Optimal Control
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Lecture - 23
Analytical Solution of Matrix Differential Riccati Equation
(Similarity Transformation Approach)

Welcome class. So, in the previous class, we were discussing about the Analytical Solution of the Matrix Differential Riccati Equation. So, we have seen one approach in which we use the State Transition Matrix to determine the solution of Matrix Differential Riccati Equation.

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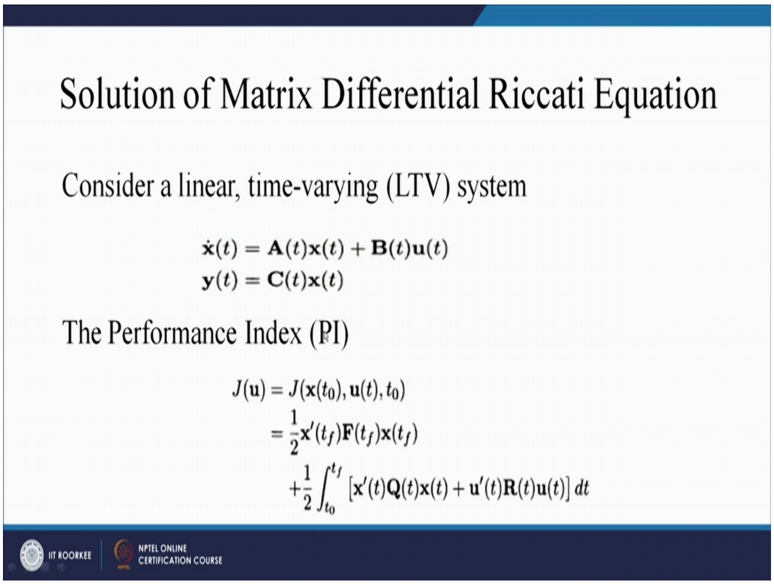

Our approach was for a given system x dot equal to x plus PI equal to C x. We are finding out the u t by minimizing the performance index as half of x prime t f f of t f x of t f plus half of integral t 0 to t f x prime t Q t x t plus u prime t R t u t. So, we have to minimize this performance index subjected to the plant. Objective is to find out the minimum value of the u.

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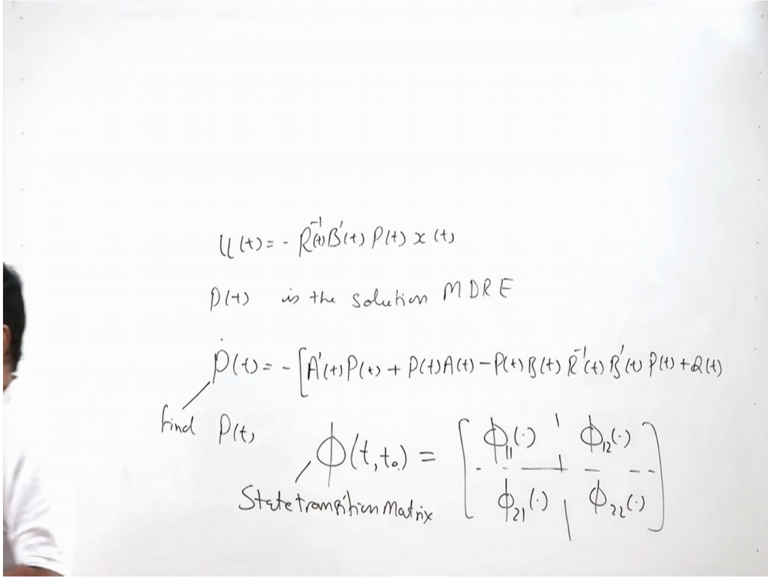

So, in finding the u basically our u is. So, u t we determine as minus R inverse function of t B prime P t x t. So, in finding the u, we have to find out the P t which is the solution of my Matrix Differential Riccati Equation, given as P dot t minus A transpose P plus P A minus P B R inverse B prime P t plus Q t. So, objective is to find P t. So, in this formation we start our solution with the Hamiltonian system given as x dot t, A t minus E t minus Q t minus A prime t x t lambda t.

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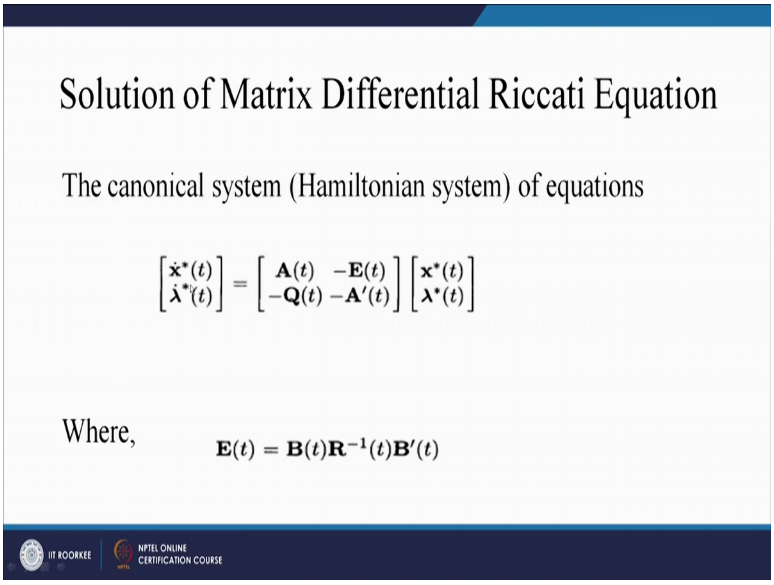

So, if we will see this is nothing, but a set of first order differential equations, similarly as given as the x dot equal to A x. So, what we do? We find out the State Transition Matrix of Hamiltonian matrix and then, we make the partition of this as we have seen in the previous class.

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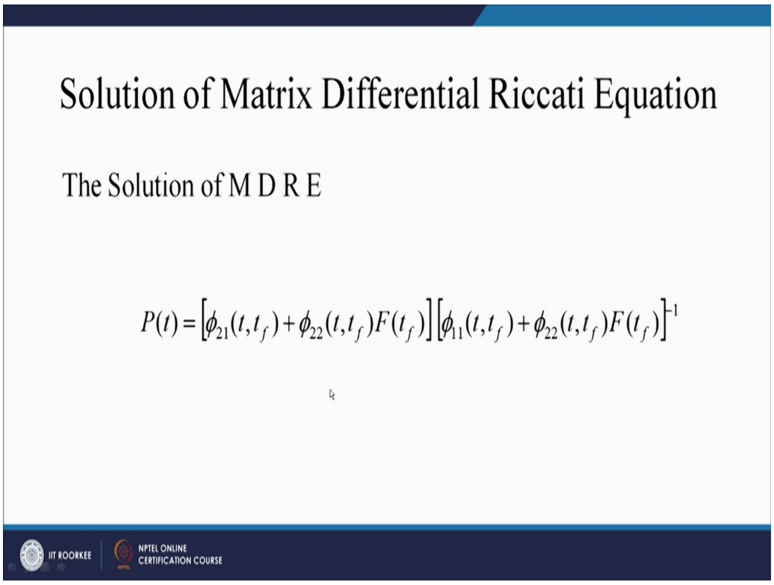

Then, we find the P t is nothing, but equal to phi 2 1 phi 2 2 phi 1 1 phi 2 2 and f of t f. What are phi 1 matrices? So, phi t t 0 is my State Transition Matrix and this we partitioned as phi 1 1 phi 1 2 phi 2 1 and phi 2 2. So, if we partition this matrix in this form, then with the help of this, we can find out P t which we have seen in the previous class.

So, first we will see the application of this taking a very simple example and see how it can be utilized to find out the matrix P t. So, take the same example which we have discussed before.

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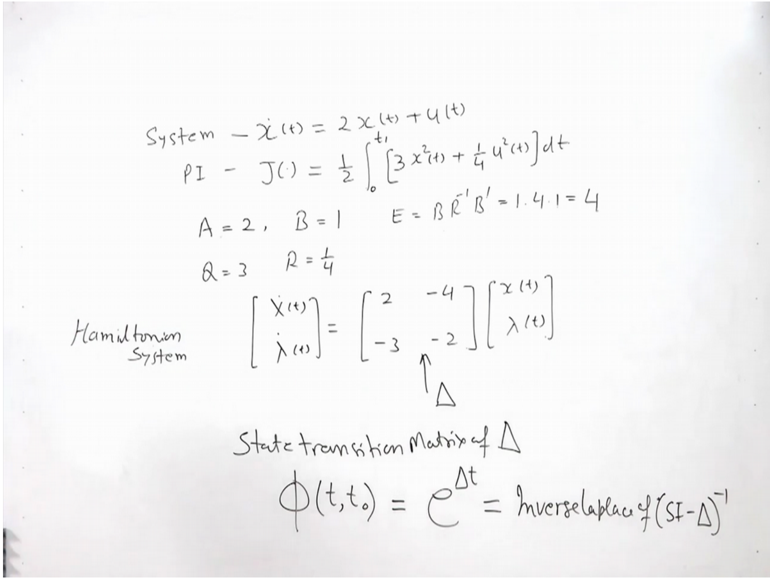

We have a plant given as x dot t 2 x t plus u t. We are taking a first order system which will give me the second order Hamiltonian which can be easily solved by the manual approach. If our system will be bigger, then we have to take to use the computer to solve this matrix.

So, my system is given by this matrix and P i is half 1 by 4 u square t d t. So, our objective is to find out the u which is minus R inverse d prime P and P is the solution of my Matrix Differential Riccati Equation. So, if I know the P, I can find out the u. So, we will take this as State Transition Matrix Approach. So, first we have to define what will be my Hamiltonian matrix. To define the Hamiltonian matrix, I must know; what is my matrix A. In this case is x dot is 2 x nothing, but 2 B equal to 1 u t, then I need the E. E is nothing, but B R inverse B prime. So, this is B is 1 R inverse is, sorry R is, if you will see here Q is 3 and R is 1 by 4. So, 1 into 4 into 1 which is, so E is nothing, but my 4; so if I will write my Hamiltonian matrix x dot t lambda dot t, this is equal to nothing, but my A is 2 minus E. So, minus 4, then minus q minus 3 and A transpose this is 2. So, minus A transpose is minus 2. So, this is nothing, but my Hamiltonian system or Hamiltonian matrix. So, we call it note matrix, this is my Hamiltonian system and this multiplied with x t lambda d.

So, for this Hamiltonian matrix delta, we have to find out the State Transition Matrix. So, State Transition Matrix of delta we will represent as phi t t 0 which is nothing, but my E to the power delta t and we can write as the inverse Laplace of SI minus delta inverse. So, delta is given here. So, if I will take, so minus delta inverse of that this will give me nothing, but my State Transition Matrix. What we can write for phi which is nothing, but my E to the power delta t and we can write as the inverse Laplace of SI minus delta inverse.

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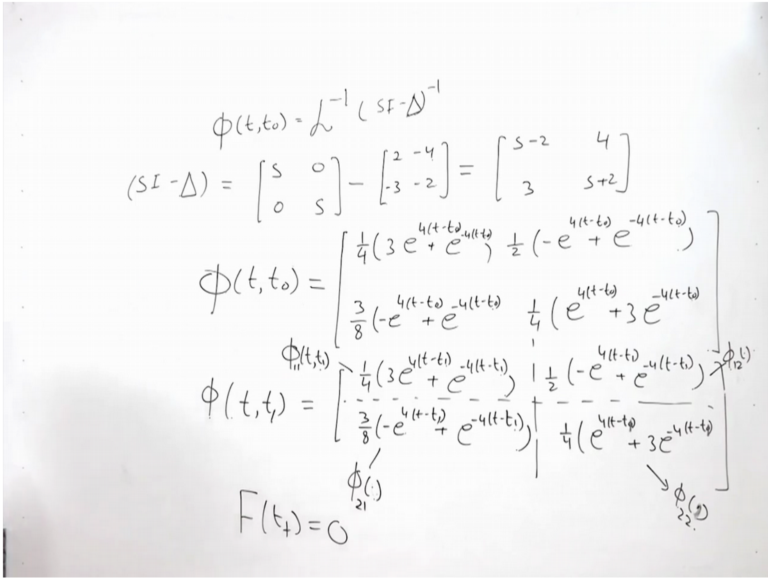

So, delta is given here. So, if I will take SI minus delta inverse of that, this will give me nothing, but my State Transition Matrix. What we can write for this is nothing, but my inverse Laplace of SI minus A inverse and what is my SI minus A. So, this will be nothing, but my s 0 0 s, sorry it is not A; we are using SI minus delta. So, this will also be my SI minus delta which we are using there so, SI minus delta; delta S 2 minus 4 minus 3 minus 2. So, if I will write this is simply my S minus 2 4 3 and S plus 2, so I have to take the inverse of this matrix which by the standard procedure we can do this. We take the inverse of this and then, take the inverse Laplace of this.

So, if we will follow this, we can write phi t t 0 as I am directly writing the result. So, is 1 by 4 3 e to the power 4 t minus t naught. The second element is 1 by 2 minus e to the power 4 t minus t naught plus e to the power minus 4 t minus t naught. This is my first row and second row is 3 by 8 minus e to the power t naught plus e to the power minus 4 t minus t naught. Second element is 1 by 4 4 t minus t naught plus 3 e to the power minus 4 t minus t naught.

So, this is my State Transition Matrix when we are evaluating it from t 0 to t. So, if you will recall this is State Transition Matrix we have to determine. So, phi t 2 t f if I will see say as j we have taken t 1, so we will take this as t 0 to t 1. So, we can write this matrix simply as just these terms I have to reverse. So, it is same matrix like 1 by 4 3 e to the power 4 t minus t 1. I have missed one thing. This is plus e to the power minus 4 t minus t naught. So, this I have missed. So, I have added this term here 3 to the power 4 t minus t 1 plus e to the power minus 4 t minus t 1. This is my first term. Second term is 1 by 2 minus e to the power 4 t minus t 1 plus e to the power minus 4 t minus t 1 is my second term. 3 by 8 minus e to the power 4 t minus t 1 plus e to the power minus 4, sorry t minus t 1 we are doing t minus t 1 and 1 by 4 e to the power 4 t minus t 1 plus 3 e to the power minus 4 t minus t 1.

So, this is the same matrix here, but here it is t 0 to t, then here we have t 1 to t. So, this State Transition Matrix we have partitioned as the four elements. The first will me by phi t t 1, sorry phi 1 1 t t 1 1. This is phi 1 2 phi 2 1 and this will be my phi 2 2 and if I will see what is my matrix p, this is phi 2 1 phi 2 2 into f phi 1 1 phi 2 2 into f. What is f of t? F in this case F t f is 0 because we have not considered any terminal cost in defining j.

So, these are the values of phi 1 1 5 1 2 phi 2 1 phi 2 2 . So, I can directly write what will be the solution of the Matrix Riccati Equation. So, as my F of t f is 0, so my this term and this term will be 0.

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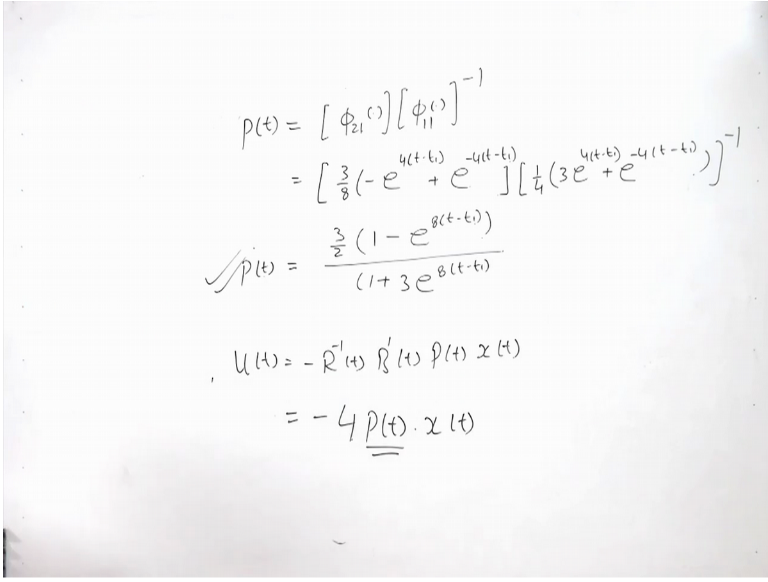

So, my P t is phi 2 1 multiplied with the phi 1 1 inverse. So, P t, I can directly write as phi 2 1 phi 1 1 inverse and what is my phi 2 1 3 by 8 minus e to the power minus 4. So, this I can simply write as, so I am writing nothing, but phi 2 1. This is 3 by 8 this term. So, this is my phi 2 1 e to the power 4 t minus t 1 plus e to the power minus 4 t minus t 1 multiplied with phi 1 1 which is 1 by 4, this is 3 e to the power 4 t minus t 1 plus e to the power minus 4 t minus t 1 and inverse of this.

So, t f is 0, sorry F of t f is 0. So, this term and this term will not appear. So, I am only with the phi 2 1 phi 1 1 inverse. So, I can write this and if I will simplify this, I will simply get writing the final result as 3 by 2 1 minus e to the power 8 t minus t 1 divided by 1 plus 3 to the power 8 t minus t 1. So, this will be my nothing, but P t. If P t is known, I can directly write what my u t is. This is minus r inverse B prime P t x t R inverse is 4 B prime is 1. So, this is nothing, but minus 4 P t into x t, where P t is given by this equation. So, this will be my u t. So, directly I can implement by close loop control. So, by this example we have seen the application of determination of analytical solution of Matrix Differential Riccati Equation utilizing the State Transition Matrix approach.

Next we will discuss another approach which is based on the Eigen Value and the Eigen Vector Analysis. So, my another approach for the analytical solution of the matrix DRE is based on Eigen Value and Eigen Vectors.

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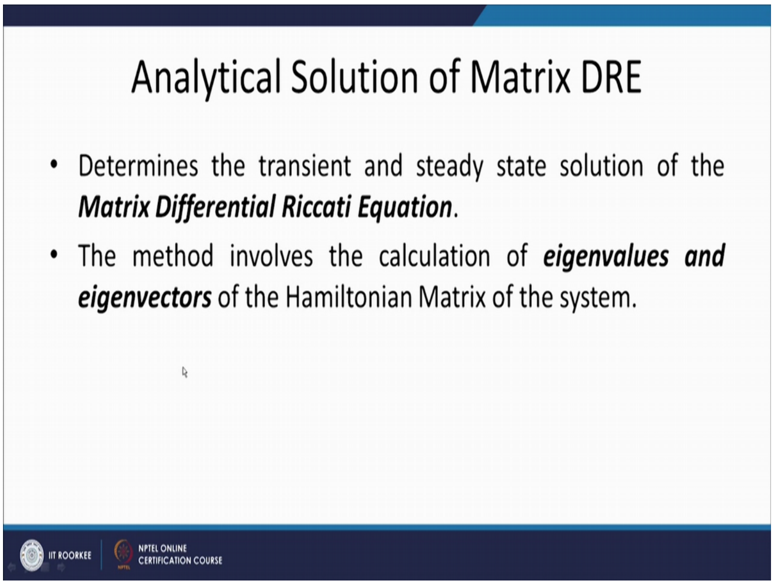

Again we use the Hamiltonian matrix to obtain the solution. In this approach, we can find out the transient and the steady state solution of Matrix Differential Riccati Equation and this method involves the calculation of the Eigen Values and Eigen Vector of the Hamiltonian matrix or Hamiltonian system. We say as we know what my Hamiltonian system is.

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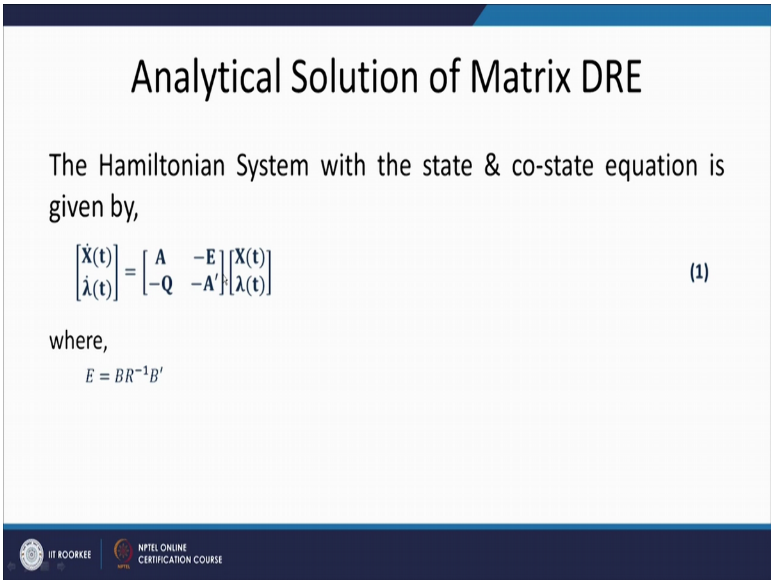

We already have discussed this X dot lambda dot t is A minus E minus Q minus A prime X lambda, where E is nothing, but B R inverse B prime.

So, again first we are discussing say here AEQA, all are the function of time. So, we are taking the time varying system, but for the simplicity this t has been dropped from the matrix A E Q A prime close loop L Q R gain. We can determine simply as using the transformation lambda t equal to P t X t which we have seen in the derivation of our Riccati equation.

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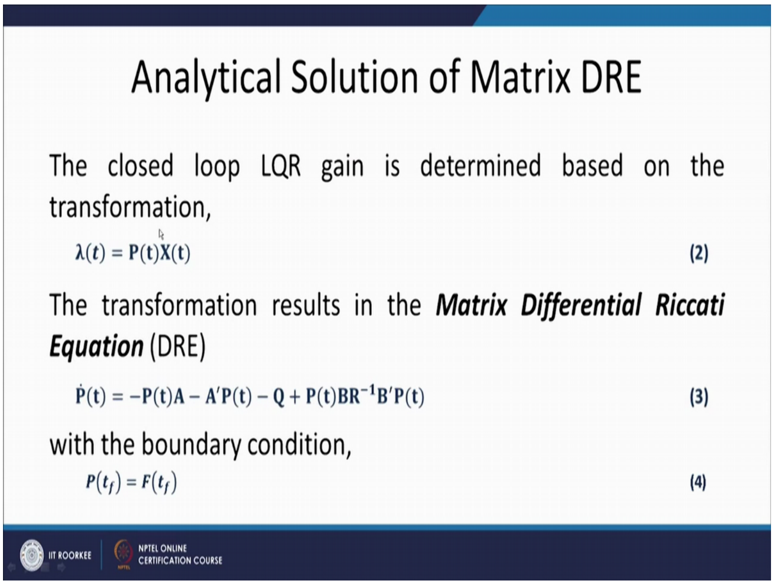

So, a transformation matrix is lambda t P t X t and my matrix differential equation I am writing as P t A minus A prime P Q P B R inverse B prime P with boundary condition is P t f equal to F of t f.

So, to determine the solution of Matrix Differential Riccati Equation means we are trying to obtain the value of P t which can be expressed in terms of the Eigen Values and the Eigen Vector of my matrix, Hamiltonian matrix.

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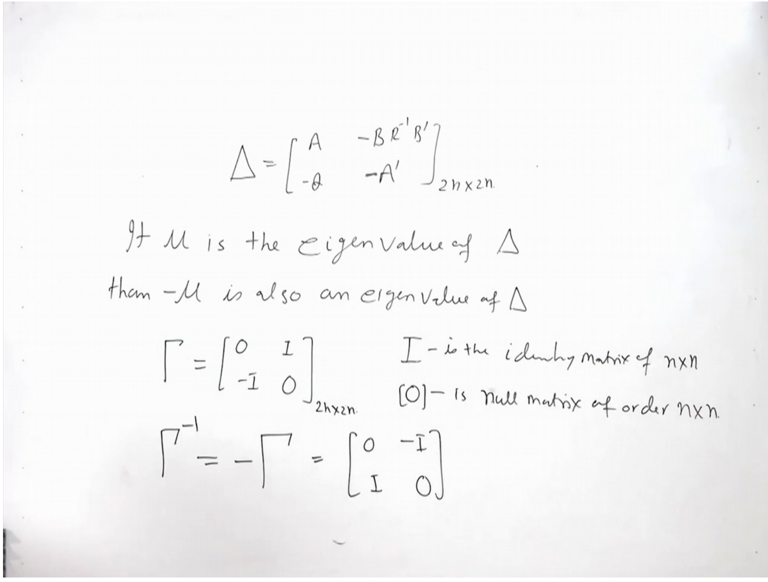

So, with the defining our Hamiltonian matrix says A either I can write E or I write B R inverse B prime minus Q A transpose. So, all these expression I have dropped out. The t term that is only for the convenience or easy representation of this, otherwise my matrix A B R Q, they all are time varying. They all will depend on t.

So, in this we give a statement if mu is the Eigen Value of delta, then minus mu is also an Eigen Value of delta. So, this we can prove considering a transformation matrix which we are saying as tau given as the 0 I minus I 0, where I is the identity matrix of order n cross n and 0 is the null matrix of order again n cross n. So, because my Hamiltonian matrix is 2 n cross 2 n matrix, where n is the order of my system.

So, we also consider this transformation matrix as 2 n cross 2 n. The property of the tau is if I will take the inverse of this which I can simply verify, this is nothing, but the minus times of tau. This means this is nothing, but 0 minus I I 0. Now, this transformation matrix is utilized to prove this condition which we are saying if mu is the Eigen Value of delta, then minus mu is also an Eigen value of delta, where delta is my Hamiltonian matrix. So, we will see this proof in the next class. This class I will stop here.

Thank you very much.