Optimal Control

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Lecture - 21
Infinite Horizon Regulator Problem (Continued)

Welcome friends. We are discussing about the Infinite Horizon Regulator Problem. So, we have discussed the 2 cases, one the time varying case and the other was the time invariant case, in time invariant case.

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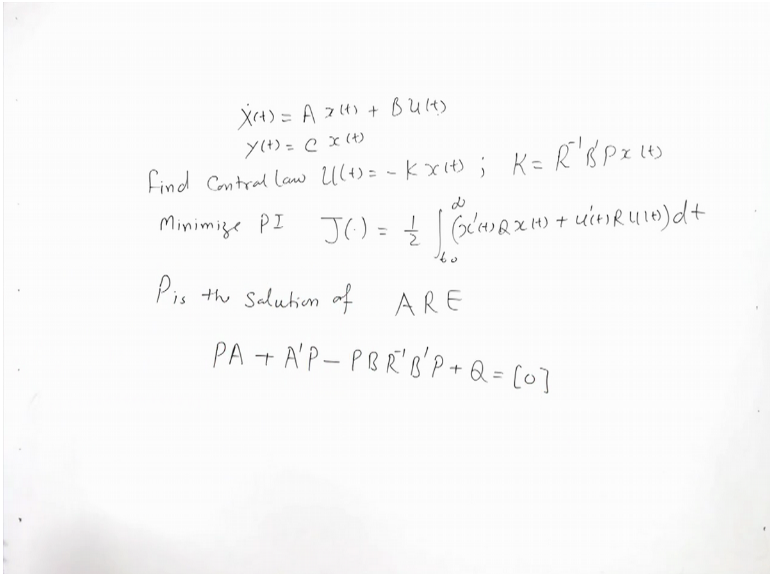

Our problem was for a given system, A x t plus B u t and y equal to C x t. We have to find the u, find control law, u equal to minus k x t, where k equal to r inverse b prime P x t. So, objective is to find the U t equal to minus K x t which will minimize the performance index, minimize the PI and PI is my J 1 by 2 t 0 to infinity x prime Q x plus u prime R u d t.

So, in LTI system, Linear Time Invariant systems A B C Q R are the constant matrix. We have to find out the control law u equal to minus k x, where k is r inverse b prime P x t and P is the solution of algebraic Riccati equation which is P A plus A prime P minus P B R inverse B prime P plus Q equal to a null matrix depending upon the size of the P null matrix you can select.

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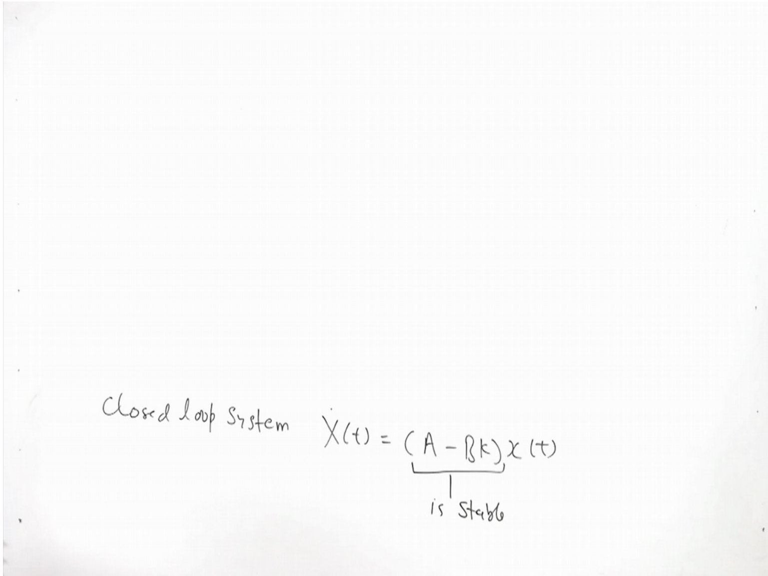

So, this is my algebraic Riccati equation which we have to solve. If we can find out the p, we can find out the control law which give me the close loop system will be now X dot t equal to A minus B k x t. So, whatever be the original system A, if my system is completely controllable, I can control using the k all the state and my matrix A minus B k is a stable matrix. So, A minus B k is stable means all Eigen values of a minus B k lie into the left half of or all Eigen values will have a negative real part.

So, now today we will prove using the Lyapunov theorem that A minus B k is a constant matrix. So, my close loop system is x dot equal to A minus B k x t and we have to prove that a minus B k is stable using Lyapunov Stability Criteria.

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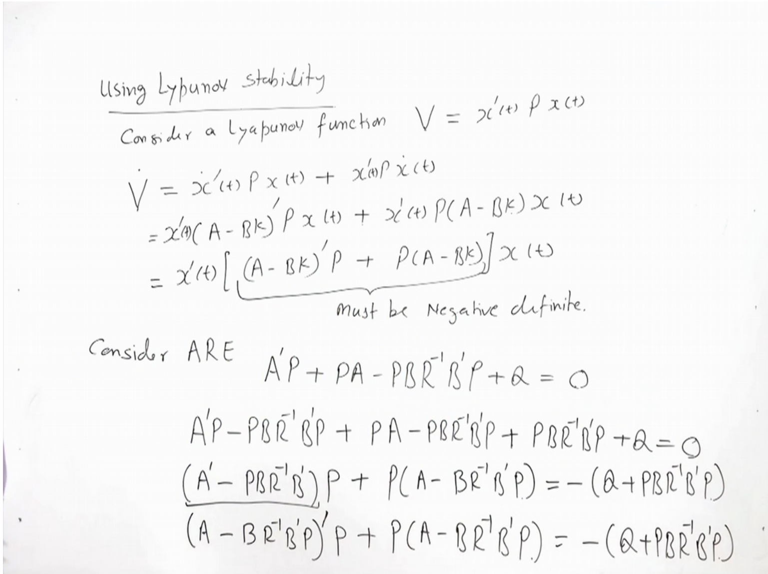

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So, this we will prove using the Lyapunov stability. Let us consider the Lyapunov function. So, by the definition of the Lyapunov function, let us say V is x transpose P x t as we have d1 in the time varying case.

Similarly, we are selecting Lyapunov function; P is a symmetric positive definite matrix x prime x gives me the squares. So, this satisfies the definition of the Lyapunov function. So, Lyapunov function should be a positive definite function. So, it is positive definite and it approaches to 0. As x approaches to 0 and x is equal to 0, V will be 0 otherwise my V is a positive definite function. So, if V is my Lyapunov function, by the Lyapunov it is stability criteria. We know if V dot is negative definite, then V said that my system is stable. So, close loop system we have x dot equal to A minus B k x t.

So, next we will try to find out the V dot. This is x dot prime t P x t if we will differentiate this considering P, now B A constant matrix. So, there is more derivative for the P that will be 0 plus x prime P x dot of t. So, x dot of t and x dot for a close loop system is a minus B k. We take this x dot as A minus B k transpose P x t plus x transpose P A minus B k, sorry this is a minus B k x t. So, I basic will have x transpose of t because once I will take the transpose of the x prime, this means I am taking the transpose of A minus B k x t. X t will come here, then A minus B k transpose that will appear in this. So, my expression will be and also, my x t is here. So, the overall explanation I can write as x transpose of t A minus B k transpose P plus P A minus B k x t. So, if A minus B k is to be a stable matrix, then A minus B k P plus P A minus B k must be negative definite.

So, now we have to show that a minus B k prime P plus P A minus B k is negative definite and this we can show to, this we start consider A R E to show that v dot to be negative definite. This means A minus B k prime P plus P A minus B k must be negative definite to prove this. Let us consider the algebraic Riccati equation as A transpose P plus P A minus P B R inverse B prime P plus Q equal to 0. So, what we are doing? We are a prime P minus P B R inverse B prime P. This I take it 1 side plus P A minus P B R inverse B prime P. This I will also take it here. Now, what I have to do? I have subtracted 1 P B R inverse B prime P. So, I will add 1 P B R inverse B prime P plus Q equal to 0. So, I did nothing. I have added and subtracted this term and rest we kept same. So, this is not going to change and what we are writing, we are writing A prime P. We are taking from out minus P B R inverse B prime P from these first 2 terms. Next term P, I am taking here P A minus B R inverse B prime P equal to minus Q plus P B R inverse B prime P.

So, let us see what we are doing? We are writing this term, the first term. This term we are writing a minus taking the transpose of this dB R inverse B prime P and whole transpose. So, if I will take the transpose of this, I will get a transpose, then P B R inverse B prime and P here plus this I kept the same P A minus B R inverse B prime P equal to minus Q plus P B R inverse B prime P.

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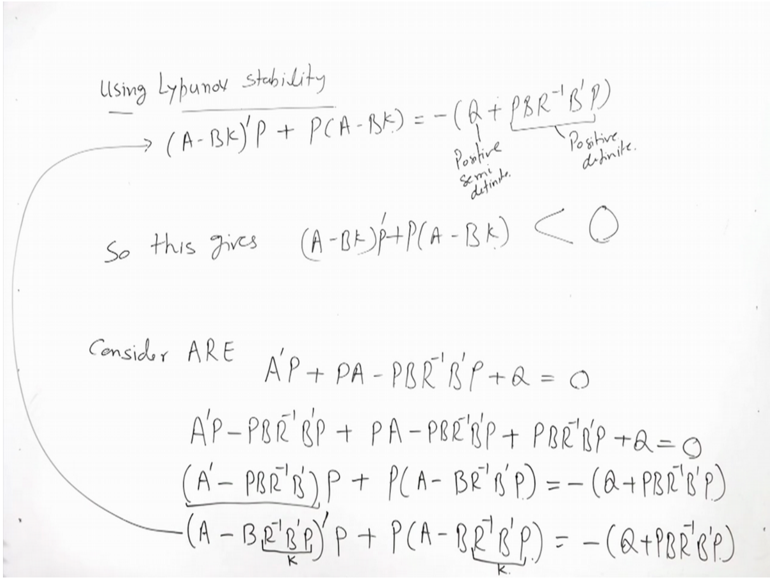

So, now in this you will see B R inverse B prime P R inverse B prime P. This is nothing, but my K R inverse B prime P, this is nothing, but my K. So, the whole equation I can write as from this, I am writing as A minus B K transpose P plus P A minus B K which is coming here, this equal to minus Q plus P B R inverse B prime P.

So, what was my matrix A minus B K prime P plus P A minus B K. This I have to prove as to be negative definite. So, my this matrix is a minus B K prime P plus P A minus B K. So, left hand side Q positive semi definite, my this whole term P send this by P is positive definite B. B prime will give me the square term, R is positive definite. So, this second term always will be positive definite matrix. Positive definite this means is with negative sign. So, this shows, this gives me A minus B K plus A minus B K prime P plus P A minus B K is less than 0 means A minus B K is negative definite matrix.

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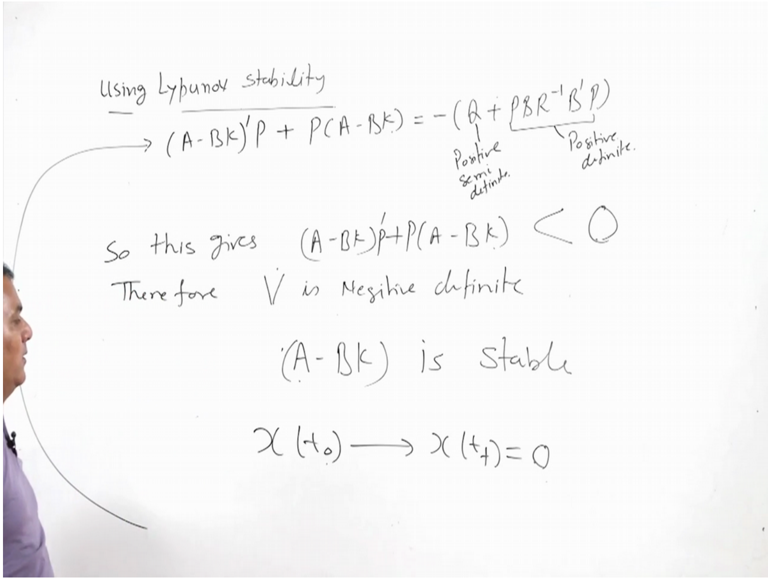

So, therefore I can say because my V dot is negative definite, so now V dot is negative definite. So, my A minus B K is stable. So, by this we can prove that the close loop system x dot equal to A minus B K, x t my system matrix is A minus B K will always be a stable matrix whatever be the condition of a subjected to my system is controllable. If my system is controllable, this means I can shift my state from x t 0 to 0. So, what we are doing? We are shifting our state x t 0 to x t f which in this case is 0 because my A minus B K always is stable matrixes have all Eigen values with negative real part.

So, in the next let us take an example to show what we have studied about the linear quadratic regulator system for linear time invariant case.

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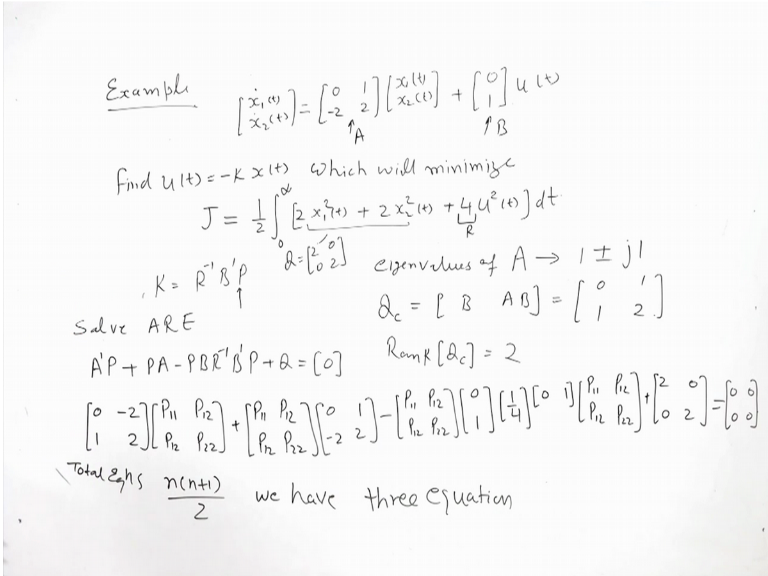

So, in our example we consider a LTI system, second order x 1 dot x 2 dot t equal to 0 1. So, this is my system, this is my matrix A, this is my matrix B. Objective is to find u t which is equal to minus k x t which will minimize. J equal to that we are taking 0 to infinity 1 by 2, 2 x 1 square plus 2 x 2 square t plus 4 u square t d t. So, we have to find the u t which will minimize this. So, u equal to minus K x as we know K is nothing, but R inverse B prime P. So, we have to find out the P.

Now, in this case if we will see Eigen values of a are 1 plus minus J 1. So, this means my original system is a unstable system. So, my first step is to design a control of u equal to minus K x. My system should be a completely controllable system. So, we check the controllability by matrix Q C which is nothing, but B A AB and this is equal to 0 1 1 2 and the rank of Q C is 2. So, my order of the system is 2. The controllability matrix Q B A B must have the rank equal to the order of the system, then I said my system is completely controllable. So, it is a controllable system.

So, next we will see if we will design A k, such that my system is optimally control means my close room matrix A minus B k will be a stable matrix. So, for that we have to find out P. So, to find the P, solve A R E which is a transpose P plus P A minus P B R inverse B prime P plus Q equal to matrix. So, what is my a prime? So, I am writing this 0 minus 2 1. So, this is my A prime. P is a symmetric positive definite matrix. So, I will select the symmetric matrix as P 11 P 12 P 12 P 22. So, this is my first a transpose P plus

P A, P 11 P 12 P 12 P 22 A is 0 1 minus 22 minus P B, B is 0 1 PBR inverse. What is my R? My; this 4 is my R and from here I can find out my Q as 2 0 0 2. So, PBR inverse is 1 by 4 B prime 0 1. So, PBR inverse B prime P 11, P 12, P 12, P 22 plus Q and this must be equal to your 0, 0, 0, 0, anal matrix.

So, now what we know if we will solve this taking P as a symmetric matrix, total number of equation is n into n plus 1 by 2. So, total equation will be n into n plus 1 by 2. Here n is 2. So, we have three equations.

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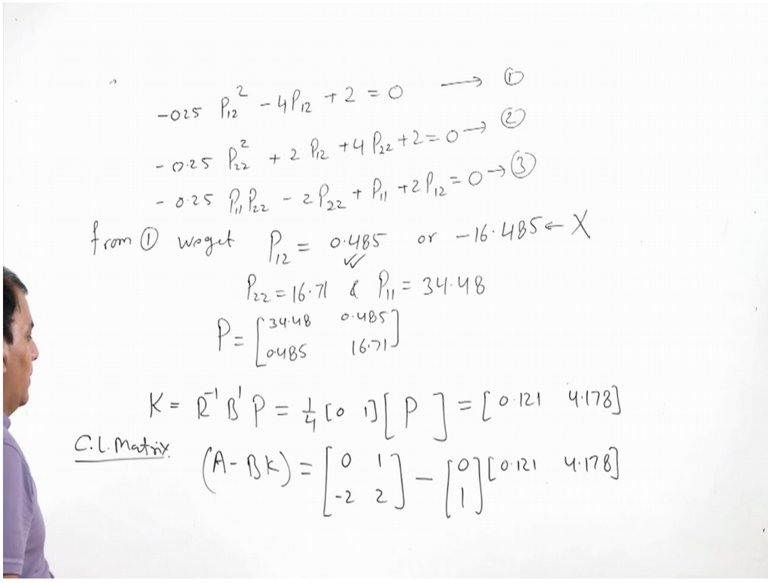

So, if we will solve this, we get the three equations. So, I will get the first equation by this. Second is this is 4 P 22 plus 2 equal to 0 and the third equation is minus 0.25 P 11 P 22 minus 2 P 22 plus P 11 plus 2 P 12 equal to 0.

So, if I will solve this equation, I will get these three equations. So, these are the non-linear algebraic equation which has to be solved. Sometimes we have to use the iterative methods, sometimes direct solution is available like in this case, we can solve the first equation. It is a quadratic equation. So, we get the 2 values of the P 12, out of which we have to select that value which will give P to be a positive definite. So, if I say equation 1 2 3. So, from 1 we get the 2 values for the P 12 is 0.485 and the another value is minus 16.485 because P 12 we retain this value because this give me the P to be positive definite and this value we discard. So, this we will not take with P 12, 2 be the 4.85, we got P 22 as 16.71 and P 11 as 34.48.

So, you can solve these equations. So, my first step will be to solve from equation 1. I will find the value of P 1 to for each value. Basically I will have the value of the P 22 and P 11. We have to select those values which is giving me P 2 be a positive definite. So, by this I can select my P to be 34.48, this is 0.485 and 16.71. So, once the P is known, I can find out what actually will be my U. So, to find the U, we can write down the K as R inverse B prime P. So, if I will take this value r inverse is basically by 1 by 4 B prime is 0 1 and P we already find this, P and if we will solve this, we will get k to be 0.121 4.178. So, this is the value of K.

Now, what will be my close loop matrix? My close loop system matrix will be a minus B k and this I can write as A is my 0 1 minus 2 2 minus B 0 1 and k, I will take 0.121 4.178.

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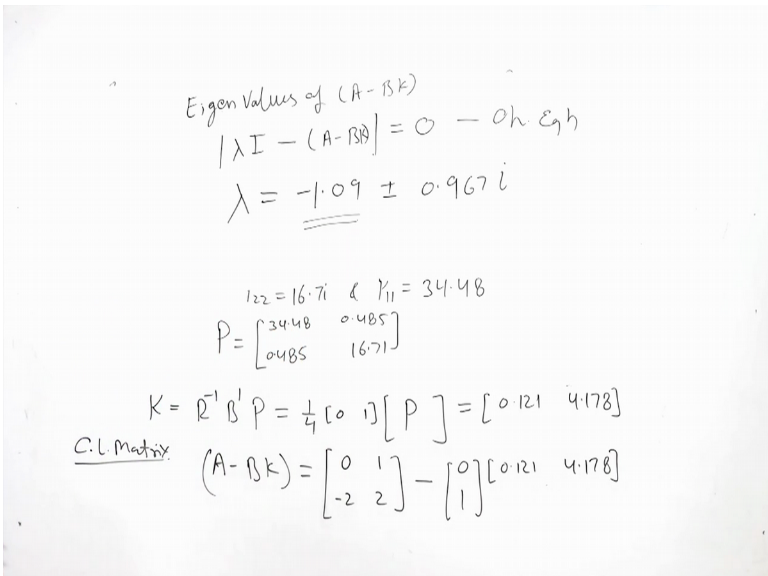

So, this matrix can be solved and if I will find out and then, find the Eigen values of A minus B k which will be nothing, but lambda I minus A minus; so determinant of lambda I minus A minus B k that must be equal to 0. This is my characteristic equation and the roots of my characteristic equation will give me the value of lambda. For this case this is minus 1.09 plus minus 967 i and we can see my Eigen value will have the negative real parts. So, my close loop system is stable while my original system was unstable, but by implementing the optimal control law, we find that my close loop system will be a stable system.

So, I stop here for this class. In the next class, we will discuss about how to find out or say how to solve the A R E and matrix differentially Riccati equation using some analytically approach.

Thank you very much.