Optimal Control
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Lecture – 19
Linear Quadratic Optimal Control Systems
(Optimal Value of Performance Index)

Welcome class. In this class we will discuss about the how to determine the optimal value of a performance index. In the previous class we have discuss about the salient feature of matrix Riccati equation, as well as we have also discussed what is a sufficient condition in which we find if del 2 H by del u square is R and R is positive definite.

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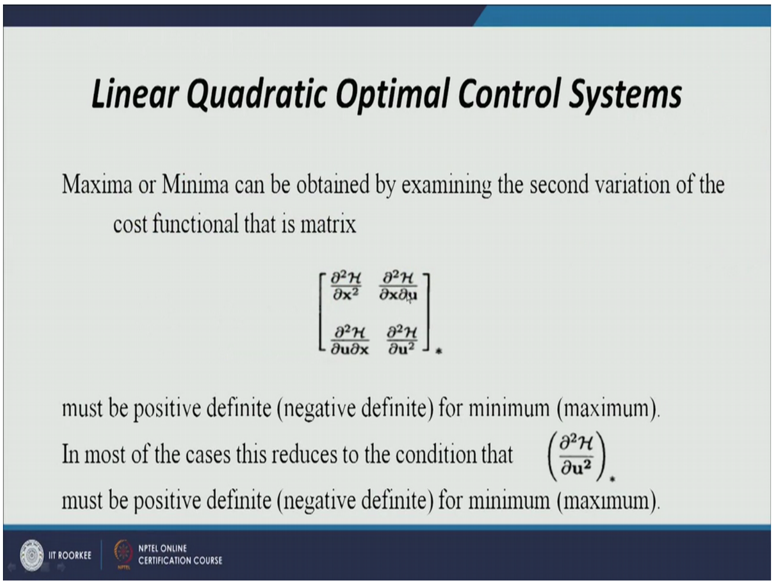

So, my matrix del 2 H by del x square del 2 H by del x del u, del 2 H by del y del u del x, and del 2 H by del u square must be a positive definite. With respect to a linear system if we will see my, this value is Q this half diagonal values are 0 and this value is r.

So my, this matrix is positive definite as Q is a positive semi definite and R is positive semi definite. So, we can say if we have del 2 H by del u square a positive definite, this means my we are minimizing the performance index in solving the optimal problem.

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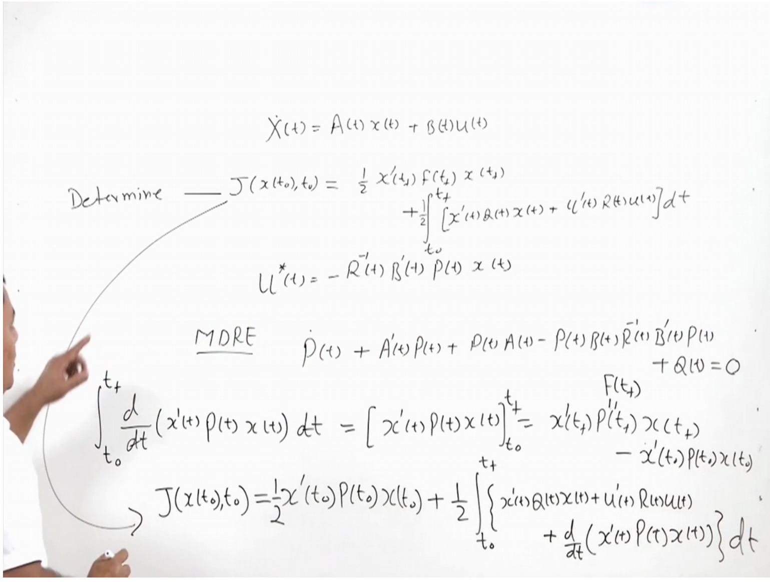

We will see how to find out the optimal value of the performance index. My problem here is I have a system x dot t as A t x t plus B t u t, and I have to find out the optimal u which will minimize my performance index, which will defined as half of x t f f of t f sorry x prime t f of t f x of t f, plus integral t 0 to t f. It is my finite time regulator problem, and this is x prime Q x, plus u prime R u d t and half we use here for our simplification.

So, my objective is to determine the optimal value of the u which will minimize the performance index, given by j x t 0 t 0 with. So, my optimal u which we have find out that is minus R inverse, B prime P t, x t where P t is the solution of my matrix differential Riccati equation, which is so matrix, differential Riccati equation is what that is P dot t is given as minus. So, we write it in a different way. We try to simple equation directly, P dot t plus a transpose P plus P a minus P B R inverse B prime P t, plus Q t equal to 0. This is my matrix differential equation. So, this is my optimal u to determine this u, I require P and P can be find out by the solution of given matrix differential Riccati equation.

So, in today's class my objective is if this values what is the; I want to determine j. So, to find the solution of this problem, we just consider d by d t of x transpose t, P t x t, into d t. What we can write for this. We can write for this because d t will be cancelled out. So, this is d of x transpose t P t x t with the limit t 0 to t f.

So, this is x transpose t P t x t, which can be written as here sorry this is t 0 to t f x of t f P of t f x transpose, of t f P of t f x of t f minus x transpose t 0, P t 0 x of t 0. So, this term will have these 2 values. So, in the given performance index; so this j, I am writing in a different manner I am writing, j x t 0 t 0 as I am adding this term x transpose t 0, P t 0 x of t 0. And see I am writing this half, plus 1 by 2 t 0 to t f, this term I will write s h x transpose Q x, plus u transpose R u. And I am adding this term d by d t in this relation.

So, I am writing this and in this we are adding d by d t of x transpose P x, plus d by d t of x transpose P x d t. Now, what this will have if I will explain this integral. So, this will give me x transpose of P t x t f P t f x t f. This term will be cancelled out and P t f is nothing but my f of t f. So, j written in this form or written in this form is the same value. This j of this j written in terms of the x prime f x t f I can also write in this given form.

So, performance index we can write in this form here. So, if I will explain the integral of d by d t of x transpose t P t x t, my this term will cancelled out and I will left only with the x transpose t f P t f x, of t f where P t f is nothing but my f of t f. Now to determine the value of the j we have to evaluate what we will have in this integral. So, we will separate out that term of the integral.

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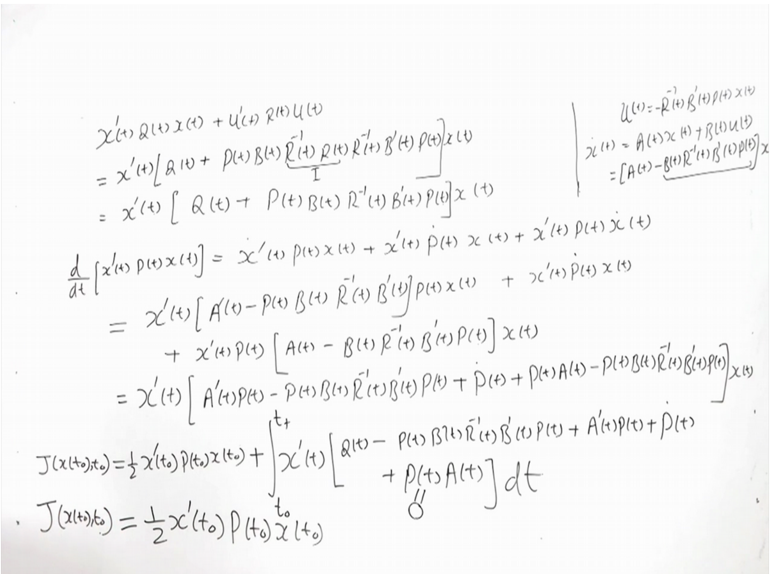

Let us first see; what is the value of x transpose Q t x t plus u transpose R u.

So, here u we can write as, so we can take u t as minus R inverse B transpose P t x t. So, if I will take u this, so value of this I can write as x transpose of t, Q t plus if I have to take the transpose of this so x transpose will be the first term then P d R inverse. So, this will be my x transpose I have taken outside. So, this will be nothing but P B R inverse for the u transpose x. We have taken then this side this R t, then u t again directly I will write this so R inverse B transpose P t, and this I will write within this bracket.

So, this is equal to this one in this one R will be i. So, if I will write this is nothing but my identity matrix i. So, I can write x transpose t Q t plus P B only one R inverse is left out, R inverse B prime P t x t. So, the first 2 term x prime, Q x u prime R u, I can write in this form Q t plus P B R inverse B prime p.

Now, if I will explain d by d t of x t transpose P t, x t this is nothing but my x dot transpose t, sorry P t x t plus. So, first term I am taking the derivative of this second taking the derivative of p, this is P dot t x t and the third will be plus x transpose t P t x dot of t.

So, d by d t of this can be written in this form, and here what is my x dot. If we will see x dot of t, is A t x t plus B t u t where u t is R inverse B prime P t x t. So, this is nothing but A t minus B R inverse B prime P t x of t. So, this is my x dot t. So, I will place my x dot t in the first and the last term.

So, this I can simply write as in place of x dot. So, if I am taking the x dot transpose. So, this I can write as x transpose of t because I have to take the transpose of the whole term this whole term means x transpose plus sorry multiplied with the transpose of second term and the transpose of this term is a transpose t minus the transpose of this term will be once you will transpose this so my first term comes out to be P t P t is symmetric. So, is transpose remains the same B transpose will become B R again in symmetric remains same and B will become the B transpose.

So, this is my x dot transpose multiplied with P t x t second term, I left it same x transpose t P dot of t x t plus, the third term x transpose of t P t and in place of x dot t I will place the value of this x dot which is A t minus B R inverse B transpose P t and x t.

Now, in the next term we will write the whole expression in between x transpose of t and last x t. So, I will take x transpose of t from here. So, this is left by and t we are multiplying into this in the first equation, this is a transpose t P t minus, P t B, R inverse B transpose p. So, by expanding this I will got this; this x we will take some outside the bracket. So, by expanding the first term we got this from here we got only the P dot of t, from the second term plus from the third term x transpose taken outside P, I will multiply plus P a and minus P, sorry P B R inverse B transpose P t, and last term will be my x t.

So, if I will see my first 2 term are in terms of the x transpose x, and d by d t term I am also writing in terms of x transpose of x; so to get the integral second term. Now, what I can write for this j. Now what I have to do? I have to at this term plus this term with d t. So, once we will add these 2 terms. So, we will get j as my first term was one by 2 x t 0 P t 0 x t 0 transpose of the first term, plus integral t 0 to t f and these 2 term I have to add. And this I am adding in this way x transpose of t, I will write here from the first 2 term I will have, Q t plus P B R inverse B transpose p.

So, I am getting this term from here and from here if I will see plus a transpose P once I will write. So, this will be the 2 times twice of P plus I will write first this as the P dot. Next term will be a P which I am getting from sorry this will be P t A t P a and minus 2 P B R inverse B transpose P t and x t d t.

So, if I will further explain this. Now, you will see this is the 2 P B, R inverse B prime P B R inverse B prime p. So, in a way what I can write I can remove this, and this I can make as minus, and I can write this into d t. So, now, what you see what I will have in this, P transpose sorry P dot t plus a transpose P a minus P B R inverse B prime P plus Q t and this is nothing but my matrix differential equation which value is 0. So, all this value is nothing but 0; so half P t 0 x t 0.

So, I will get j optimal value of performance index is nothing but half of x transpose t 0 P t 0 x t 0. So, what we have done once we will see.

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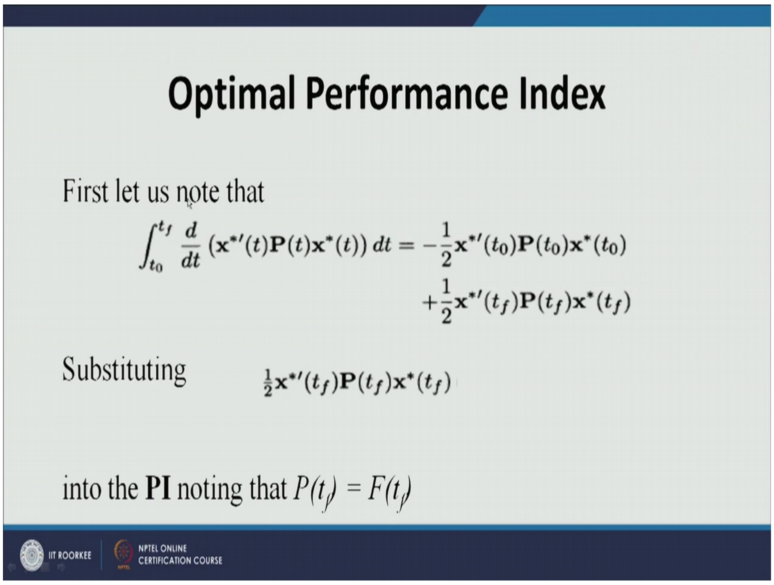

We have taken the d by d t of this expanding as x transpose t 0 P t 0 x t 0, with condition that P of t f equal to f of t f. So, this means if I will replace this so this is showing me nothing but my terminal cost.

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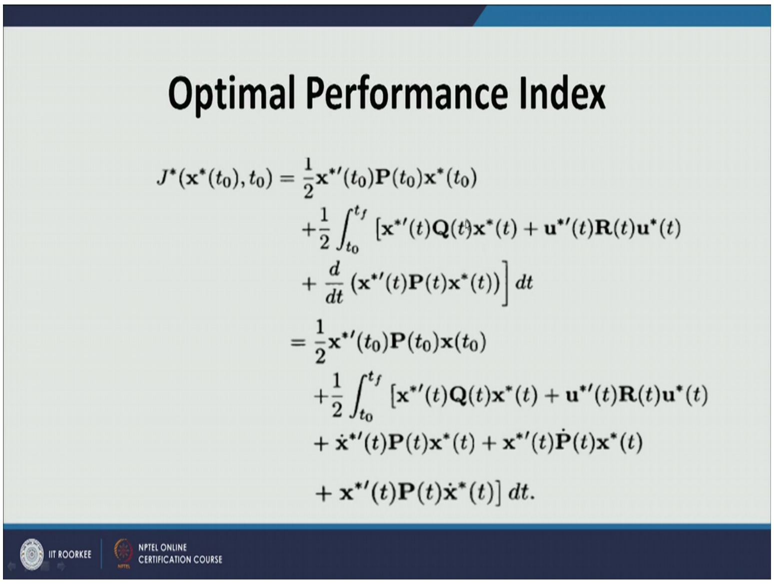

In my performance index, I am substituting I am adding this half of x transpose t 0 P t 0 x t 0 and including d by d t of x transpose P x into the integral part.

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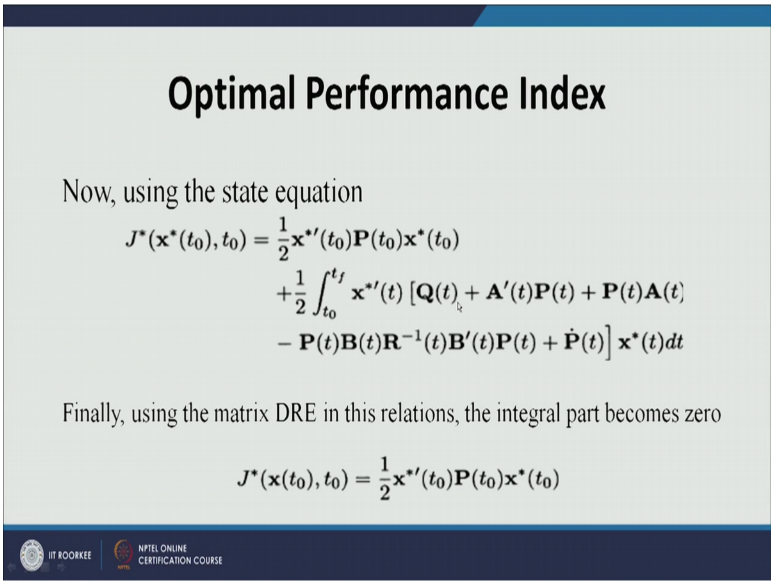

By expanding this as we have seen here we got nothing but half of x t 0 P t 0 x t 0 plus half of x transpose Q plus a transpose P, p a minus P B R inverse B prime P plus P dot x t. And if you will see this is nothing but my matrix differential Riccati equation which value is 0. So, my this whole term will become 0 some left with the initial cost as j x t 0 t 0 as half of x transpose t 0 P t 0 x t 0.

So, this expression gives me nothing but my optimal value of the objective function. Now as we have solve this from t 0 to t f. Similarly, I can solve it from t to t f for any intermediate point.

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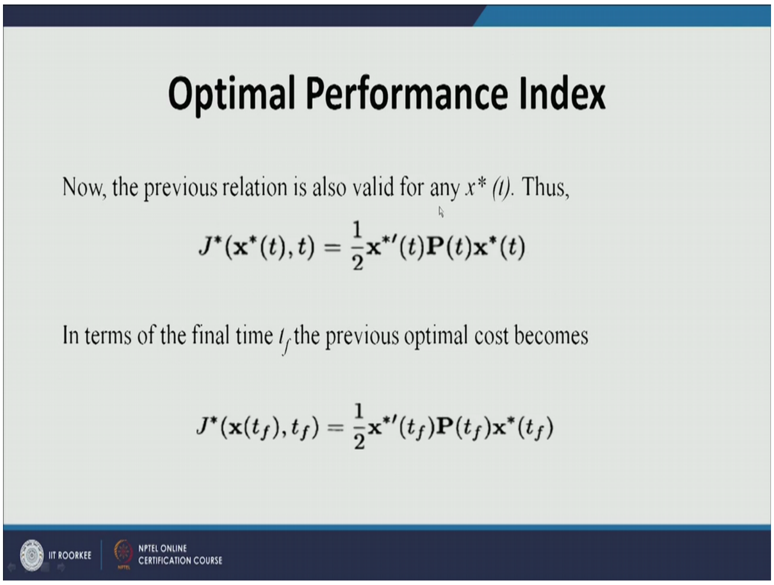

So, if I am solving it for t to t f my optimal cost will be x prime t P t x t, because t is varying from t 0 to t f even I can also express my optimal cost as half of x transpose of t f P t f x t f. So, by this we can determine the optimal value of my objective function, and as my second variation is 0, R is positive definite. So, this means this optimal cost is nothing but the minimal cost of my performance index.

In the next class we will start with the infinite time l Q R system. So, I stop my discussion for this class here.

Thank you very much.