**Optimal Control**
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**Lecture - 22**
**Analytical Solution of Matrix Differential Riccati Equation**
**(State Transition Matrix Approach)**

Welcome friends to this class and in this class, basically we will study the approach how we can find out, how we can solve the MDRE, Matrix Differential Riccati Equation or ARE.

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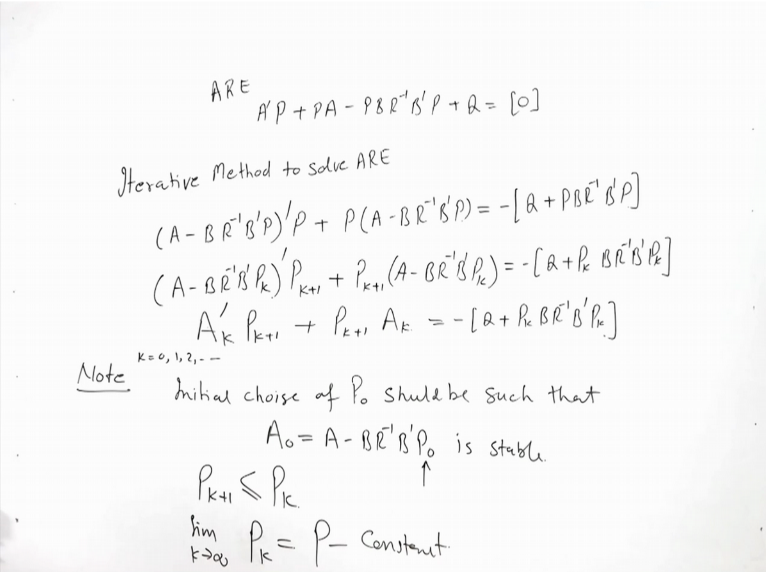


So, first if you recall in the previous class, we have seen that my algebraic Riccati equation is A transpose P plus P A minus P B R inverse B prime P plus Q equal to 0. So, this is my Algebraic Riccati Equation and the nature of this equation is, it is a non-linear algebraic equation and the solution of this equation is not direct. Sometimes we will have the multiple solutions.

So, today we will see how to solve this Riccati equation. So, the first approach we take is an iterative approach. The iterative method to solve this algebraic Riccati equation say if you recall this A transpose P, we can write in the form A B R inverse B prime P transpose P plus P A minus B R inverse B transpose P. So, once we are proving A minus B K to be stable, then you can recall we have returned this ARE in this form.

Now, as an iterative approach what we say let us give the Kth iteration to the P inside the bracket and the K plus for iteration to the P outside the iteration. This means we write this as A minus B R inverse B prime P K, P K plus 1 P K plus 1 A minus, sorry A minus B R inverse P prime P K s minus Q plus P K B R inverse P prime P k. So, this is I can say is my iterative equation. So, I will start with the initial guess of P K and solve this equation for P K plus 1. So, I will get the next value of the P K.

So, I can start with K to be 0 1 2 and so on. So, we have to note down here, so what we are saying. So, we have to write this as my iterative equation. So, this means for K equal to 0 that will be by first choice, K equal to 1 when K equal to 0, I am finding P1. So, this equation I write this matrix A minus B K as I can write this A K P K plus 1 plus P K plus 1, sorry A K prime P K plus 1 P K plus 1 A K as minus Q plus P K B R inverse B prime P K and I start my iteration to be K equal to 0.

So, this matrix 0; so, to converge my solution to a constant matrix of P, I have to select my initial guess should be such that so P 0, I have to guess. So, P 0 value should be such that my matrix A 0 A minus B R inverse B prime. P 0 should be a stable matrix. This means this matrix will have its all Eigen values with negative real part. So, if I will start with this, then I have to ensure that P K plus 1 is less than or equal to P K and as limit K approaches to infinity, my sorry P K approaches to P A constant matrix.

So, we can see this may be my one of the approach to find out P, we will set an iterative equation as A transpose K P K plus 1 P K plus 1 A K equal to minus Q P K B R inverse B prime P K. We start with the first iteration at K equal to 0. This means we first have to find out A minus B R inverse B prime P 0 and we select P 0 with my initial guess, such that my anode matrix is stable. I have to ensure P K plus 1 is less than equal to P K, then as K approaches to infinity, so after the few iteration we find that my P value is constant value. As P will become constant, we will stop the iteration and that will be my choice for P which is my Riccati coefficient. So, by this method we can find out the P K.

My another approach is as given by the title of this lecture is Analytical Solution of Matrix Differential Riccati Equation using State Transition Matrix Approach. So, by this State Transition Matrix Approach, we can solve the Matrix Differential Riccati Equation or Algebraic Riccati equation.

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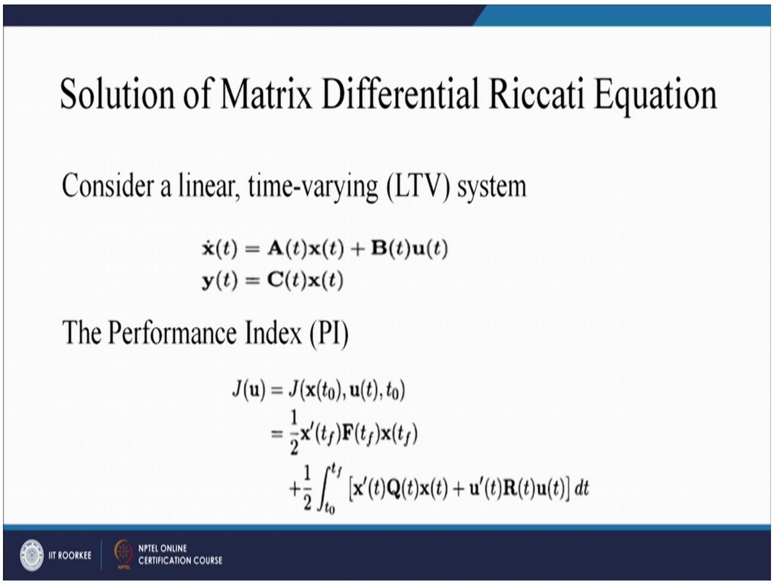


So, first we will see to find the solution of the Matrix Differential Riccati Equation. If you recall we have considered a system x dot equal to a x plus B u, y equal to C x t which is and performance index has half of x prime t f f of t f x of t f x prime Q x plus u prime R u.

So, this means first we will consider the case of a time varying system. So, time varying system finite time regulator problem, we are seeing in which we will encounter with the Matrix Differential Riccati Equation and its solution we have to find out. So, you can recall.

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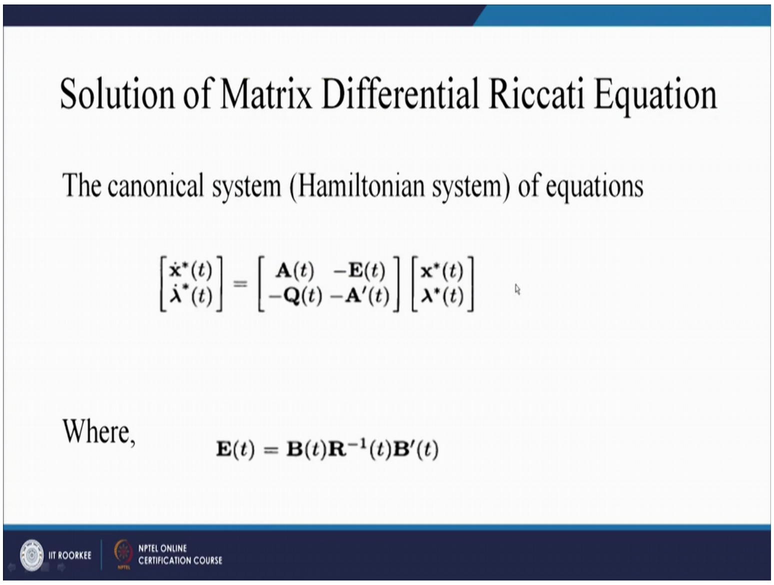


Once we are solving this system by the Hamiltonian approach, we first write the Hamiltonian, then write the control equation del u by, sorry del h by del u equal to 0 which give me the control relation and state and the costate state equation.

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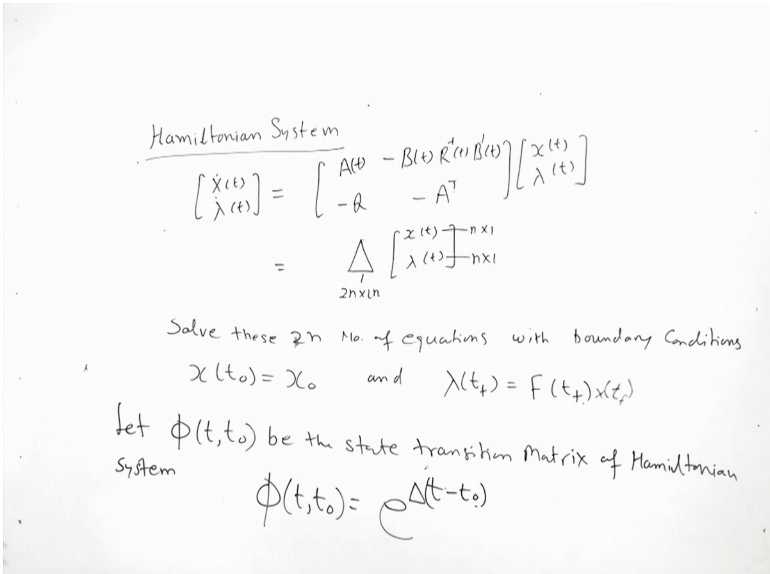


So, the state and the costate equation I write in as a Hamiltonian system which is, so I am writing my Hamiltonian system or Canonical system in the form of x dot t lambda dot of t which is the set of 2 n equation is A minus e e is B R inverse B prime. So, here we write A minus B say it is a time varying case. So, all my matrices are function of time B R inverse, d prime is my e minus Q minus A transpose lambda t, sorry x t lambda t. So, now if we will see my; these 2 equations, so this Hamiltonian system if I know x 0 and lambda 0. So, this system I can directly solve in terms of the State Transition Matrix.

So, I write this matrix as my Hamiltonian matrix delta and this I will take it as x t lambda t and what is the delta. Delta is nothing, but a 2 n by 2 n matrix because x is n cross 1 lambda is also n cross 1. So, the size of my Hamiltonian matrix is 2 n by 2 n. So, if you will say this is the first order 2 n first order differential equations which we have to solve subjected to the initial condition as solve these 2 n number of equations with boundary condition as what the conditions, we have x t 0 as say x 0 and lambda t f as F of t f x of t f.

So, in case of the state, we are given with initial condition, but in case of the lambda, we are given with the final terminal conditions. So, with this terminal condition, our objective is to solve these 2 n number of the equations. As we know with this the solution of this equation can be written as to solve this. Let phi t t 0 be the State Transition Matrix of Hamiltonian system. This means we are considering phi t t 0 to be e to the power lambda t. So, if I have my Hamiltonian system, the State Transition Matrix of Hamiltonian system is defined as the phi t t 0 which is equivalent to lambda, sorry delta t minus t 0.

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$$\phi(t,t_0) = \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix}$$

Say $\lambda(t_0)$ (Unknown) be the initial value of $\lambda(t)$

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \phi(t,t_0) \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix}$$

$$\begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} = \phi^{-1}(t,t_0) \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \phi(t_0,t) \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \phi(t,t_f) \begin{bmatrix} x(t_f) \\ \lambda(t_f) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t,t_f) & \phi_{12}(t,t_f) \\ \phi_{21}(t,t_f) & \phi_{22}(t,t_f) \end{bmatrix} \begin{bmatrix} x(t_f) \\ \lambda(t_f) \end{bmatrix}$$

Let us partition see we write phi t t 0 in partition form as phi 1 1 phi 1 2 phi 2 1 and say phi 2; 2. Now, say lambda t 0 which is unknown say lambda t 0 be the initial value of be the initial value of lambda t. So, what we can write the solution for the Hamiltonian system, what will be my x t and lambda t from the linear control theory. This is nothing, but my phi t t 0 x of t 0 lambda of t 0.

So, this means we have considered that the lambda t 0 is known, then I can write the solution in this form by this. If I will write let by this if I will determine the initial condition x t 0 lambda t 0 which is nothing, but my phi inverse t t 0 x t lambda t and phi t t 0, I can write as nothing phi t 0 t. So, this is the identity of the State Transition Matrix phi inverse t t 0 is phi t 0 t which I can write in terms of the x t lambda t.

So, now, if we will select the t 0 to be any given time and t to be my final time, so I can write my solution in terms of the final time this means I can also write this as x t lambda t as phi t 0 is my t 2 t f x of t f lambda of t f while phi t t f, I can write as phi 1 1 t t f phi 1 2 t, t f phi 2 1 phi 2 2 t, t f multiplied with x of t f lambda of t f.

So, I can write my x t lambda t in terms of the t f by this expression. So, I will keep the last equation.

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$$x(t) = \phi_{11}(\cdot)\, x(t_f) + \phi_{12}(\cdot)\, \lambda(t_f)$$

$$x(t) = \left[\phi_{11}(\cdot) + \phi_{12}(\cdot) F(t_f)\right] x(t_f)$$

$$x(t_f) = \left[\phi_{11}(\cdot) + \phi_{12}(\cdot) F(t_f)\right]^{-1} x(t)$$

$$\lambda(t_f) = F(t_f)\, x(t_f)$$

$$\lambda(t) = P(t)\, x(t)$$

$$\lambda(t) = \phi_{21}(\cdot)\, x(t_f) + \phi_{22}(\cdot)\, \lambda(t_f)$$

$$P(t)\, x(t) = \left[\phi_{21}(\cdot) + \phi_{22}(\cdot) F(t_f)\right] x(t_f)$$

$$P(t)\, \cancel{x(t)} = \left[\phi_{21}(\cdot) + \phi_{22}(\cdot) F(t_f)\right]\left[\phi_{11}(\cdot) + \phi_{12}(\cdot) F(t_f)\right]^{-1} \cancel{x(t)}$$

Finite time case

$$P(t) = \left[\phi_{21}(\cdot) + \phi_{22}(\cdot) F(t_f)\right]\left[\phi_{11}(\cdot) + \phi_{12}(\cdot) F(t_f)\right]^{-1}$$

for Infinite time $F(t_f) = 0$

$$P = \left[\phi_{21}(\cdot)\right]\left[\phi_{11}(\cdot)\right]^{-1}$$

Now, if I will expand this relation in terms of the x t, what I will get phi 1 1 x t f. So, I write this as phi 1 1 x t f plus phi 1 2 lambda t f and what I have is lambda t f is f of t f x of t f. So, I will replace lambda t f by this. So, I can write phi 1 1 plus phi 1 2 and lambda t f is F of t f and x of t f. I am writing outside. So, taking by replacing lambda t f by f of t f x of t f substituting here, so I can write x t as this form in terms of the x of t f and by this x of t f. I can write as nothing, but phi 1 1 plus phi 1 2 f of t f if I have to write x of t f in terms of x t, this inverse x of t.

So, from the first equation x t we have expanded from here as phi 1 1 x phi 1 2 lambda we replace the lambda t f by a f and then, we are writing x t f in terms of the x t, then we expand lambda t which is by this, we will get phi 2 1 x of t f phi 2 2 lambda of t f. So, by expanding lambda t by this expression, we get this phi 2 1 x of t f phi 2 2 lambda of t f lambda of t f. We are replacing by this f of t f x of t f and you recall what were their substitution lambda t, we have taken as P t x t. So, lambda t we are writing as P t x t and here phi 2 1 plus phi 2 2 in place of lambda t f, we have f of t f. This is multiplied with x of t f.

Now, x of t f, I can write it in terms of the x t from here. So, I can write this as phi 2 1 plus phi 2 2 f of t f. In place of x of t f, I write phi 1 1 plus phi 1 2 f of t f inverse x t and this is nothing, but my P t x t x t. We can cancel out from both side. So, my P t is nothing, but phi 2 1, sorry plus phi 2 2 f of t f phi 1 1 plus phi 1 2 f of t f inverse.

So, this Matrix Differential Riccati Equation and this Riccati coefficient can directly be find out by this, where phi 1 1 phi 1 2 phi 2 2 phi 2 1 are nothing, but the partition of my State Transition Matrix. So, if I know my State Transition Matrix, I can partition it as phi 1 1 phi 1 2 phi 2 1 phi 2 2 and if this is known to me, I can directly find what will be my Riccati coefficient. Riccati coefficient is known to me. This is for time varying case or we can say this is for the finite time case for infinite case for infinite time. Basically my terminal cost will become 0. So, f of t f will be 0. So, for infinite time my P nothing, but will be phi 2 1 phi 1 1 inverse.

So, this is the another approach by which we can find out the Riccati coefficient matrix P t for time varying case or sorry, for the finite time regulator case and P for the infinite time regulator case.

So, I stop it here for this lecture. In the next lecture, we will take up one example based on this approach and one another method to find out the analytical solution of this Matrix

Differential Riccati Equation which is based on the similarity transformation and particularly on the Eigen value and Eigen vector approach.

Thank you very much.