Optimal Control
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Lecture – 20
Infinite Horizon Regulator Problem

Welcome friends, in the today's class which is on the infinite horizon regulator problem. In the previous class we have discussed about a time for the time varying system, we are trying to develop the optimal control law u, which will minimize the given performance index in a finite time. So, that is call the finite time optimal control problem, in which in a specified time we have to determine the optimal control, which will minimize the performance index as well, as will satisfy our plant equations. So, plant equation in that case will work as a constraint to my optimal problem.

Today we extend our discussion on the infinite horizon problem. So, if we will tend the time tf up to infinity.

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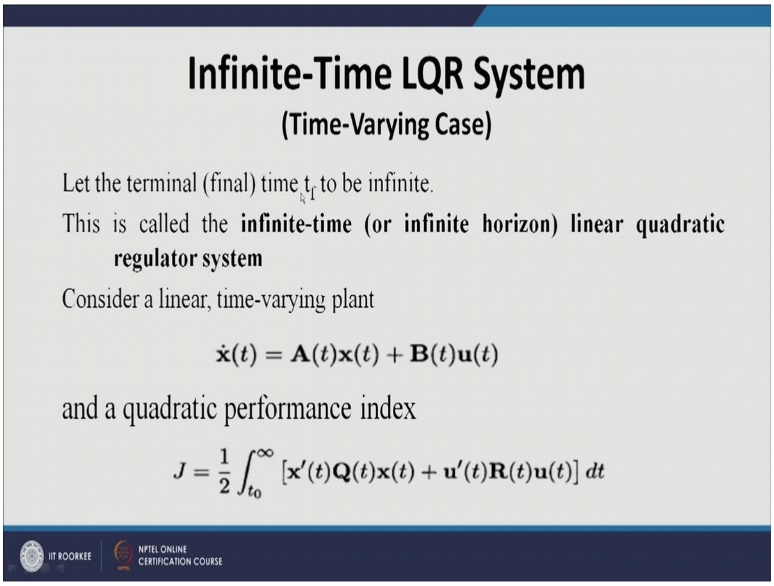

So, in performance index my time is varying from 0 to infinity, in that case my problem is called the infinite time linear quadratic regulator system. So, we can also call it the infinite time or infinite horizon.

So, we will take the 2 case in this, for the solution of this problem the first one is the time varying case in which my A and B matrix are time dependent and they are changing with respect to time. So, we can formulate our problem in a manner that for a given plant which is x dot t equal to A t x t, plus B t u t my objective is to determine the optimal value of the u which will minimize the given performance index.

So, if we will see this performance index we do not have the terminal cost term normally.

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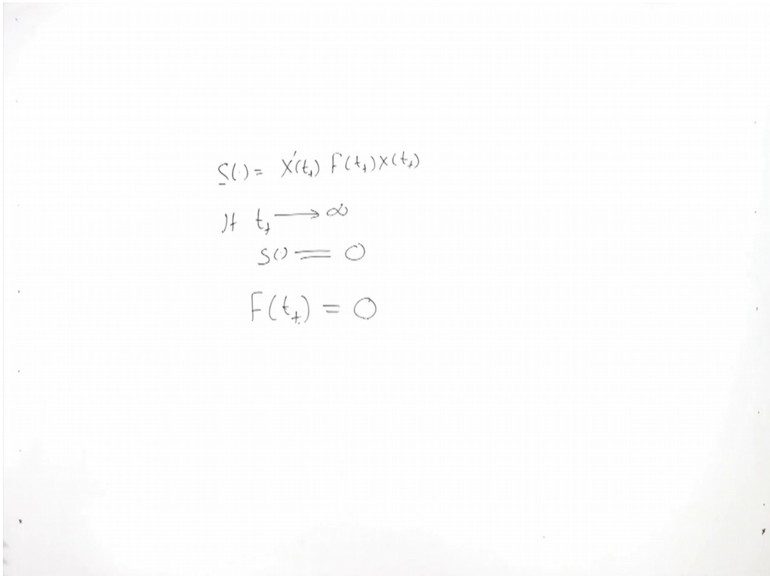

We are defining our terminal cost as for a linear regulator problem this is x transpose tf F of tf x of t F. Now if tf approaches to infinity if my tf is approaching to infinity in that case this s as no meaning. So, we consider s is equal to 0 this means we say F of tf nothing, but we consider it to be 0.

So, in this case my terminal cost has no engineering meaning. So, we are making this terminal cost to 0. And my performance index is simply defined as j equal to half. Now my tf is approaching to infinity. So, my limit will be t 0 to infinity x prime Q x plus u prime R u as A and B is my time varying system matrix and the input matrix. So, Q and R are also selected as the function of time. So, they are also varying with the time.

Now, this problem as we have solved the previous case with the same step defining the Hamiltonian del h by del u equal to 0, then state and the costate equations we will solve. So, you will find there is no change in the derivation, accept my tf is changing to infinity.

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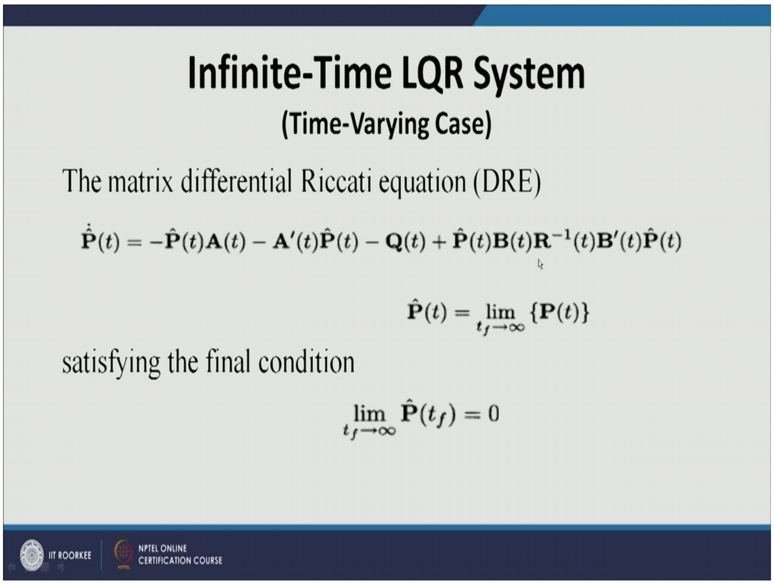

So, if I will go through this I will simply reach to my matrix differential Riccati equation, but in this matrix differential Riccati equation what is changing my terminal condition is changing.

So, P hat of t P hat of t what we have taken we have taken the this is as a variable P hat, this is related to the Pt which is my Riccati coefficient which we have considered for the finite horizon problem and P dot t is my Riccati coefficient which I am considering for a infinite horizon problem. So, this Pt is related to the Pt as limit tf tends to infinity, I will get this P t.

So, my problem for the infinite horizon is I have the same equation which we have write for the finite horizon case in finite horizon time varying case, this equation again is a matrix differential equation as shown by this equation, the only point is changing here the terminal point their my P tf was F of t f.

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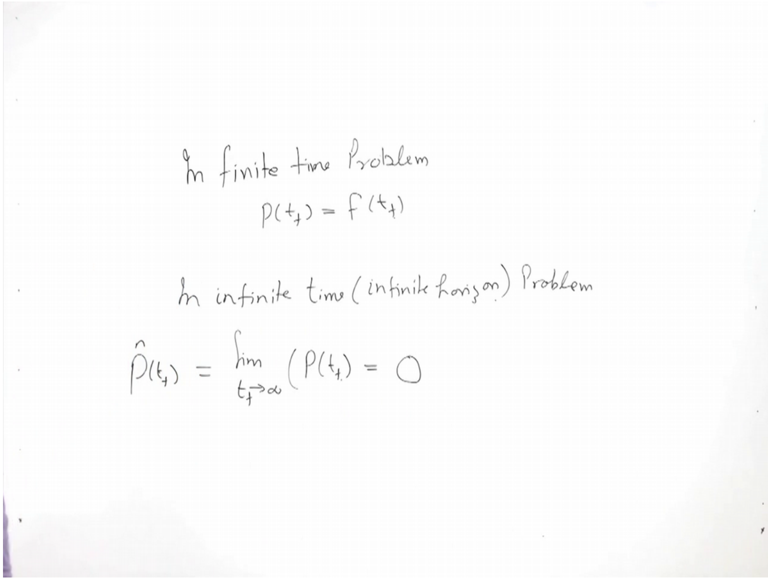

So, in finite time problem, I have P tf equal to F of t F. In infinite horizon problem I am writing infinite time or infinite horizon problem, what I will have limit tf tends to infinite of this P of tf. This will be my nothing, but P hat of tf because this is P tf is h of tf which is 0. So, my this value will be 0. So, as tf approaches to 0 my P tf will be 0. So, to find out the optimal control.

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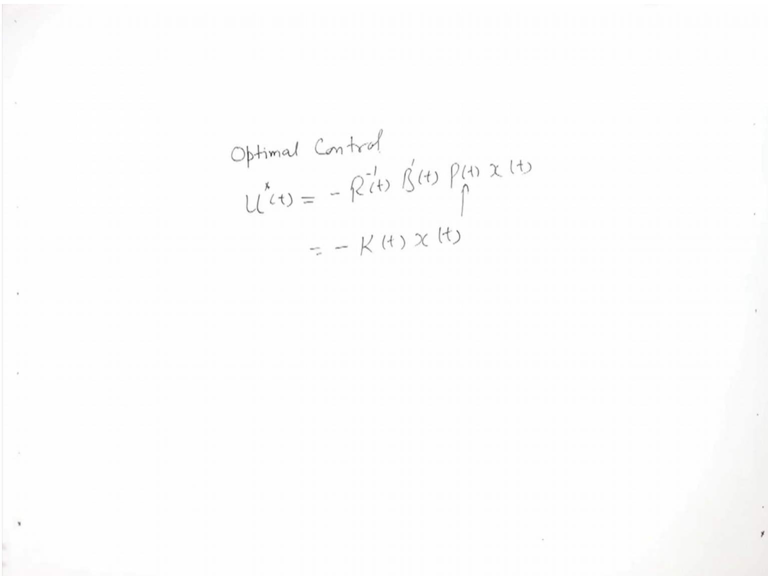

So, in this case also my optimal control will be, u star t minus R inverse B transpose Pt x t. And Pt is nothing, but the solution of my this given equation. So, by solving the Riccati equation I can find out the Pt if Pt is known I can find out the control as minus K t x t and this control can be utilize to control the system.

My second cases the time invariant case. In time invariant case my matrix A and B are the constant matrix.

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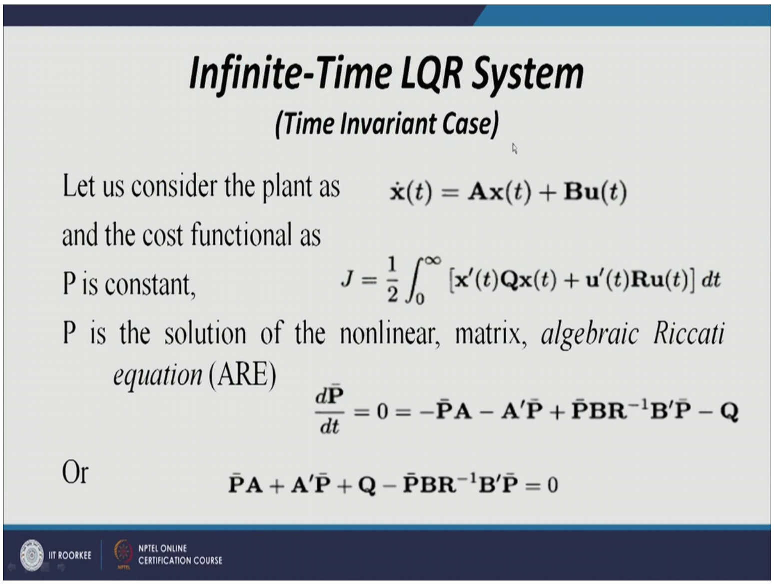

So, time invariant problem we can formulate as for the given plant x dot t which is A of x t B of u t A and B are the constant matrix of proper dimension objective is again to determine the optimal value of the u which will minimize the given performance index, which is defined as half of 0 to infinity, x transpose t Q x t plus u transpose t u sorry R u t. And in this case it is a time invariant case Q and R are also selected as a constant matrix with the condition that Q is positive semi definite and R is positive definite matrix.

So, if I will solve this equation, I can find out the u t as minus R inverse B transpose P x t.

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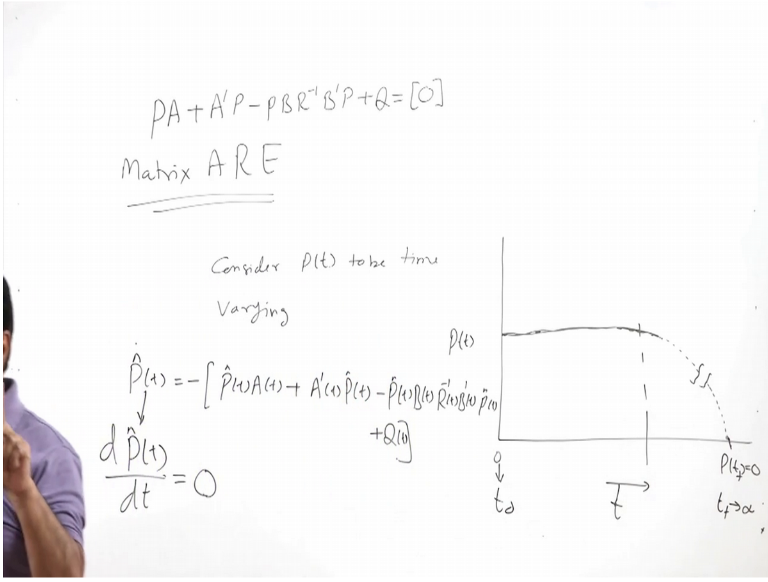

So, here as my R and B these are the constant matrix. So, I have to check what is the nature of my P matrix. Now what actually the P is? So, we have considered the consider Pt to be time varying. And what we are doing, we are basically starting our solution as P tf to be 0. What we have seen in the first case. At P tf f of tf is 0 we are not considering any terminal cost. So, P F tf will be 0, if this is the value of Pt and this is the time, as we are considering as tf approaches to infinity. So, I am starting my solution from this point. So, this point will move forward. So, as tf will approaches to infinity my A B Q and R matrix are constant. If Pt to be the time varying, then my Riccati equation is P hat dot t as minus PA plus a transpose P minus P B R inverse B transpose P plus Q.

So, now if we will see this P will depend on A B R and Q, Q is also time varying for time invariant case my A B R Q this all matrices are the constant matrix. So, if these all are the constant. So, naturally my P hat matrix will also be the constant. If this is constant this means the P dot is d P by d t. This will be 0. So, this d P by d t, I can make it to 0 and my equation will become PA plus a transpose P minus P B R inverse B prime P, plus Q equal to 0.

So, nothing, but what is the time varying equation that is going to change into a constant equation, my P will also be a constant because my A B R Q matrices are constant. So, I will get in place of the matrix differential Riccati equation, I will get matrix algebraic Riccati equation. So, this is nothing, but my matrix A R E.

Now, as we are saying at P tf my terminal cost is 0. So, as my solution will move on this up to the t 0. And this is my if 0 I will take it as my t 0. So, nothing, but for a when my solution will become constant. So, this is the constant range of the Pt as I will move towards the P 0. Because you can recall that we will solve the matrix Riccati equation with the condition that P tf is known to me, in time varying case the P tf is nothing, but my F of tf, for finite horizon. For infinite horizon F of tf will be 0. So, my P tf will become 0. So, my solution will Pt solution will be this one.

So, this means once I am solving this equation as soon as this will approach towards the t 0 point. So, I will get a constant value for the p. So, a infinite horizon problem we will have matrix algebraic Riccati equation to determine. And what is my matrix algebraic Riccati equation is which is called the matrix A R E PA plus a transpose P minus P B R inverse B prime P plus Q equal to 0.

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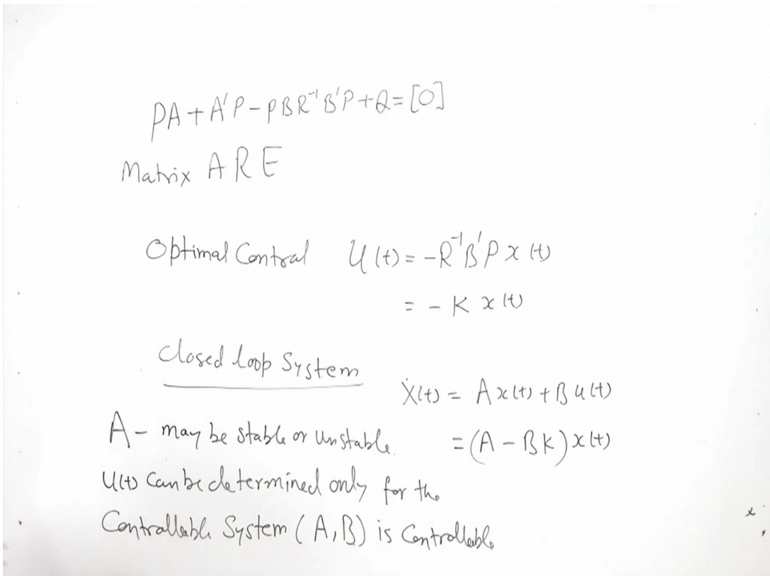

So, if I can determine the P by this then my optimal control is nothing, but u t which is R inverse R is a constant. Now B is also a constant P is also a constant and x t, which simply I can write as u equal to minus K x of t where K is a constant vector sorry constant matrix. So, if u is minus K of x t my system was as we have considered x dot equal to A x plus B u where A and B are time invariant. So, my closed loop system will be x dot t, this is A of x t plus B of u t, which we can write as a minus if I will place the u as minus K x, B K x of t. So, this is my closed loop matrix A is my original system matrix.

Now, the first question is whether A to be stable or unstable. So, A may be stable or unstable. Now second question is can we determine the u for all kind of the system? U t can be determine only for the controllable system. So, means pair A B is controllable.

So, if we will recall what we said in a finite horizon problem our condition was even if the system is not controllable, we can find out the optimal or say the value of the u and x, because my time is finite. So, any finite value I will get by the solution of my plant equation, for a given value of the optimal for a given optimal u, but for a time invariant case, if my system is not controllable this means there are the few states which are uncontrollable, in which you do not have any effect on the few state of x. If the more related to them is unstable. So, my system will also be unstable.

So, this means I do not have effect on the modes related to the few states. So, I am not able to control the system. If my system is not controllable if my system is controllable this means I can change, all by states from if they are initial value to some final value. As tf is infinity, this means all these final value must approach to the 0 for my regulator problem. This means if my system is controllable this means my matrix for a closed loop system, it should be a stable matrix.

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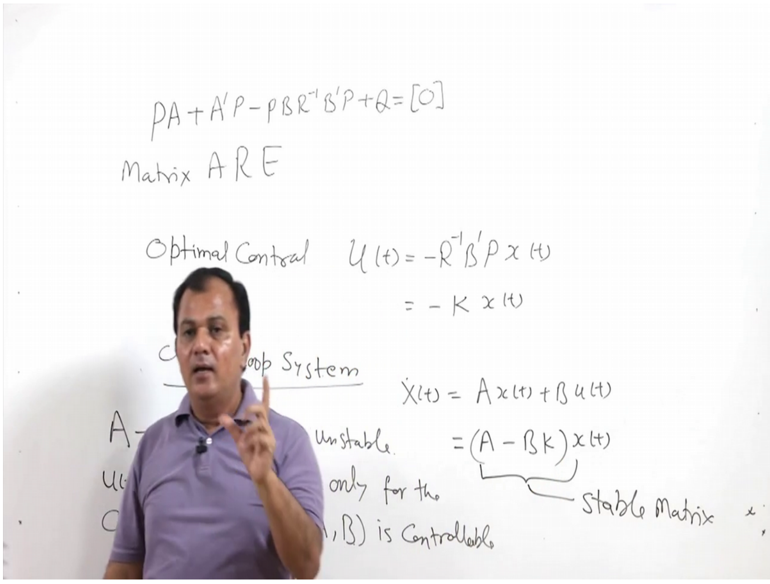

So, what actually we are doing my original system is x dot equal to A x plus B u and where A B is controllable. This means I can transfer all the state of my x for original system from t 0 to origin, because there is no boundation on the time t. So, I am not worried about where the how much time it will take, but I can ensure my all state will reach to 0, if my a minus B K matrix is stable.

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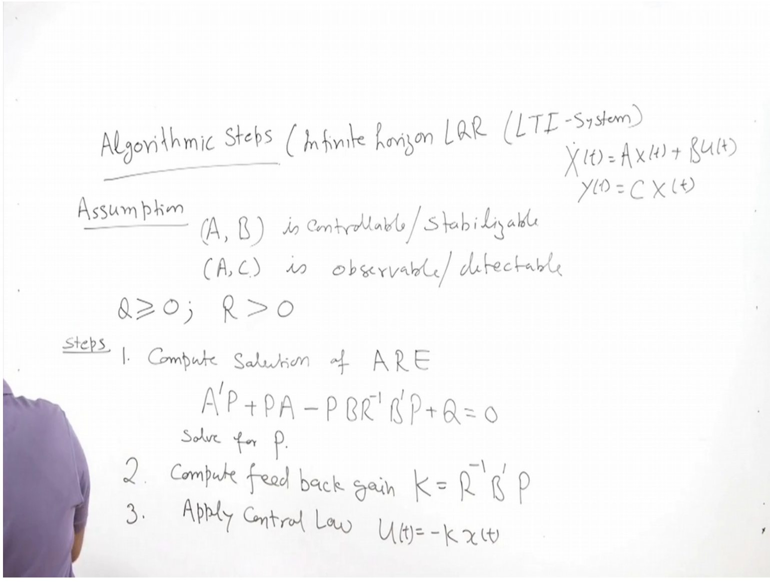

So, next we will see the algorithmic steps which we have to take to develop such a system. So, we will see what is my algorithmic steps. And my problem is infinite horizon infinite horizon L Q R which is time invariant for time invariant case. So, time horizon LTI system linear time invariant system.

So, we are considering infinite horizon LTI system. So, what will be my algorithmic step. So, my assumption is A B is controllable or at least is stabilizable. And AC is observable or at least detectable. So, stabilizability and the detectability are the inferior properties of the controllability, in stabilizability we said that the modes which are uncontrollable. These modes are is stable in nature then we said the system is stabilizable. And similarly my AC P R is observable or at least detectable. Detectable means I have the certain states which cannot be estimated. In observable system we are estimating all the states, if we are not able to estimate all the states we are able to at least measure these states.

So, in a detectable system if my states are not measurable directly they must be sorry if my states are not cannot be estimated. Then these state must be measurable. Then we said my system to be the detectable. So, I am considering for a linear LTI time invariant system, my which is defined as x dot equal to, So, the system is x dot t equal to A x t plus B u t. And y t equal to c x of t my A B pair is controllable or stabilizable, AC pair is observable or detectable.

We have selected the Q to be positive definite, sorry positive semi definite and R to be positive definite matrix. So, my steps are compute solution of arithmetic Riccati equation means a transpose P plus PA minus P B R inverse B prime P plus Q equal to 0. So, solve for P. If P is known, compute feedback gain is my K which will be constant this is R inverse B transpose p.

The third step is apply control law my control law is u equal to minus K x of t.

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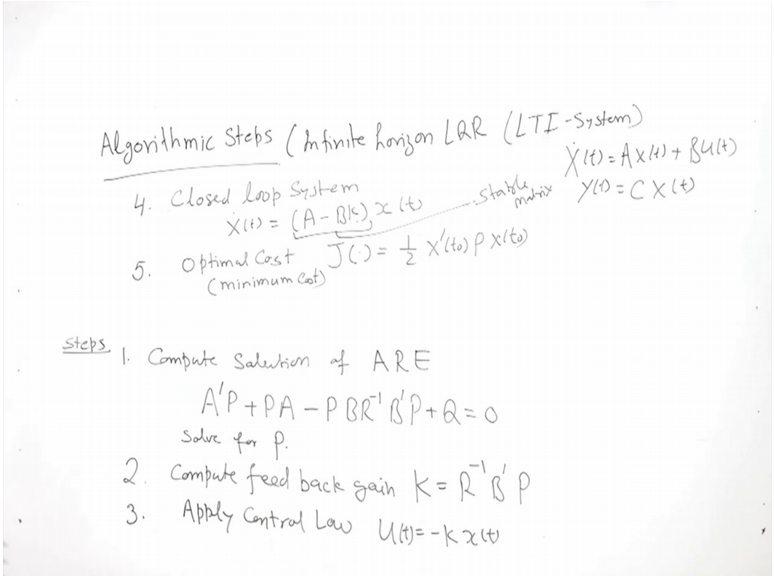

So, you can write u also as u of t equal to minus K x of t. Now closed loop system, and the closed loop system is x dot t is a minus B K x of t and optimal cost of my performance index j is half of x t 0 P x transpose of t 0 x of t 0. Where so, this is optimal cost is nothing, but my minimum cost.

So, these are my algorithmic steps these 5 steps, I have to follow I will compute the arithmetic Riccati equation for P, I will compute the back sorry feedback gain K as R inverse P prime P, I apply the control law u equal to minus K x t in my given system my close loop system will be a minus B K x t and the optimal cost or the minimum cost will be half of x transpose t 0 P x t 0, which we have seen. In this now next we will show that my closed loop matrix is a stable matrix. So, if a minus B K is stable my system is controllable this means whatever be the; I can transfer my all initial state x t 0 to the 0 if a minus B K is a stable matrix.

So, next we will show a minus B K is a stable. So, that we will take in the next class, I stop this class here.

Thank you very much.