Optimal Control
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Lecture - 24
Analytical Solution of Matrix Differential Riccati Equation
(Similarity Transformation Approach) (Continued)

Welcome class. Today's lecture we are continuing from our previous lecture discussion which was on the Analytical Solution of Matrix Differential Riccati Equation. So, we are utilizing the eigenvector and the eigenvalue concept to determine the solution of the matrix differential Riccati equation.

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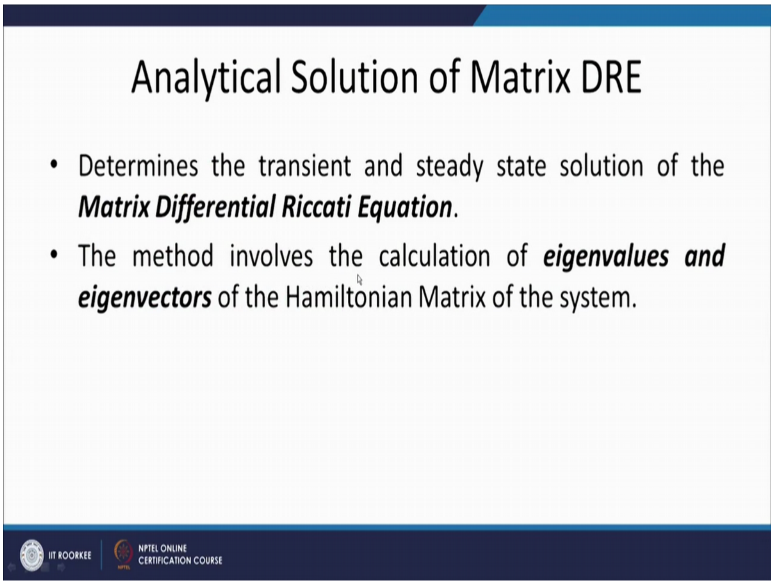

So, this method as we have discussed determine the transient and the study state solution of the matrix differential Riccati equation and this method involves eigenvalue and the eigenvectors of the Hamiltonian Matrix of the system.

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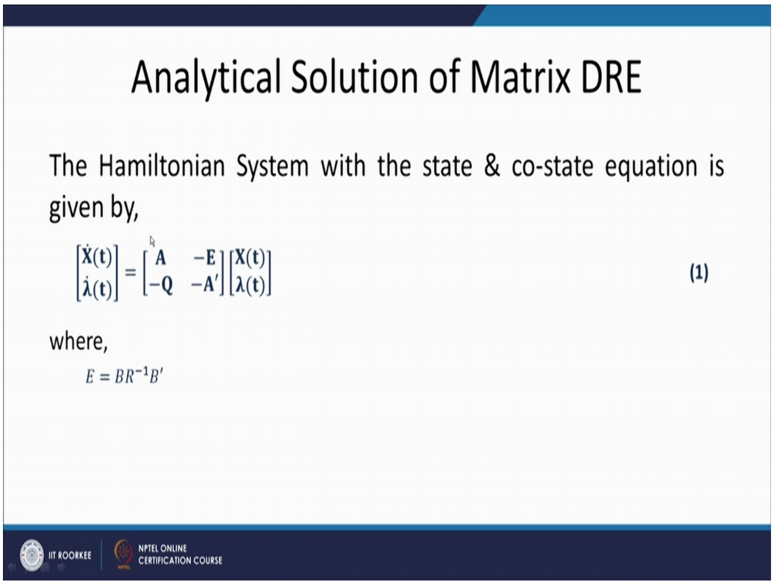

As we know we can define the Hamiltonian Matrix for a system which is time varying and linear. So, for a linear time varying system we are defining the Hamiltonian System which is nothing but this X dot t is my state equation and lambda dot t is my costate equation. So, if I am clubbing this state and the costate equation I am getting the Hamiltonian Matrix in which E is nothing but my BR inverse B prime.

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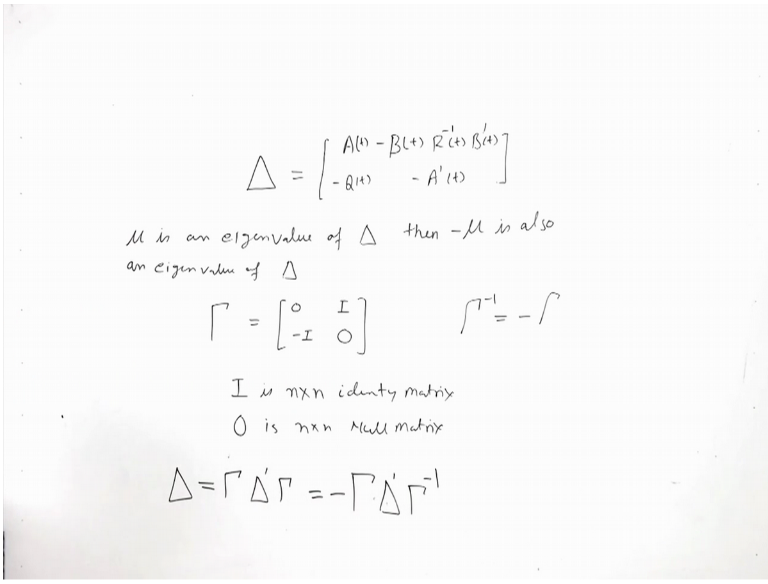

So, if we will see my Hamiltonian Matrix is A minus (Refer Time: 01:54) A t B R inverse B transpose minus A t minus A transpose. This is my Hamiltonian Matrix. For this matrix if you will say mu is an eigenvalue of delta then minus mu is also an eigenvalue of delta.

This statement is true for delta that we will prove just now, and to prove this we consider a transformation tau which is I minus I 0. And this tau will have the property, the tau inverse is nothing but minus tau which we can easily prove if tau is have they structure like 0 I minus I 0 then tau inverse is nothing but minus tau. In this case my I is n cross n identity matrix and 0 is nothing but n cross n null matrix. So we are considering the tau, tau inverse is minus tau. If we will use the transformation tau delta inverse tau is nothing but giving me the delta.

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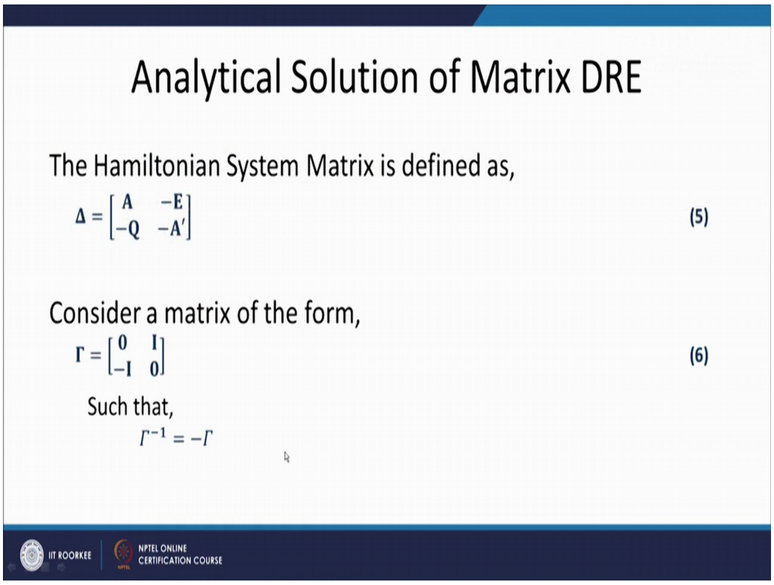

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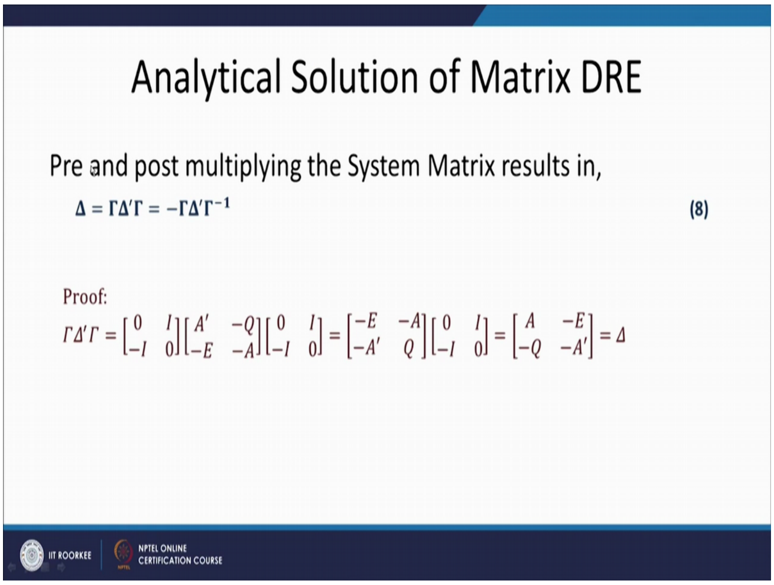

My Hamiltonian Matrix is tau lambda prime tau inverse; tau inverse we know the tau so simply I can also write this as minus tau delta prime tau. And this you can see as we have proved this my tau is 0 I minus I 0, this is representing my delta I did a mistake, if pre and post multiply with this then I get minus tau inverse. So, in this case delta is tau; delta prime tau and tau I can write as the tau inverse minus of tau inverse. So, I can represent this when this following manner.

So, if I will multiply this we can see that tau delta prime tau is nothing but by delta. So, with this transformation our objective is to prove that if mu is the eigenvalue of delta then minus mu is also an eigenvalue of delta.

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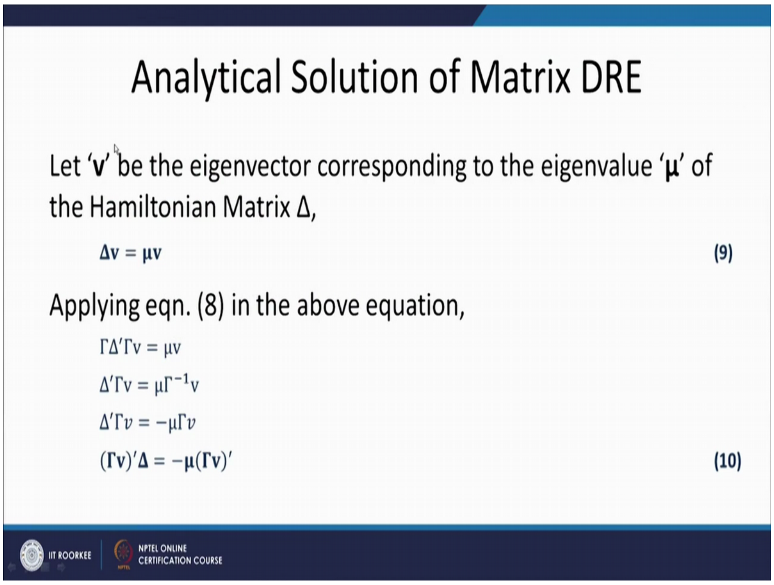

So, what we are considering? We are considering v to be n eigenvector corresponding to mu of Hamiltonian Matrix delta. So, if v is the eigenvector and mu is the eigenvalue of matrix delta then by definition of eigenvector.

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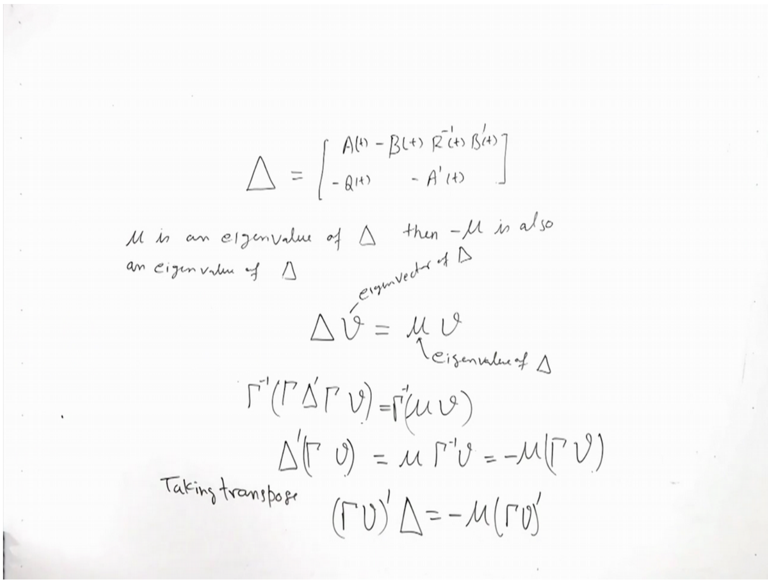

We will have delta v as mu v, where v is the eigenvector of delta and mu is the eigenvalue of delta. So, this is by the definition of the eigenvector we can write del mu equal to mu v.

Now, this delta we are replacing by tau delta prime tau. So, this is this delta I am replacing by tau delta prime tau v and this as such mu v. This I will pre multiply with the delta inverse this whole equation are multiplying with delta inverse here; this sorry tau inverse on both the side. So, my this side will give me lambda prime tau v equal to so mu is the scalar, so I can write it outside tau inverse v. And this whole I can write, so what I am doing just club these two. And tau inverse I can write as minus mu tau v. So, the tau inverse is minus of tau. And I will also club and consider this.

I will take the transpose of this equation. Now, taking transpose what we get? Del v prime, sorry tau v prime delta equal to minus mu because it is a scalar and tau v prime. So, we can treat this tau v as my another eigenvector of delta. So, by the definition now minus mu is nothing but an eigenvalue of y delta.

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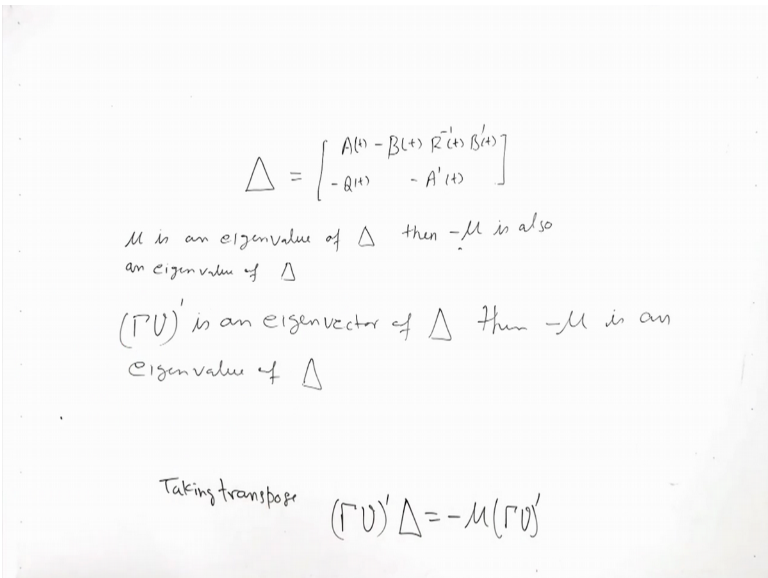

So, in this equation if we say tau v prime is an eigenvector of delta then minus mu is an eigenvalue of delta. So, we can say if mu is an eigenvalue of delta then minus mu is also an eigenvalue of delta. So, this is proved here.

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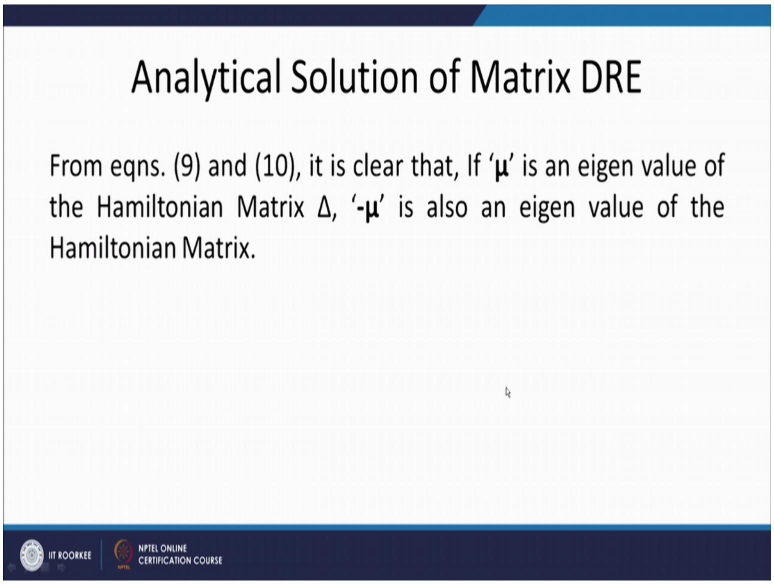

So, pre multiplying by tau inverse I get delta prime tau v as mu tau inverse v; tau inverse we are replacing by the minus tau. So, we are treating this as my eigenvector. And can state from 9 and 10; if mu is an eigenvalue of Hamiltonian Matrix delta then minus mu is also an eigenvalue of the Hamiltonian Matrix delta. Meaning here is: the Hamiltonian Matrix will have one set of the eigenvalues and another set of the eigenvalue is just opposite sign of the first set.

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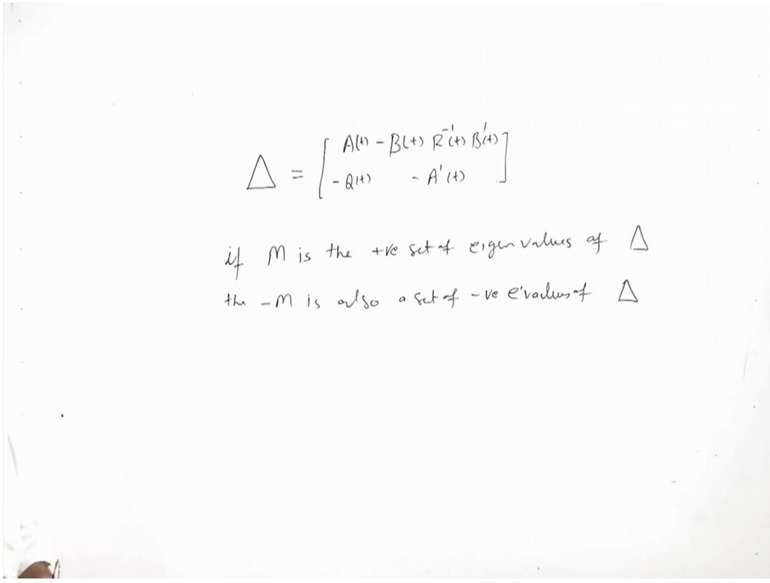

So, we can say- if M is the positive set of eigenvalues of delta then minus M is also a set of, I can say the negative eigenvalues of delta.

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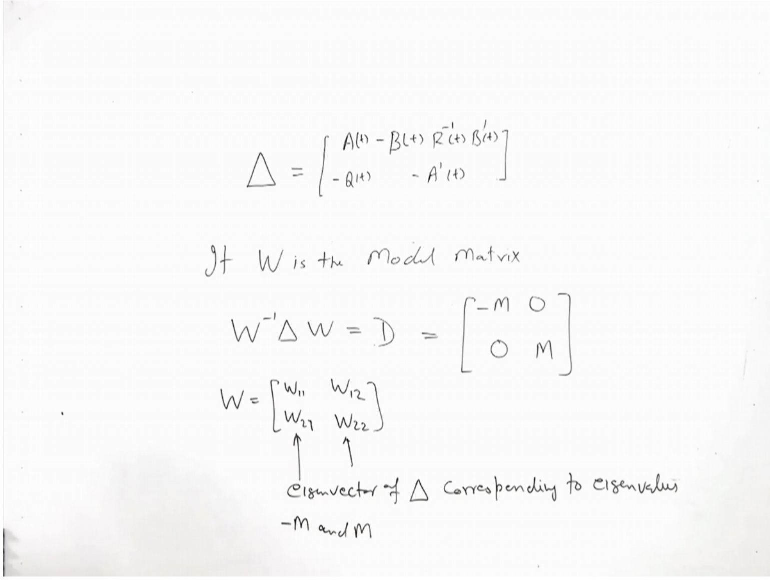

So, if I will use the similarity transformation on delta I can convert this delta into the diagonal set of eigenvalues in Jordan canonical form. So if, W is the model matrix then the transformation W inverse delta W is D. And D is nothing but can convert delta matrix into the diagonal form. Where, M is my one set of the eigenvalues then minus M will be another set of my eigenvalues.

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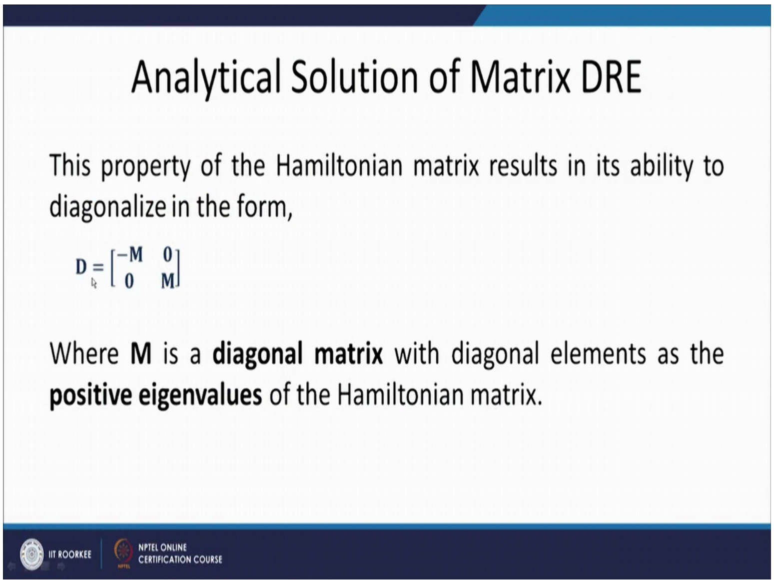

So, we are considering this D which is nothing but the set of eigenvalues of delta in diagonal form, where if M is one set then another set will be the minus M. So, they can be placed in the diagonal form using the simulative transformation where W is my model matrix.

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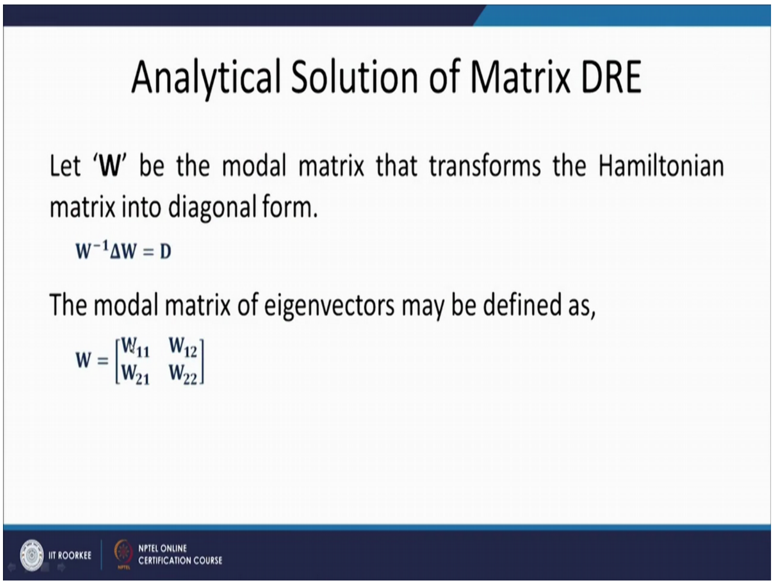

So, we can consider this model matrix as W 11 W 12, W 21 W 22: in this the first two terms, sorry the each column will represent the eigenvectors of delta. So, this means if we are considering W 21 and W 22. So, each column represent nothing but these are the eigenvectors of delta corresponding to eigenvalues M and sorry minus M and M. So, if I say if these are corresponding to minus M eigenvalues, this will correspond to the M eigenvalues. So, corresponding to each eigenvalue we can find out the eigenvector corresponding to that particular eigenvalue and this can be arranged in the following manner: let the first columns represent the negative eigenvalues, the next columns will represent the another eigenvalues.

By this we can generate our model matrix W which with the transformation as W inverse delta W will give us the diagonal form of the delta matrix in which all diagonal elements are nothing but the eigenvalues of delta. So, this is my transformation.

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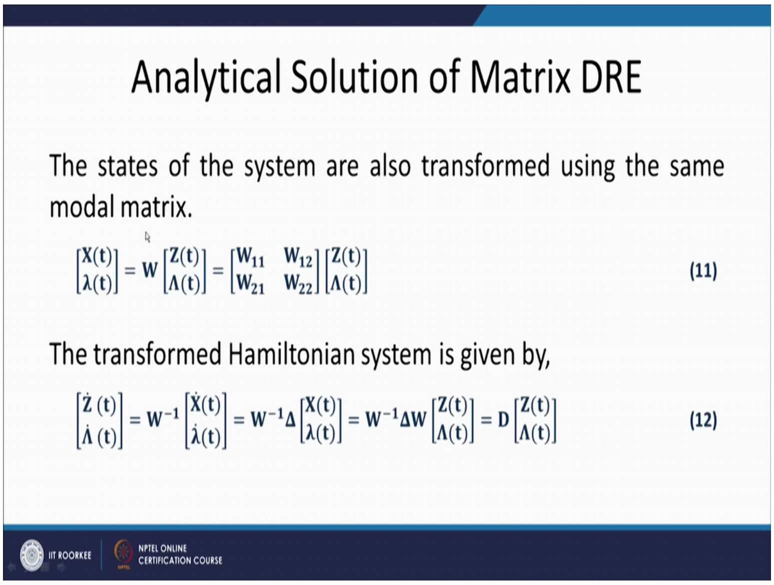

Let us take the simulative transformation as X t lambda t as W Z t del t. So, this symbol I am calling that del, and Hamiltonian Matrix I am calling the delta. So, this is del t so I have the two new state Z t and del t which is related to x and lambda s with model matrix W So, we keep this equation 11 as X t lambda t W 11 W 12, W 21 W 22 Z del t.

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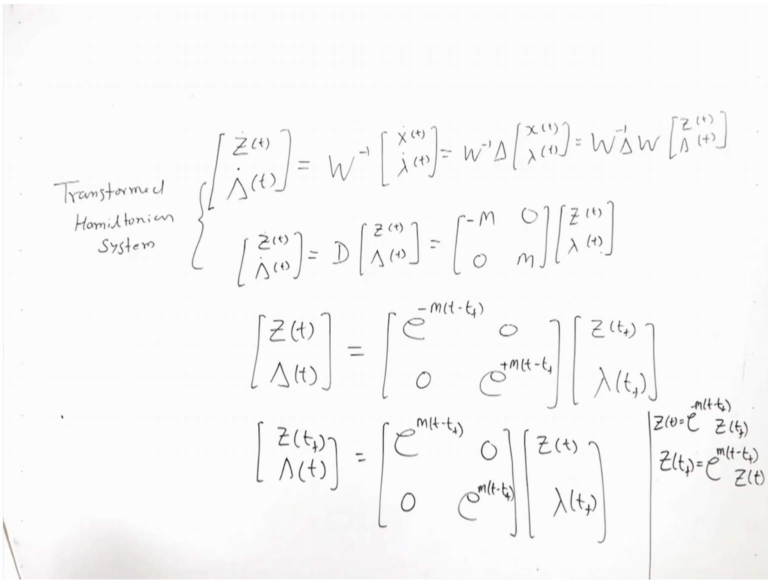

So, the transform Hamiltonian System will be what? This means my transformed Hamiltonian System I am representing as Z dot t del dot t, what I can write for this as my transformation is X t by lambda t equal to W Z t del t. So, Z dot t del dot t I can simply write as W inverse X dot t lambda dot t. So, this I am writing as X dot t lambda dot t- X lambda is nothing but my this is delta X t lambda t. And again we will apply the transformation so this is nothing but W inverse delta W Z t del t.

Or we are simply writing this as Z dot t del dot t which is nothing but my D diagonal matrix Z t del t and D is what as we have know this is nothing but my diagonal matrix, so my transformed Hamiltonian System so this is nothing but my transformed Hamiltonian System. This system we are simply representing as Z t del t as minus M 0 0 M Z t del t where M is nothing but my all diagonal term.

So, if I know this then I can directly write the solution of this equation, which is nothing but E to the power minus M t, but I know the some terminal conditions. So, my terminal condition may be the initial conditions or the end point condition. Normally lambda t f is my p t f X of t f and p t f is F of t f. So, my terminal condition if it is known to me so I will write the solution of this equation in terms of the final time this means I am writing the solution of this as Z t del t and this I can simply write as minus t minus t f in terms of the terminal time 0 0; sorry this is plus M t minus t f this is my Z of t f lambda of t f, that we can have

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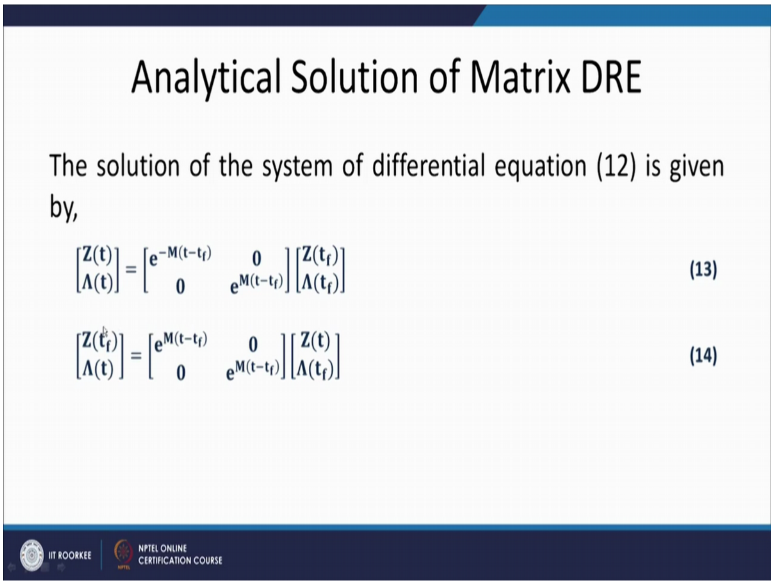

So, we can see it here: we can write this solution in terms of the Z t f and del t f. So, this can further be transformed in. So, simply by manipulating the variables on left and the right side I can write this solution in terms of the Z of t f del of t f. Purposefully we are doing this, later we will utilize this set- I am want to write this in Z of t f del of t. So, this I want to write in terms of Z t and lambda t f. So, I can see my last equation remain the same, but the first equation if I will write this is nothing but my Z t equal to E to the power minus M t minus t f Z of t f, and now we are writing Z of t f in terms of Z so this is simply Z t f as E to the power M t minus t f Z of t. So, my this term will be changed to E to the power M t minus t f and this term remains same E to the power M t minus t f.

So, this solution I am writing in terms of the equation number 14: Z of t f del of t f in terms of the Z of t del of t f. So, we will keep this result and see next.

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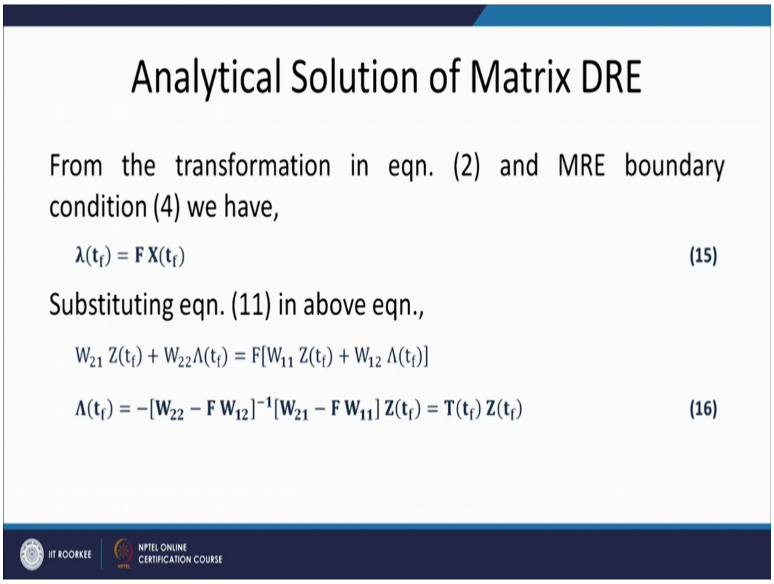

We know from the terminal condition my lambda of t f is nothing but equal to F of t f X of t f.

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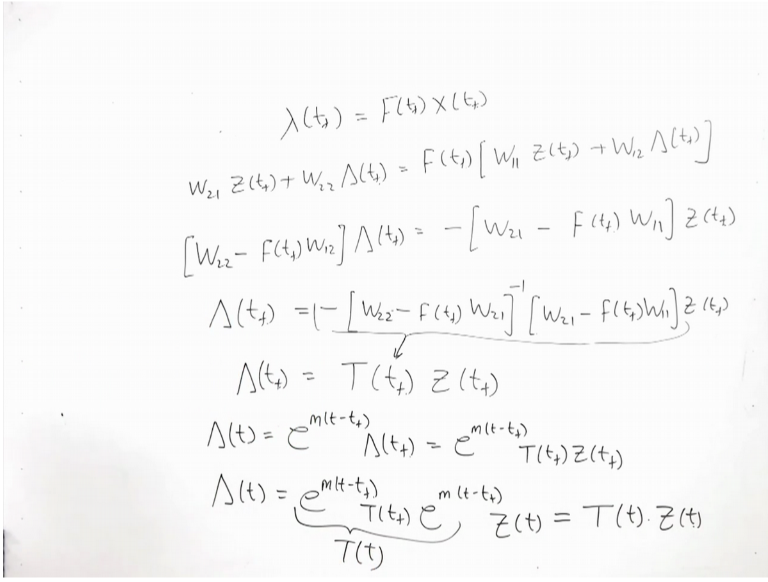

So, we are using the transformation lambda t f as F of t f X of t f.

Now see: from my equation 11 I can write x and lambda in terms of the W 11 W 12, W 21 W 22 with Z and del. So, if I will write at the terminal point X of t f is nothing but W 11, Z of t f W 12 del of t f. And similarly lambda t f I can write as W 21, Z of t f W 22 del of t f. So, in this transformation I am replacing this del as; this is W 21 plus W 22 del t f. So, first I am writing the second equation del t f W 21 Z t f W 22 del t f. Then similarly x I can also replace, so this is F of t f and x will be W 11 Z t f plus W 12 del t f.

So, both side I can collect the terms with del t f and Z t f my objective is: this I am writing from equation 11 to represent del t f in terms of the Z of t f. So, from this equation I am collecting the terms with del t f first. So, this is W 22 minus F of t f W 12 and this is nothing with del t f and this is sorry; and this side we will collect the terms related to my Z of t f. So, this I am writing as if I will take it here W 21 minus F of t f W 11 Z of t f.

Now, pre multiplying this with the inverse of this, so I can simply write del t f as minus pre multiplying with this W 22 minus F of t f W 21 inverse multiplied with W 21 minus F of t f W 11 Z of t f. And this whole I am writing as T of t f Z of t f, where this whole along with the negative sign this we are taking as T of t f. So, this relation gives me a condition which is equivalent to my; see this is relating lambda with X through F of t f and this relation is relating my the transformed state variable now which is del T t f and the Z t f. So, this is giving me the relation between these two.

So, this del of t f I am representing simply as T t f Z of t f where T t f is nothing but minus this whole term- I am taking as the T of t f.

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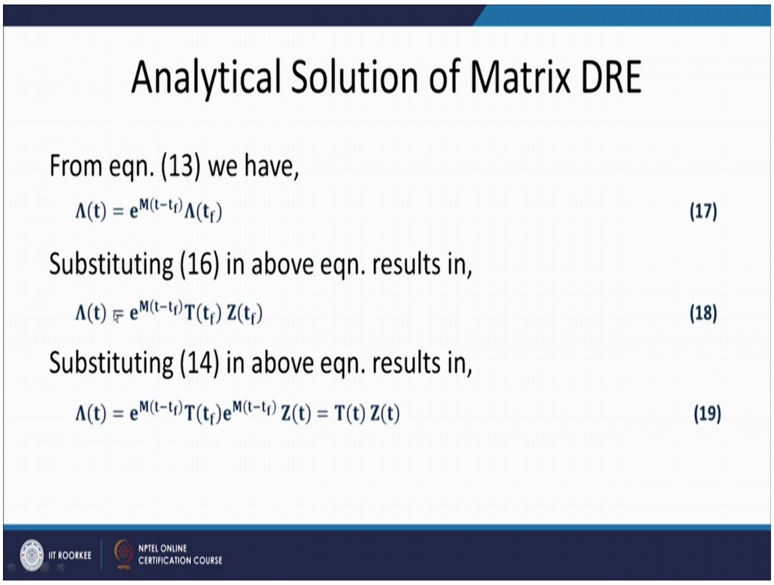

Now, see by this relation we are writing the value of del t. So, we can either 13 or 14 I am writing this del t as E to the power M t minus t f del of t f. So, this means now my relation is in between del of t f and Z of t f. So, I will write first del t as the next step I am writing this del t which is M t minus t f with del t f. So, this from equation 13 or 14 we have written from this equation we have written and del t f from the previous relation we know is t f t f Z of t f. So, E to the power M t minus t f and del t f I am replacing with T t f with Z t f.

And this Z of t f again from my equation 13 and 14; so this Z of t f I can represent in terms of the z t. So, this del t I can simply write as E to the power M t minus t f T of t f E to the power M t minus t f and this Z t f is in terms of my Z t. At this I simply representing as T of t into Z t; like this is the relation at the terminal point, this is the relation at any given point where T t is nothing but given by this expression which is T of t.

Now, we have the two transformation one is delta t f as T f t f Z of t f where T of t f is given by this expression and another is del t is T of t of Z t where T of t is given by this expression. So, we have established a relation between del t and Z t as T of t, where T of t is represented by my E to the power M t minus t f T of t f E to the power M t minus t f.

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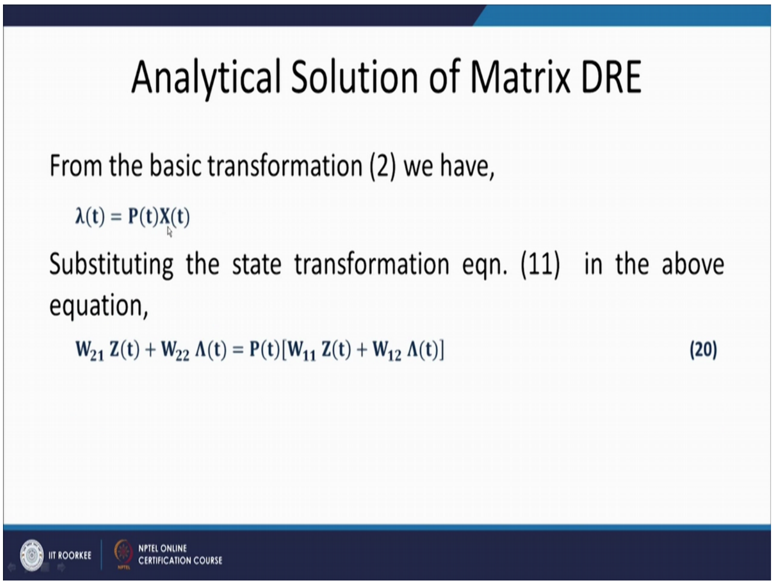

So, another next again we will take the transformation as we know lambda t equal to p t x t.

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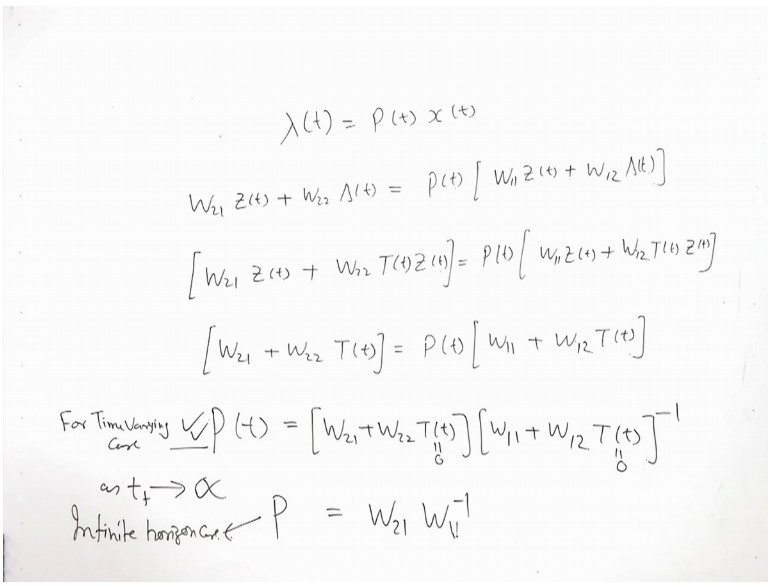

So, in this case as we know we can represent this lambda t as p of t X of t. So, this is the relation between lambda t and X t at any given time key. Again from equation number 11 which giving me the expansion of I can represent my lambda t in terms of my transformed state variable z and del. So, first I will write for the lambda t as W 21 Z t plus W 22 del t and this is p t and X t is as I can see from here my X t is W 11 j t W 12 del t plus W 21 del t.

In this expression I know what is my relation between del and Z t; that previously we have proven that my del t is nothing but T of t n z t.

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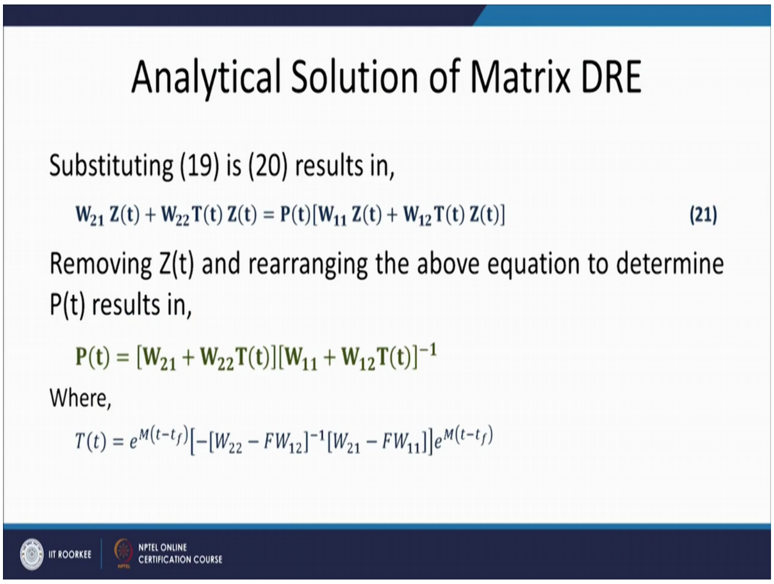

So, I will replace transform this del t, so W 21 Z t plus W 22 this I am transforming into T of t Z of t. Similarly on this side, I am also transforming this del t in terms of the Z t plus W 21 T of t and Z of t. Now Z of t from both side I can take it out and can cancel out. So, this is nothing but I am left with W 21 plus W 22 T of t as p of t W 11 plus W 21 T of t, and p t can be represented as if I will post multiply the whole with the inverse of this term. So, p t can simply be taken as W 21 W 22 T of t W 21 plus W; sorry, this is W 12 this is W 12, this W 12 T of t inverse.

So, by this we can find out the Riccati matrix coefficient p t with the help of my model matrix which is nothing but the set of eigenvectors which we are dividing into the or we are partitioning into; partitioning at sorry, this W 11 we are taking the inverse on this side W 12 inverse. We are partitioning is W 11 W 12, W 21, W 22. So, if we can partition this model matrix we can directly determine what will be my matrix Riccati coefficient. So, we are writing this p t as W 21 W 22 T f t, W 11 W 12 T f t whole inverse of this where T of t is we are representing by this term.

Now, in this you will you will see as t f approaches to infinity; so this is for time varying case. As t f approaches to infinity I can see this here my E to the power M t minus t f this will approach to 0. So, this means my t f t this term will become 0 as t approaches to infinity, this means in this my this term is going to be 0 and this term is going to be 0. So, for infinite horizon my Riccati coefficient matrix is nothing but W 21 W 11 and also this will be a constant matrix P.

So, this is for the time varying case, so this result will be for the infinite horizon case. So, by this approach we can find out the analytical solution of matrix differential equation either for the time varying case or for the time invarient case- for the time varying, finite time horizon and also for the infinite time horizon.

So, this lecture we stop here, and in the next lecture we will discuss about the frequency domain interpretation of the LQR system.

Thank you very much.