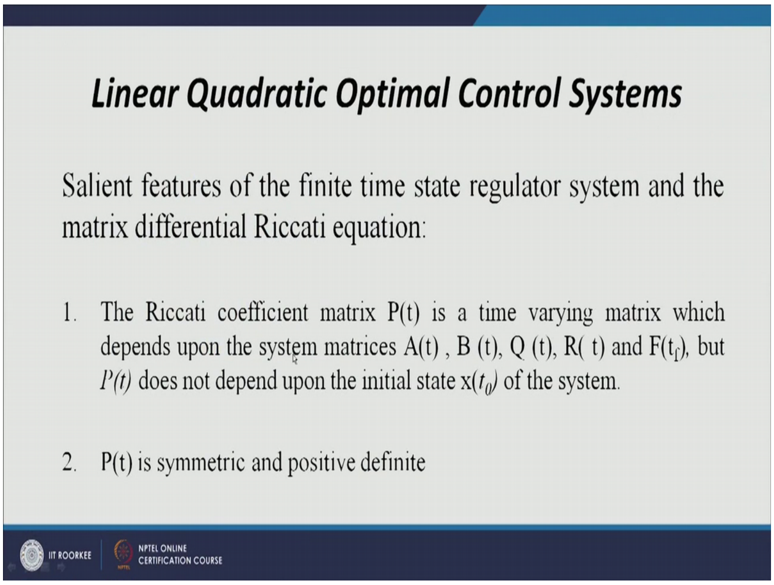
Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture – 18
Linear Quadratic Optimal Control Systems (Continued)

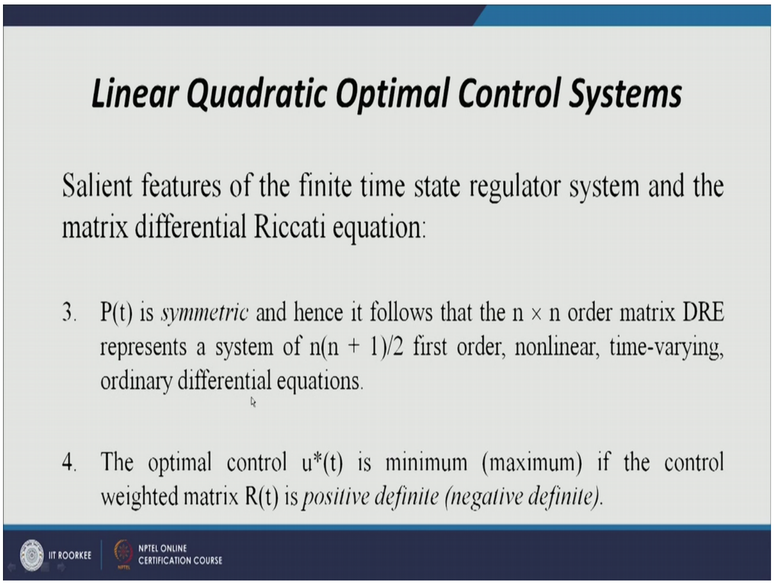
So, welcome friends to this class we are discussing about the linear quadratic optimal control system. And particularly we are discussing the problem of the LQR linear quadratic regulator system. In the previous class we are discussing about the some important feature of LQR system in which the Riccati coefficient matrix Pt depends on A B Q R and F.

(Refer Slide Time: 00:48)



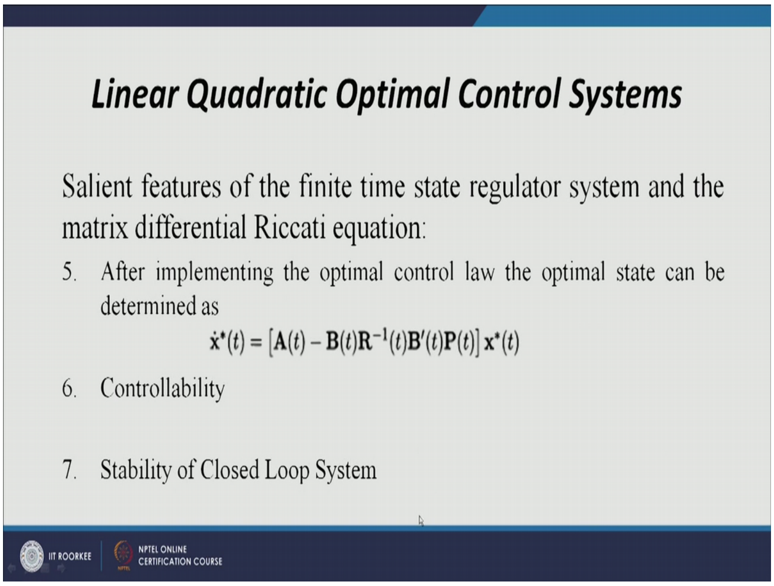
But it does not depend on the state Pt is symmetric and positive definite.

(Refer Slide Time: 00:56)



Because Pt is symmetric so we have the total number of the matrix differential Riccati equation is n into n plus 1 by 2 where n is the order of my system. And for u star to be minimum my R t should be a positive definite matrix.

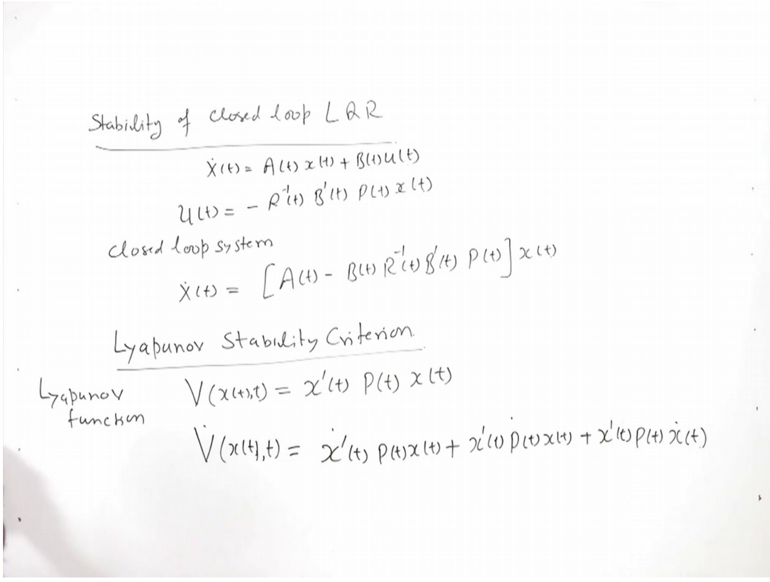
(Refer Slide Time: 01:18)



We have a closed loop implementation has At minus Bt R inverse P prime Pt. A finite time regulator need not to be controllable and we are discussing about the stability of a closed loop system means we are discussing the stability of my closed loop system, which is defined as x dot t equal to At minus B R inverse B prime P t.

So, this system is stable, that we can show using the Lyapunov theorem.

(Refer Slide Time: 02:04)

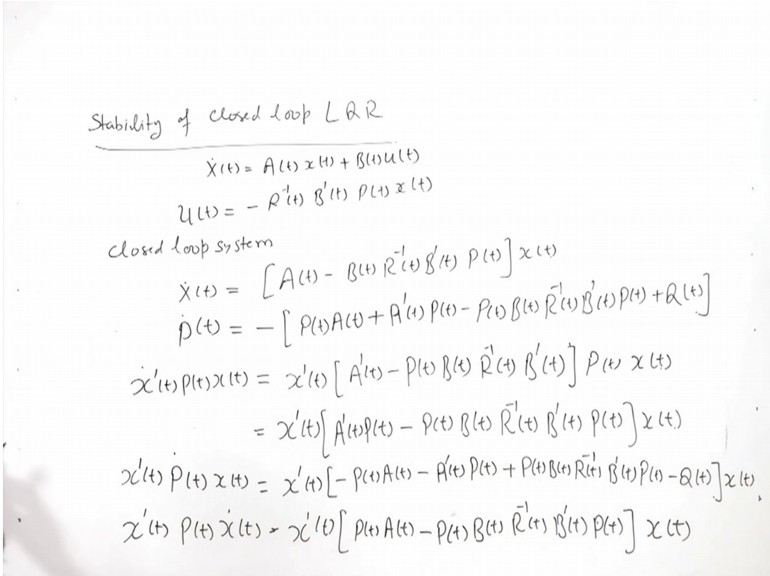


So, we will discuss about the stability of closed loop linear quadratic regulator. So, as we have shown that my given system is X t equal to At x t plus Bt u t. And u t is nothing, but my minus B prime t Pt x t; where Pt is the solution of my matrix differential Riccati equation. With this my closed loop system is my closed loop system will become x dot t is At minus B R inverse B prime Pt x t.

So, we have to show that my closed loop system is a stable system. So, what is my Lyapunov stability criterion? So, this we will prove using the Lyapunov stability criteria. So, direct approach of the Lyapunov we can apply in which first of all we have to define my Lyapunov function. So, I can take my Lyapunov function as V x t t equal to x transpose t Pt x t. Because P is asymmetric positive definite matrix and by the definition of my Lyapunov function, I can select my Lyapunov function quadratic in nature for a linear system. Where P is a positive definite symmetric matrix, and here this P nothing, but I can select what is my Riccati coefficient Pt. That I can use in Lyapunov function, to define my Lyapunov function here. What is my Lyapunov criteria if I have a positive definite Lyapunov function then V dot x t t must be less than 0; if V is this and V dot is less than 0. So, I can say my system is a stable system.

Now, prove this V dot to be less than 0, if my V is x prime P x. So, V dot will be x dot prime t Pt x t, plus x prime t P dot of t x of t, plus x transpose of t Pt x dot of t. So, this V dot will have the 3 term first is x dot Pt x t, second is x transpose P dot t x t, and the third term is x transpose t Pt x dot of t, x dot of t is x dot I will take as a minus Bt R inverse B prime P into x t and I know what is my P dot P dot t is nothing, but minus.

(Refer Slide Time: 08:04)



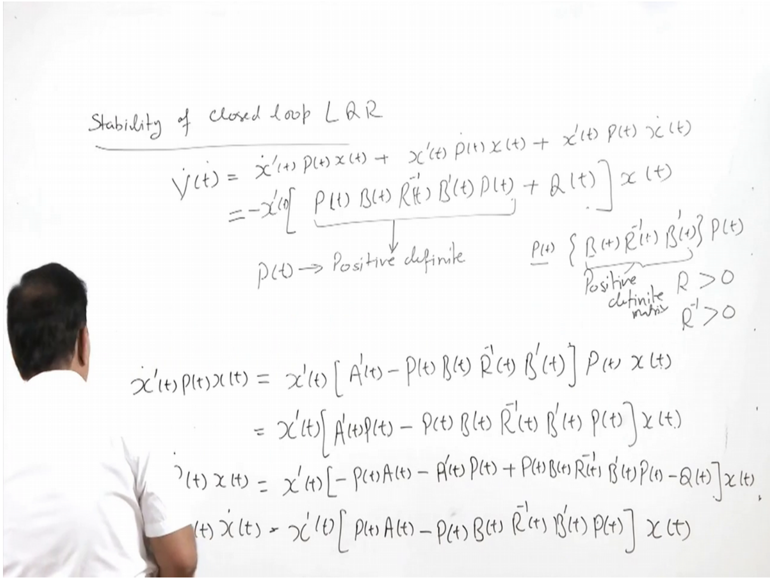
Sorry Pt P a, plus a transpose P minus P B R inverse B prime P plus q this is my P dot, so x dot and P dot if I will place this equation. If I will place this, individually we will find the 3 terms. So, my 3 terms are x dot prime Pt, x t will be what means I am taking the transpose of this matrix.

So, x dot transpose is my taking the transpose of the whole this this is giving me x transpose of t, transposing this a transpose of this minus in this transpose first I will get the P p is a symmetric. So, transpose remain the same B transpose is nothing, but B R is again symmetric. So, this will remain R inverse t and B will B transpose t. So, this is the P B R inverse P prime, and this x we have taken here multiplied with Pt x t. Pt if you will multiply. So, the whole time I can write as x transpose of t a transpose t Pt, minus Pt R inverse of t B transpose t Pt x t.

So, this I can write as this expression. My next term was sorry it was x transpose P dot of t and x t. So, here I will place the value of the P dot. So, this is x transpose of t and P dot I will write as minus P a minus a transpose P. This minus will become plus P B R inverse B transpose Pt minus Qt into last term x nothing, but my x t.

Last time will be x transpose t Pt x dot of t, x dot of t I will take this and multiplied with this p. So, what this value will come I write this x transpose, s h P, I am multiply in this equation. So, this is Pt At minus, sorry P B R inverse B transpose P, I am multiplying into this and x t. So, if we will see these term I am writing in between x prime and x. So, once we will at these term. So, the overall term I can write this as.

(Refer Slide Time: 13:46)

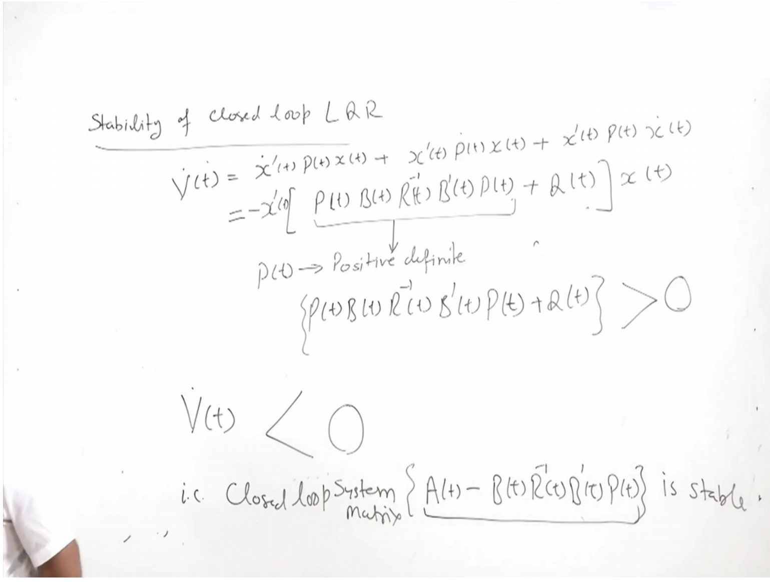


So, my V dot of t was x dot t, x prime dot Pt x t plus x transpose t Pt x t, plus x transpose of t, Pt x dot of t. So, to get my V dot I have to write all these term. If I am adding all these term, so we can see it here a transpose t Pt this will cancelled out by this, P a will cancelled out by this and P B this is negative this is negative sorry this is negative positive will cancelled out. So, I will left only with minus P B R inverse B prime Pt, my plus Qt. So, my V dot is this and naturally I will have minus x transpose and x t.

So, in between x transpose and x t, I will have this matrix. See what actually have Pt is positive definite. This means now this is my positive definite matrix. In between I have B R inverse B prime, and Pt. What is this; this is because this is B transpose sorry B transpose. So, they will give me the square terms. So, they cannot be negative R is, if R is positive definite R inverse we will also be positive definite. This means my this matrix will be a positive definite matrix.

So, this with this multiplication this means I got my whole this matrix to be also be the positive definite. Qt we have taken as a positive semi definite matrix.

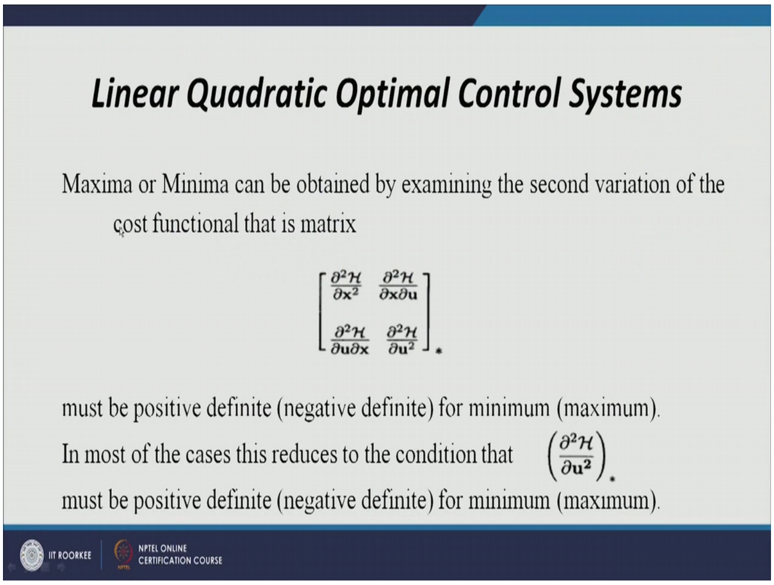
(Refer Slide Time: 17:38)



So, this means a matrix between x transpose and x. So, this means my Pt Bt R inverse B transpose Pt plus Qt. My whole this matrix is a positive definite matrix. If this is positive definite matrix I have negative here. So, I can say my V dot t will be negative definite is less than 0. This means my closed loop matrix what was this? At minus Bt R inverse B prime P t.

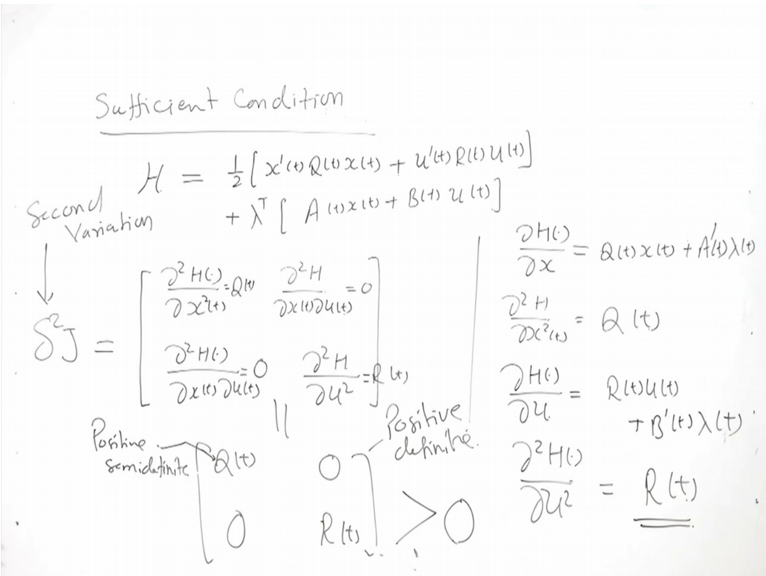
So, I can say this is my closed loop closed loop system matrix, which is this is stable. So, as we are saying my closed loop matrix will be a stable matrix. So, in a finite time regulator also whatever be my matrix A, I can say my state feedback regulator system is a stable system. So, next we will show what is my sufficient conditions.

(Refer Slide Time: 19:53)



Like we said that initially we have proved in a Hamiltonian case my second variation I can write in the form of del 2 h by del x square, del 2 h by del x del u del 2 h by del u del x del 2 h by del u square. And my this matrix should be positive definite if it is minimum and if my objective function is minimum, and it should be negative definite if my objective function is maximum.

(Refer Slide Time: 20:31)



So, for minimum I, if we will have the sufficient condition. So, to get my sufficient condition first of all I have to write my h, what is my h this is V plus lambda prime if and

V we have taken as a quadratic 1 by 2, Qt x t plus u prime R u plus lambda transpose, At x t plus Bt u t. Where, So, my sufficient condition is, I have to find my matrix del 2 h by del x t square, del 2 h by del x of t, del u of t, del 2 h by del x of t, del u of t and del 2 h by del u square.

So, we have find out the del h by del x, which is nothing, but my costate equation. And then del h by the second derivative of this. So, we have del h by del x if you will take, this is Qt x t del h by del x. We will take here independent of this and this is plus A transpose. So, this will have a transpose of lambda t. So, del 2 h by del x square is nothing, but your Qt.

So, del 2 h by del x del h by del x. So, this is if I will take with respect to u this will be 0. So, my this term will be 0. If I will say again the similar way my this term will be 0. This is my nothing, but Qt, and what is del 2 h by del u square, that we can find as del h by del u. What we are getting by del h by del u, I will differentiate this with respect to u. So, this is nothing, but my R t, u t plus B transpose t, lambda t, and del 2 h by del u square is nothing, but my R t.

So, this is nothing, but my R t. So, this matrix is nothing, but my Qt 0. So, this is my second variation. If I will see, this is my del square, j this is my second variation. So, second variation is equal to del 2 h by del x square and nothing, but Qt 0 0 R t. Now what is the state of this, if Qt is Qt is a positive semi definite matrix R t is a positive definite matrix. So, this means this matrix is a positive definite matrix. So, this is positive definite. So, this means in LQR, my this matrix is positive definite this means we are solving the LQR problem for minimization of the j. If this matrix is greater than 0 or if the sorry if this matrix is less than 0 means it is negative definite than our condition will be for the maxima.

So, LQR problem normally we solve for the minima, we are minimizing the performance index and my optimal u that we are determining by minimizing the performance index. And in this matrix if you will see my q is positive semi definite. So, the definiteness of this matrix will depend on the R t. And this R t always should be positive definite for system to be the for the performance index to be the minimum. So, that is why and initially pervious review we have also shown that R required to be the positive definite because in optimal u we also have to use the R inverse.

So, in general we can say we can directly check the sufficient condition simply by checking the del 2 h by del u square. And this del 2 h by del u square should be a positive definite matrix, and if it is positive definite. So, we can say that my sufficient condition is satisfied and we are working for the minima of my system. So, in some cases we are also required to find out what is the optimal value of the performance index. So, this we will continue in the next class this class I stop here.

Thank you very much.