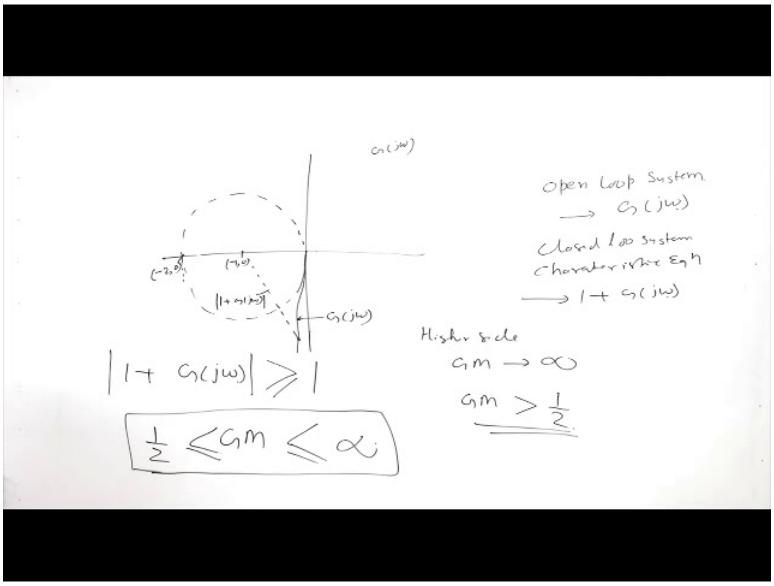
(Refer Slide Time: 12:26)



We consider a Nyquist contour in, so to draw the Nyquist plot; we are mapping my S plane into G j omega plain. So depending upon the condition, we consider a Nyquist contour of infinity radius which is covering the right of plane and we map all into the G j omega plane and analyze for the stability in G j omega plane. So, that already we know how to draw the Nyquist plot and what is the Nyquist stability criteria. For our case, our condition is; so, my system is 1 plus G j omega, mod 1 plus G j omega is greater than or equal to 1.

What is the meaning of this? This means, if I will analyze the closed loop system stability then my Nyquist plot will have the magnitude, the closed; will have the magnitude greater than or equal to unity.

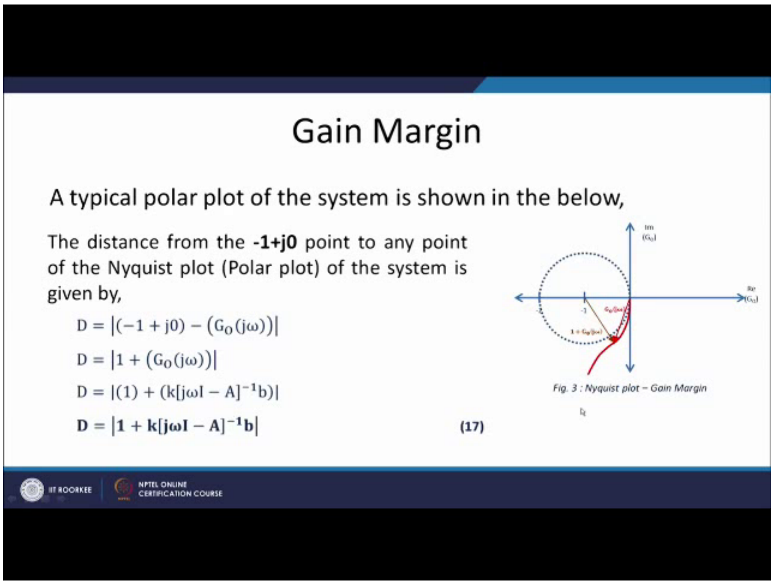
(Refer Slide Time: 14:15)



This means if I will draw; so, in G j omega plane, I am considering a unit circle with one angle 0. So, this means my Nyquist plot always away from this unit circle, my Nyquist plot cannot enter into the unit circle.

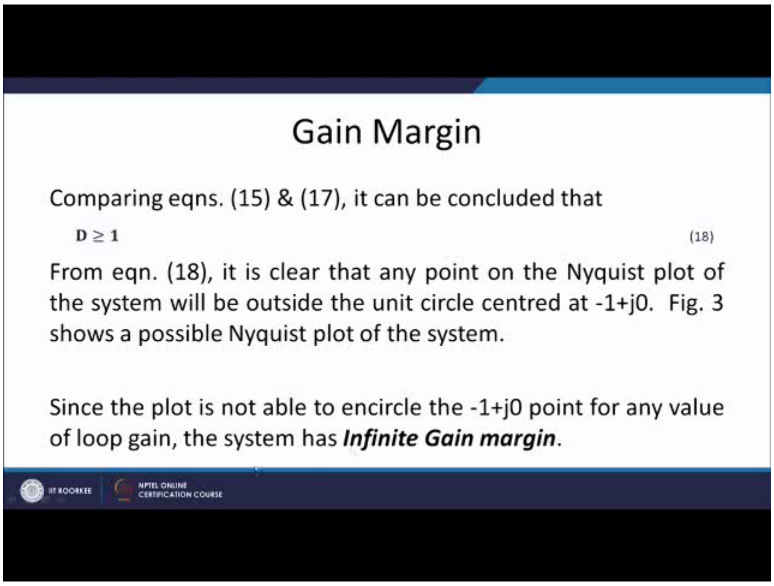
So, it always will be away from this so that the magnitude of; so, what this represent; this represent 1 plus G j omega. So, the magnitude of this G j omega always will be greater than or equal to unity at any given frequency. So, at the most it may be equal to unity and it will go away so this will be greater than or equal to unity. This means know where this G j omega plot is entering into this circle because always this distance is greater than or equal to unity.

(Refer Slide Time: 15:40)



So, we can see this figure here a red line here it showing the any G j omega plot. So, this always at the most it maybe the tangent to this, but G j omega plot always it cannot enter into this unit circle. So, if it is not entering the unit circle and condition is like this. What will be the gain margin? The gain margin of the system will be the infinity.

(Refer Slide Time: 16:11)

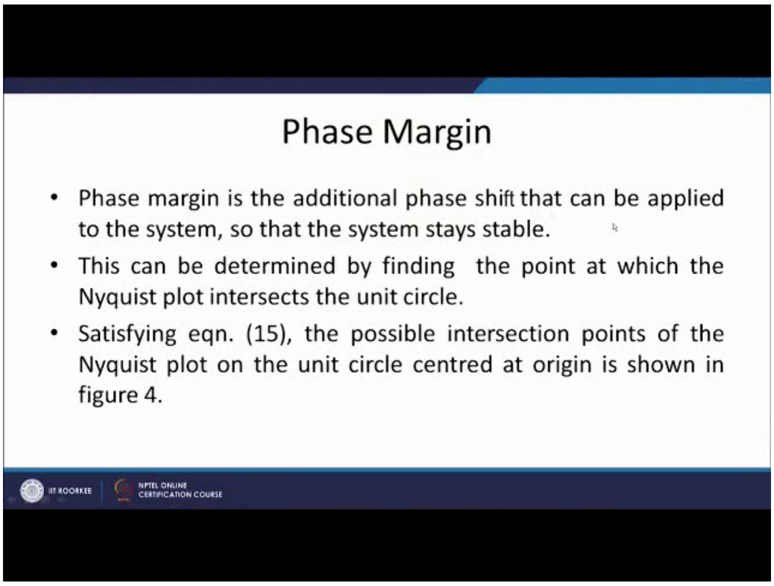


So, since the plot is not able to encircle the minus 1 j 0 point for any value of the loop gain, the system has the Infinity Gain margin, but they are may be the other condition that is a unit circle. So, my this point will be minus 2, 0. So, on the higher side; my gain margin is infinity what may be the another possibility that; my Nyquist plot at the most made read the tangent to this minus 2, 0 point. If it is at this point, so we know my gain margin should be greater than half.

So, in this system my gain margin will be greater than or equal to half at the most and this will be infinity. So, my two extreme points are here at the most by as per this condition, it can have the magnitude as unity. So, at the most my G j omega, this is my G j omega plot maybe tangent to this and can move here, it cannot enter into the circle.

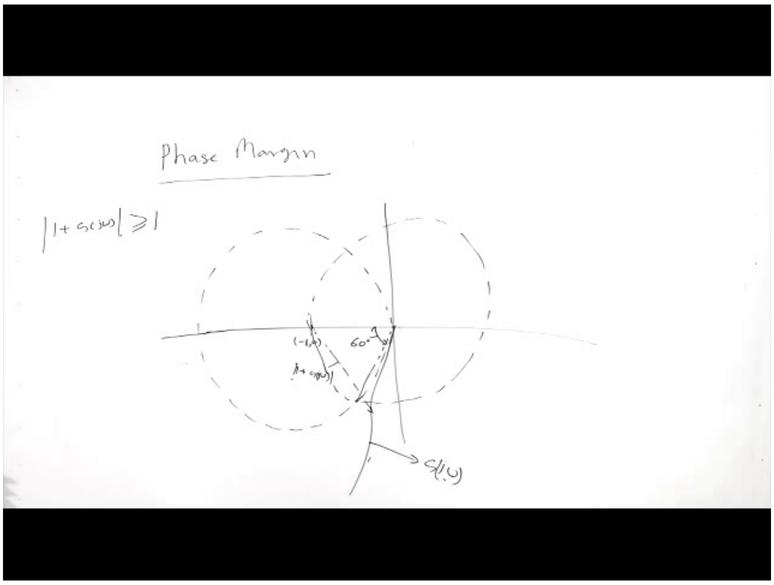
So, gain margin will be infinity; at the most it may be tangent to the minus 2, 0 point; here gain margin will be 1 by 2. So, I can say for a LQR system; my gain margin lie between half and infinity.

(Refer Slide Time: 18:31)



So, this is the case of the gain margin. What is the phase margin?

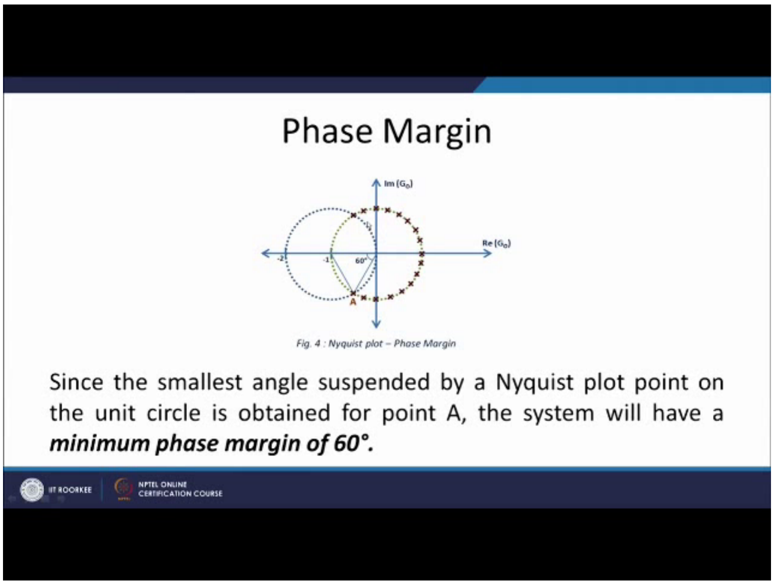
(Refer Slide Time: 18:46)



By definition phase margin is the additional phase shift that can be applied to the system, so the system stays stable and how we can find this phase margin. So, the phase margin of the system; what is my condition? This is minus 1, 0 point; I have a unit circle and somewhere this is my G j omega plot, which is giving me the gain margin is infinity. So, to get the phase margin; I draw a unit circle but now centered at the origin. So, this unit circle we have considered at the centre at the minus 1, 0 point because my condition is mod 1 plus G j omega is greater than equal to unity, so 1 plus G j omega I am finding.

So, this is representing at any instant; my 1 plus G j omega, so this magnitude from this to this is 1 plus G j omega and this is greater than unity. So, to get the phase margin; I draw unit circle centered at origin and the minimum phase margin in this system is where my G j omega will be tangent to this and this will be nothing, but my 60 degree.

(Refer Slide Time: 20:58)



So, we can understand this; so, what I will have? I will have this unit circle which is representing my condition mod of 1 plus G j omega is greater than or equal to unity. To get the phase margin, I am drawing the another unit circle; which is intersecting this at the point A, this is my minimum phase margin this means I can; if I will increase the phase, so that the system will reach to unstability. Because my system cannot enter into this unit circle, so my actual system is always will be outside the unit circle.

So where it is intersecting this, so this is the minimum phase margin of my LQR and this will be because both are the unit circle; so this is a equilateral triangle; so, my this angle will be 60 degree. So, we can say the smallest angle suspended by the Nyquist plot on, the unit circle is obtained from point A and the system will have a minimum phase margin of 60 degree. So, we have seen that in this case, in the frequency response of LQR system; we can determine what actually is the gain margin and the phase margin of my system.

My gain margin lies between half and infinity and the phase margin minimum phase margin is 60 degree. So, by this we can conclude that (Refer Time: 22:37) a optimal state feedback system is a stable system having a good stability marginal also. So, with this I stop my discussion on frequency response of LQR system by which we have seen that we can find out the gain margin and the phase margin of a optimal control system, optimal state feedback control system; which give me the gain margin as between half to infinity and the minimum phase margin is 60 degree, so we stop here.

Thank you very much.