Optimal Control

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Lecture - 25
Frequency Domain Interpretation of LQR
(Linear Time Invarient System)

Welcome class for this session. Today we will discuss about the Frequency Domain Interpretation of a LQR System, and this we will do for the linear time invarient system. In the previous class we have discussed about how we can develop the optimal control law for a time varying and the time invariant system. This optimal control law can be determined if we can find out the solution of a Riccati equation.

So, for time varying case my Riccati equation is a differential matrix Riccati equation and for the time invariant case this is a algebraic equation, which can be in both case this is a non-linear equation for time varying case it can be solved using the numerical technique. And for analytical solution also we have seen how we can determine the value of the Riccati coefficient P by the different ways. We have also discussed about the stability of the LQR system and we have shown for time varying as well as the time invariant case. In both the cases my system is a stable system.

Whatever be the condition of the matrix A, but the close loop system A minus B k always will be a stable system. So, today we will discuss about the frequency domain interpretation means what is the; if we will have the frequency response analysis and we are willing to find out the gain margin and the face margin of the system how we can find all this.

So, this we are taking for the linear time invariant system, so we will consider ALTI system as X dot t equal to A X t plus B U t.

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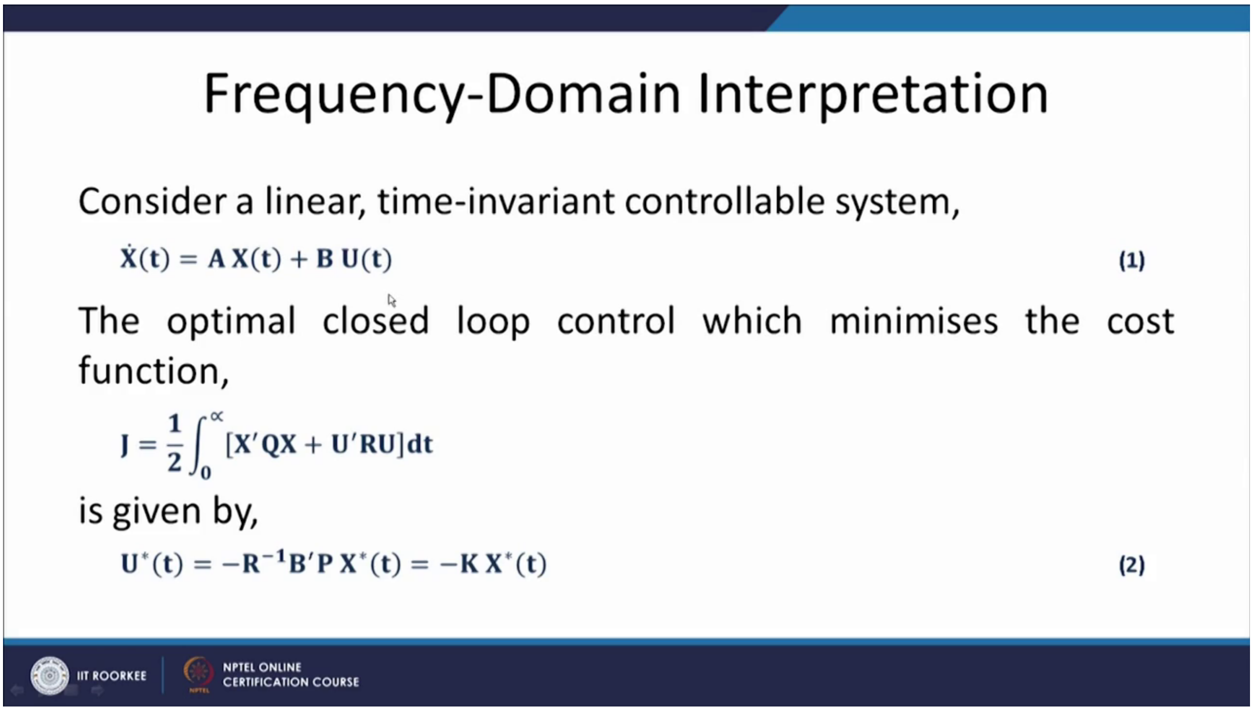

Objective is we have to determine the optimal control law U t which will minimize the performance index J as 1 by 2 integral 0 to infinity x prime Q X plus U prime RU dt. So, if we will solve this problem we know we can find the optimal control law which is minus R inverse B prime P X t, where R inverse B prime P we call the K.

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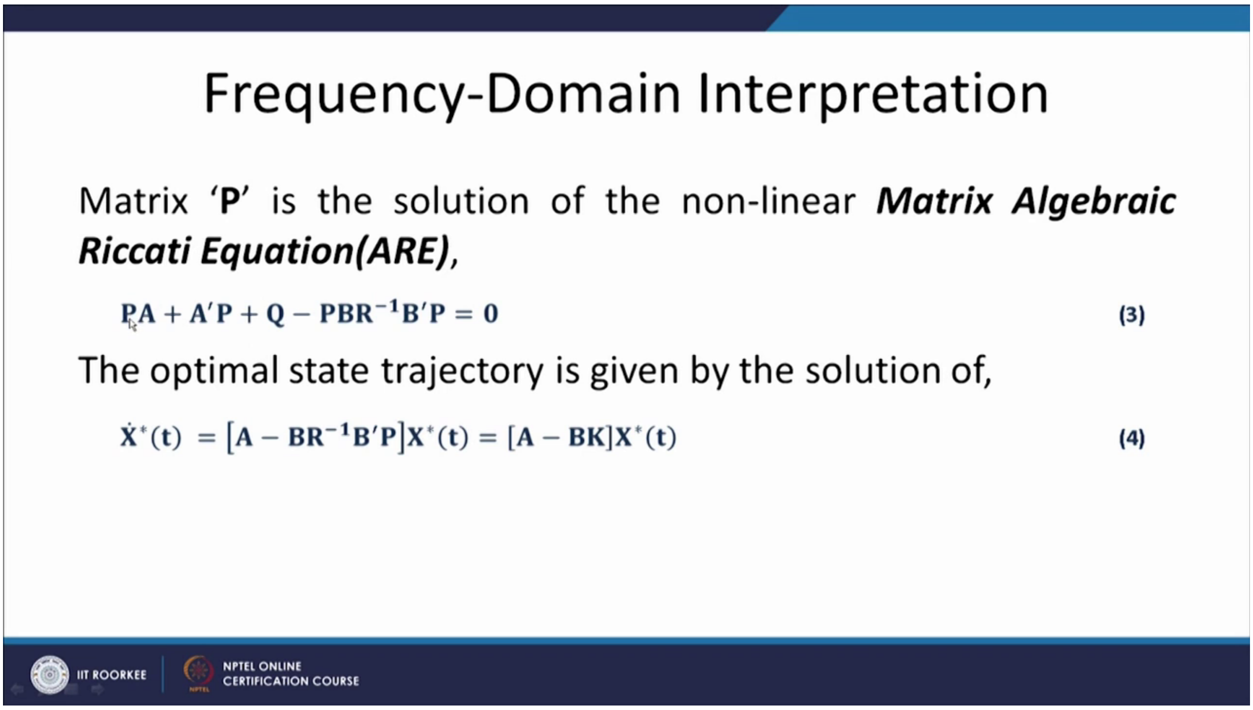

So, this is minus K X t where P is the solution of my algebraic Riccati equation which is PA plus A prime P plus Q minus PBR inverse B prime P equal to 0. So, if I can solve this algebraic Riccati equation to find the P I can find out the U which is R inverse B prime P R I know, B I know P we already have determined. So, I can find out the K. So, if I will apply the control law U equal to minus K X. So, this is A X t minus B K X t. So, A minus B k A minus B k R inverse B prime P is nothing but my close loop matrix. And we already have shown that in this system whatever be the matrix A my lose loop system always is a stable system.

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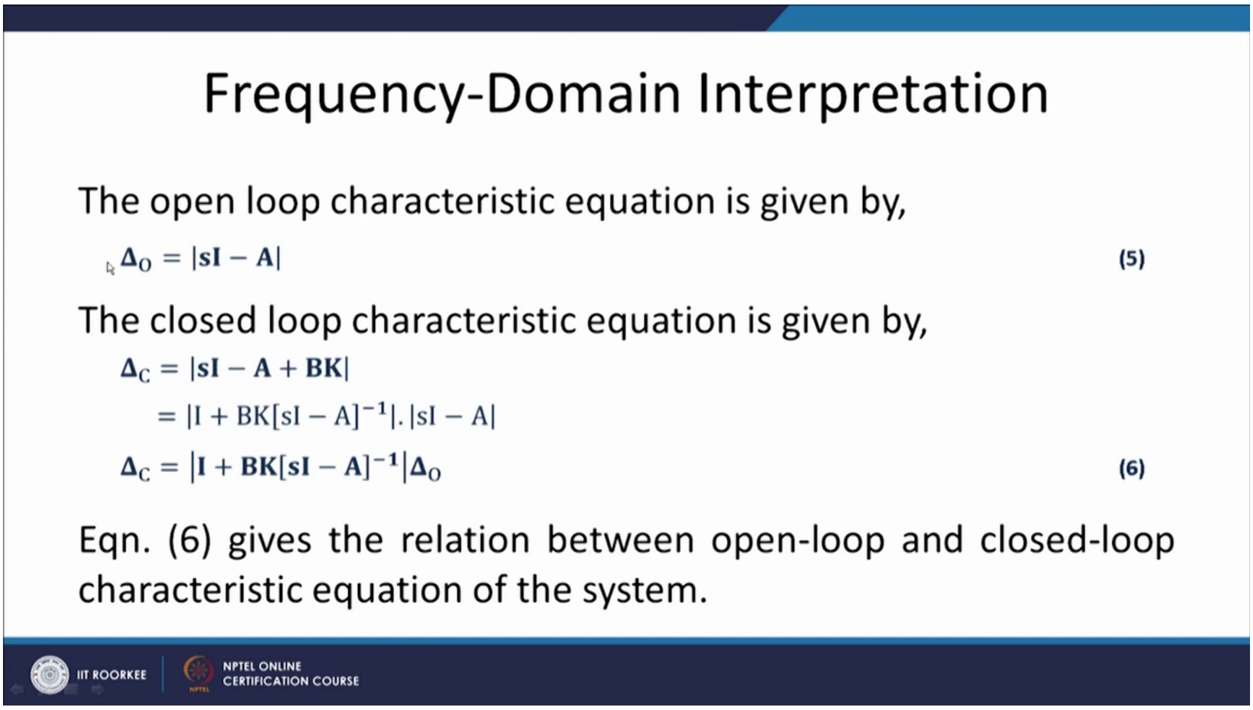

Now, if we see the characteristics equation of my close loop and the open loop system.

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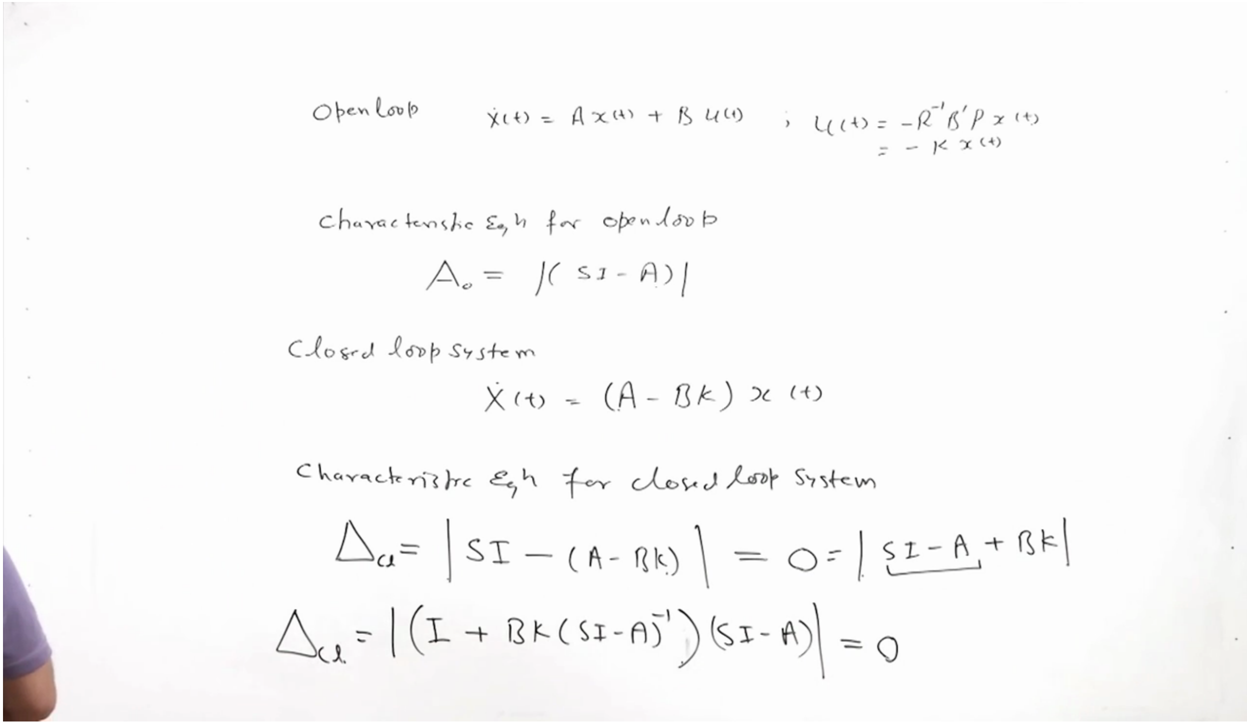

My open loop system is X dot t equal to a which is constant X t plus B U t and U t is nothing but minus R inverse B prime P x of t, and which nothing but we are saying this is U equal to minus K X of t. So, my characteristic equation for open loop system which I mention as the A naught it is nothing but; my SI minus, determinant of SI minus A.

So, this is the characteristic equation for the open loop. And what is my close loop system? My close loop system is X dot t if I will substitute U in to this, this is nothing but A minus B K X t. For close loop system my characteristic equation is nothing but SI minus in place of a I have my A minus B k matrix. So, determinant of this must be equal to 0.

So, that we are saying here if this is the open loop characteristics equation determinant of SI minus A and for close loop this is delta c this is SI minus A plus B k or we can write SI minus A minus B k in bracket. So, this equation can be written in the form as determinant of SI minus A the whole determinant must be 0. And this is my delta close loop.

So, these closes loop characteristic equation. So, what we have done? If I will explain this as SI minus A plus B k, so I am taking these two as outside. So, if will take SI minus A outside. So, this is I plus B k SI minus A inverse. So if I multiplying this, this is SI minus A this would become unity and I get the SI minus A plus BK. So, this close loop equation I can further simply.

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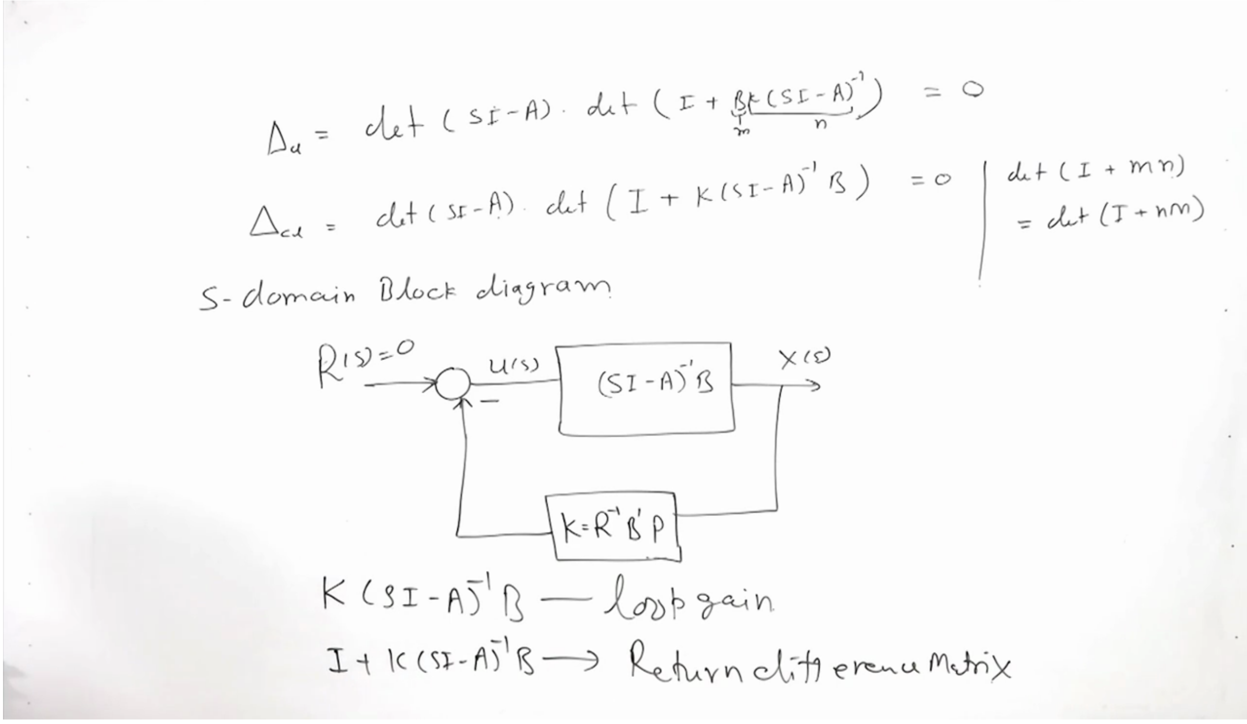

So, I am writing this close loop characteristic equation as: so in place of state line determinant I am writing the determinant of say- what is this? This is the determinant of this matrix and determinant of this matrix.

If I will explain this I can write determinant of SI minus A multiplied with the determinant of I plus B k SI minus A inverse SI minus A inverse, and this is nothing equal to my 0. So, for the second determinant we use the identity determinant I plus m n is equal to determinant I plus n m. So, if you will use this identity with considering k SI minus A inverse as my m and B as m. So, in this if we will apply this identity determinant I plus m n. So, I can write my close loop characteristic equation as determinant SI minus A multiplied with determinant I plus k SI minus A inverse B: this is equal to 0.

So, this close loop matrix we are writing as determinant of I plus B k SI minus A inverse into SI or this is my close loop characteristic equation. Where, delta 0 we are seeing SI minus A. So, this B k we have written as k SI minus A inverse B. So, this equation here delta 0 is nothing but my SI minus A this I am expanding as determinant SI minus A determinant I plus k SI minus A inverse B.

Now if we will see the s domain block diagram of the system. So, this is my plant and plant is nothing but my SI minus A inverse B. So, this is my U s this will be nothing but my X s, here we are adding the k which is nothing but R inverse B prime P, this is given here with negative sign and sorry; this we can write as capital R s which is nothing but we have taken as 0. In LQR system my R s is 0.

So, I can represent the s domain block diagram by this.

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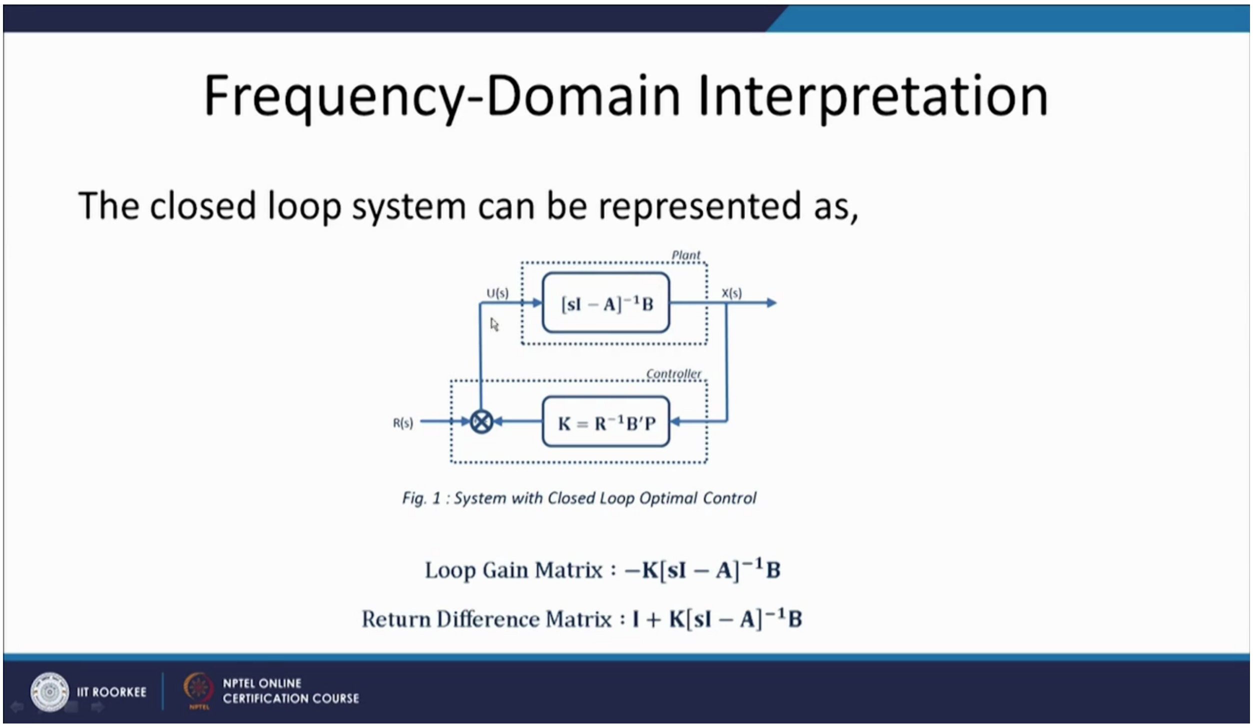

So, my overall block diagram is s I minus A inverse B, K multiplied with the K which is giving me nothing but the R s value. So, this is the relation between R s and x s. So, my if we will see K s I minus A inverse B, if I am multiplying this nothing but giving me the loop gain and if I am writing the characteristic equation this is I plus K s I minus A inverse B this is known as my return difference matrix. So, in frequency domain I can represent my LQR system as given by this block diagram, which is similar to as given here with this block diagram. Where K S I minus A inverse B is nothing but my loop gain, and if I am writing I plus k SI minus A inverse B this is my return difference matrix.

So, this is the representation of LQR system in frequency domain we can represent. Our objective here is to determine the gain and the phase margin of this system. So, system is represented here what will my gain margin and the phase margin that we are trying to find out. So, this we can prove using our A R E. So, consider arithmetic Riccati equation what is that this is my P A.

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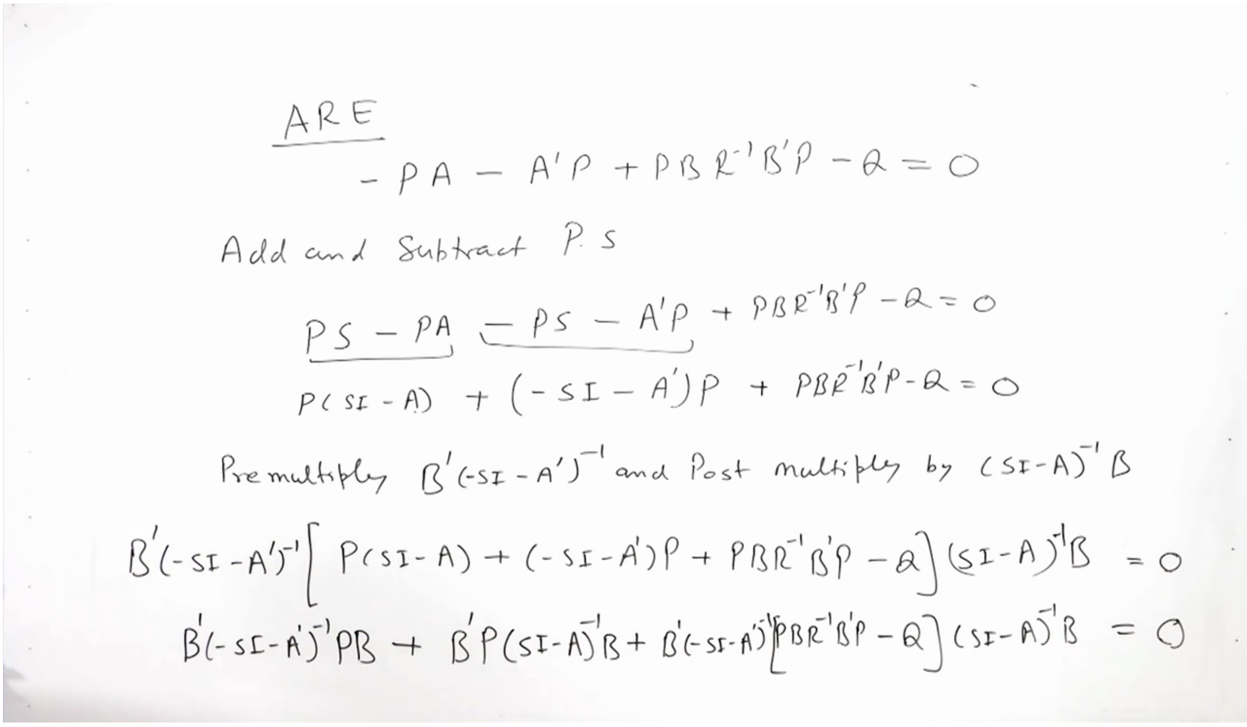

So, I write this in the form minus PA minus A prime P plus PBR inverse B prime P minus Q equal to 0, so nothing but if I will take the minus kalman. So, this will be in the standard form PA plus A prime P minus PBR inverse B prime P plus Q equal to 0. So, this is my matrix; sorry arithmetic Riccati equation.

So, in this we will add and subtract P into S. So, what we are doing we are adding P S and subtracting P S. Basically we are doing P S minus PA, and this P S I am subtracting here P S minus A prime P and this I will kept as such P B, R inverse B prime P minus Q S 0. So, if I will take P common from these two term. So, this is P SI minus A and from here I will write this as minus s I because s is a scalar function. So, I can write this as s P

S or P S minus A prime and this P. I will write on the right hand side. So, considering these two term including the negative this I am writing as P S minus A and this I am writing as minus SI minus A prime P, and this I will write as such PBR inverse B prime P minus Q equal to 0.

So, in this equation we are pre multiply with and post multiply by SI minus A inverse B. Now this equation I am pre multiplying with B prime minus SI minus A prime inverse and post multiplying with SI minus A inverse B, this means this whole equation is pre multiplied by B prime minus SI minus A prime inverse, this is P SI minus A plus minus SI minus A prime P plus PBR inverse B prime P minus Q and post multiply with SI minus A inverse B.

So, now just pre multiply and post multiply this with if I will expand this what I will get? B prime minus SI minus A prime inverse P, this SI minus A SI minus A inverse sorry SI minus A inverse will give me the I and this multiplied with B. So, this is B prime minus SI minus A prime inverse P B, with second term if I will multiply. So, minus SI minus A inverse this will give me plus B prime P this two term will give me B prime P and post multiply with SI minus A inverse B. So, these two term I will get from these two term and here, this is multiplied with B prime minus SI minus A prime inverse PBR inverse B prime P minus Q with SI minus A inverse B. So, by expanding this we are getting this term.

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$(sI-A)^{-1}$ is the state transition Matrix — $\phi(s)$

$(-sI-A')^{-1} \longrightarrow \phi'(-s)$

$$B'\phi'(-s)PB + B'P\phi(s)B + B'\phi'(-s)PBR^{-1}B'P\phi(s)B = B'\phi'(-s)Q\phi(s)B$$

Add "R" on both side.

$$B'\phi'(-s)PB + B'P\phi(s)B + B'\phi'(-s)PBR^{-1}B'P\phi(s)B + R = B'\phi'(-s)Q\phi(s)B + R$$

$$K = R^{-1}B'P \Rightarrow RK = B'P \Rightarrow K'R = PB$$

$$B'\phi'(-s)K'R + RK\phi(s)B + B'\phi'(-s)K'\underline{RR^{-1}}RK\phi(s)B + R = B'\phi'(-s)Q\phi(s)B + R$$

$$B'\phi'(-s)K'R + RK\phi(s)B + B'\phi'(-s)K'RK\phi(s)B + R = B'\phi'(-s)Q\phi(s)B + R$$

What is SI minus A inverse? SI minus A inverse is the transition matrix, is the state transition matrix and this we represent as phi s and so therefore, minus SI minus A prime inverse, I can represent as phi prime minus S. So, I will use for SI minus A inverse as phi S which is my state transition matrix and minus SI minus A prime inverse this is the transpose conjugate transpose of my state transition matrix.

So, in this time if I will write I can write this as B prime phi prime S P B plus B prime P phi S, B plus if I am writing this as B prime Phi prime minus S PBR inverse B prime P phi s B and this I am writing Q on the other side as B prime Phi prime S Q this is phi S P. So, by expanding this I am writing Q on right hand side. In this equation add R on both the side by adding R what actually we will get? We get this P B plus B prime P phi s B plus B prime Phi prime minus s PBR inverse P prime P phi s B plus R and similarly on this side B prime Phi prime minus s Q phi s B plus R.

So, now the whole equation I can represent by this. So, now, we know K equal to minus my K is R inverse B prime P or I can write this as R k equal to B prime P and K prime R equal to P B. So, I can use these relations because I know the controller gain K is nothing but my R inverse B prime P, if I will pre multiply this equation by R. So, this is nothing but R k equal to B prime P if I will take the transpose of this k prime R equal to P B. So, in this equation I will write P B as K prime R. So, B prime Phi prime minus s K prime R for the first term, the second term is B prime P which I am writing as R k phi s B plus B prime Phi prime minus s.

Now, see P B is in this equation P B is k prime R, k prime R I am writing for PBR inverse as such B prime P is R k and this I will write as such phi s B plus R and this side is B prime Q phi s B plus (Refer Time: 00:00); now in this R r inverse will become i. So, this is k prime R k. So, B prime Phi prime minus S K prime R plus R k phi s B plus B prime Phi prime minus s and this I can write k prime R k phi s B phi s B plus R and this is nothing but equal to my B prime phi prime minus s Q phi s B plus R.

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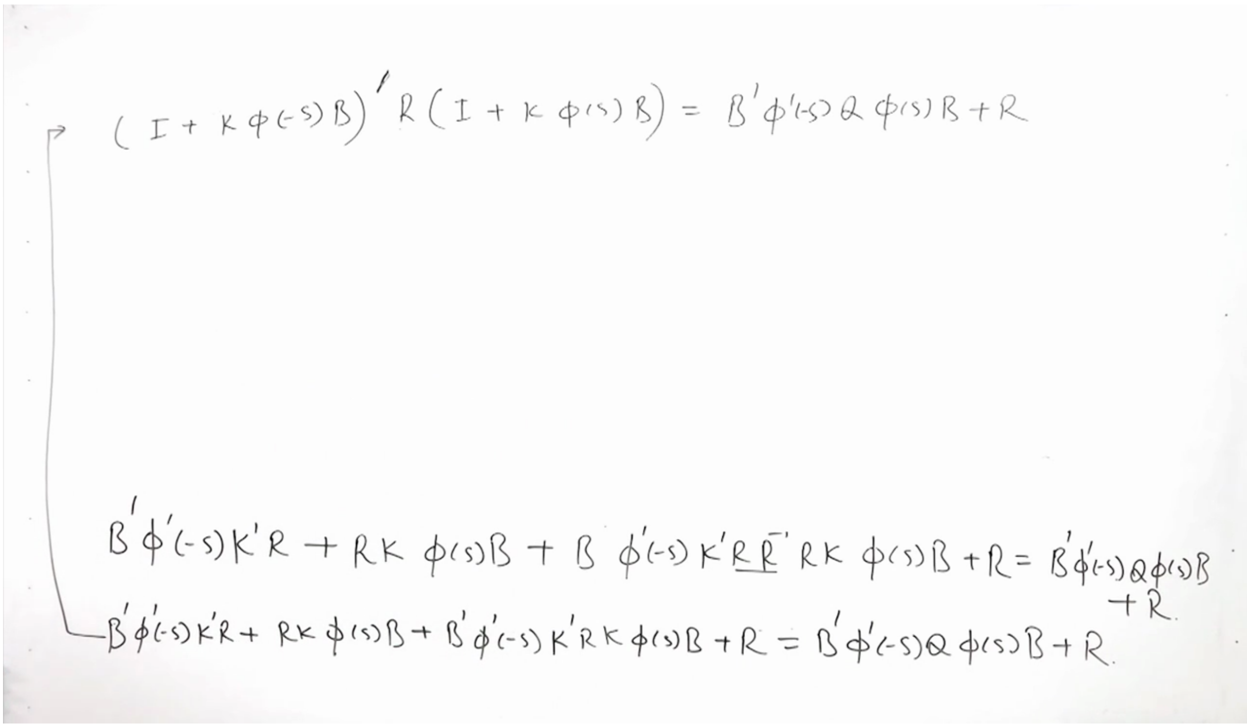

Now, see in this if I will consider the left hand side. So, this equation I can write in the form as I plus sorry phi prime minus s Q phi s B plus R. So, this equation I am writing as I plus k phi minus s B prime B whole transpose, this is the transpose multiplied with the R I plus k phi s B. So, I phi will take the transpose and explain this I will get the same term which I have on this write hand side.

So, my overall system can be represented in this particular form, which we can see here this is nothing but I plus K phi minus s B whole transpose R multiplied with I plus K phi s B equal to B prime Phi minus s.

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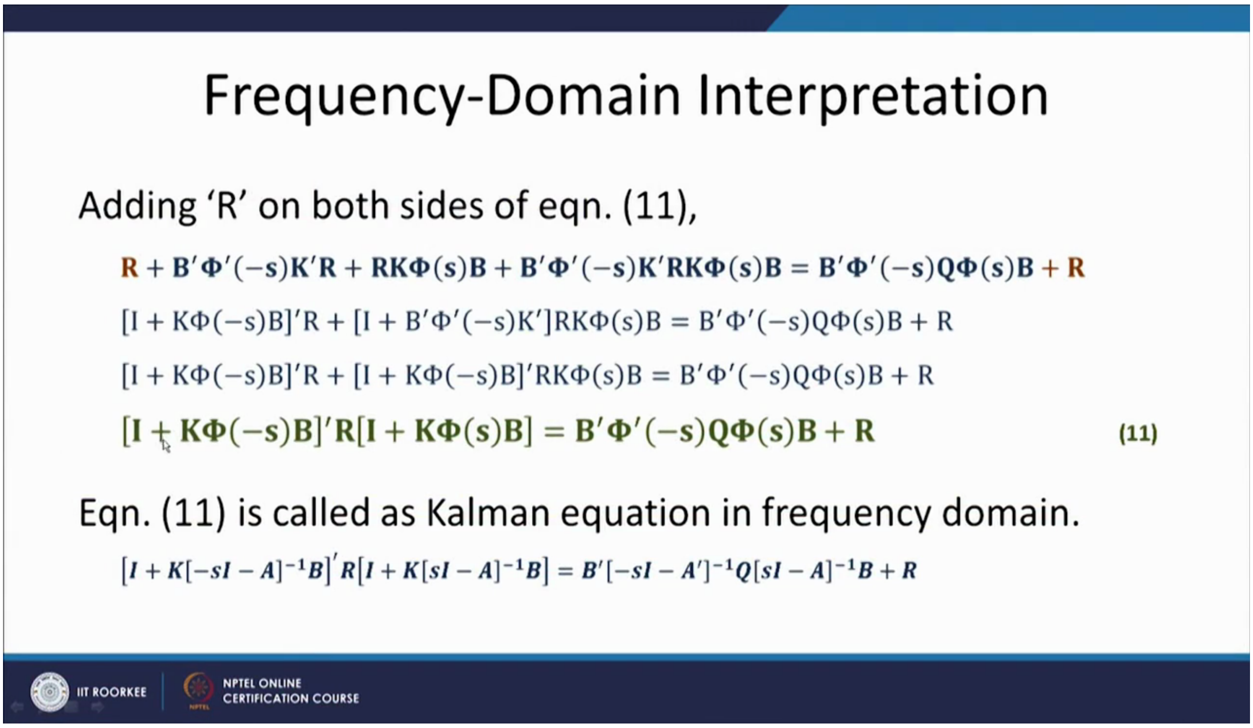

So, this equation is called the Kalman equation in the frequency domain, for LQR system this Kalman equation we can find using our a R e with the a R e we have got our Kalman equation in the frequency domain, and what is my phi minus s this is minus SI minus A inverse phi s is SI minus A inverse.

So, another way to represent this equation is I plus K minus SI minus A inverse B R multiplied with the I plus K SI minus A inverse B equal to B prime minus SI minus A prime inverse which is phi prime minus s multiplied with Q SI minus A inverse B plus R. Now in the next class we will see that how this equation can be utilized to determine the gain margin and the phase margin of a LQR system.

So this class we stop here. And further we will continue our discussion in the next class to determine the gain margin and the phase margin of this LQR system.

Thank you very much.