Optimal Control
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Lecture – 16
Linear Quadratic Optimal Control Systems

So, welcome friends. In this lecture we will continue our previous lecture which was in the linear quadratic optimal control system problem, we are trying to find out the optimal control law for a linear quadratic regulator problem.

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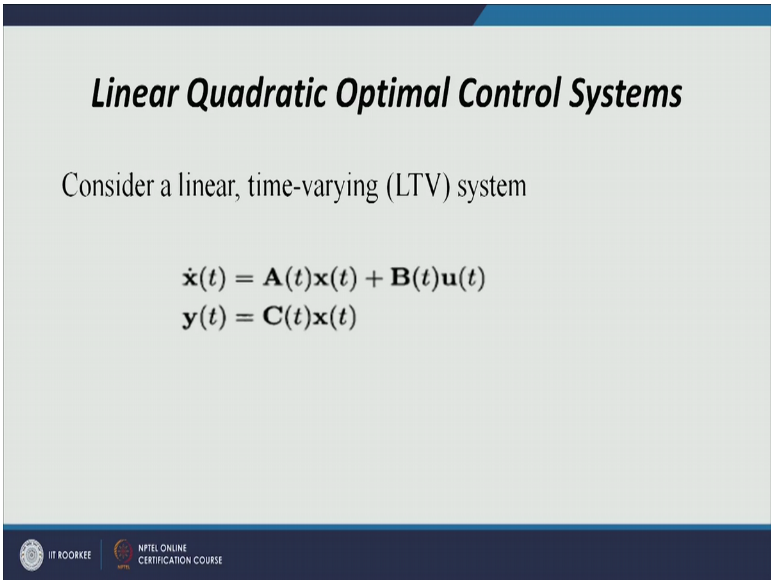

In linear quadratic regulator problem what we are doing? With a, we are given with the system x dot equal to A x plus B u, y equal to C x.

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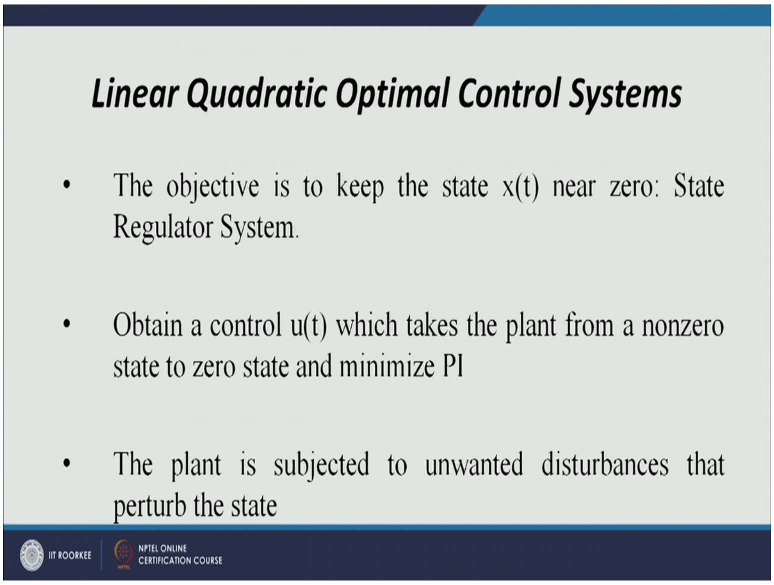

Objective is to determine the u which will take the plant from nonzero state to a 0 state and simultaneously minimize a performance index which is given as x prime t f f of t f, x of t f plus half of x prime Q x plus u prime R u.

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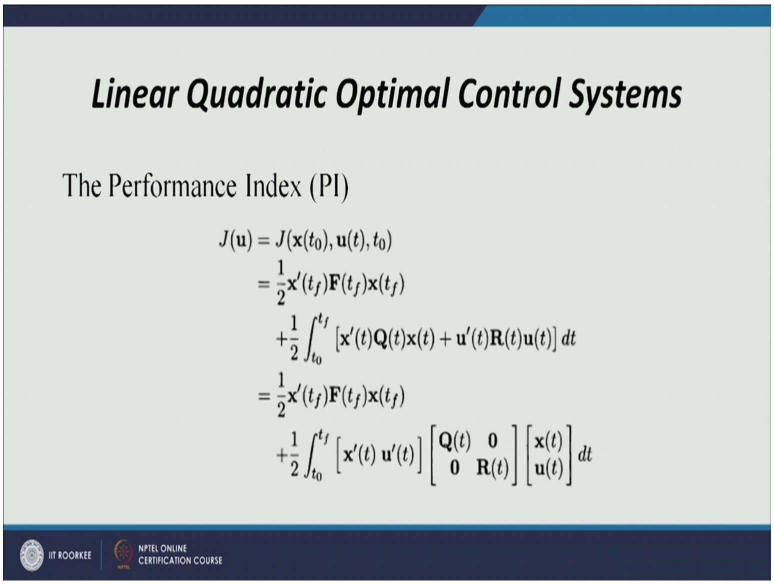

So, this problem we are solving using the Hamiltonian approach. We first define our Hamiltonian H we take del H by del u equal to 0 which give the u equal to R inverse B prime lambda t.

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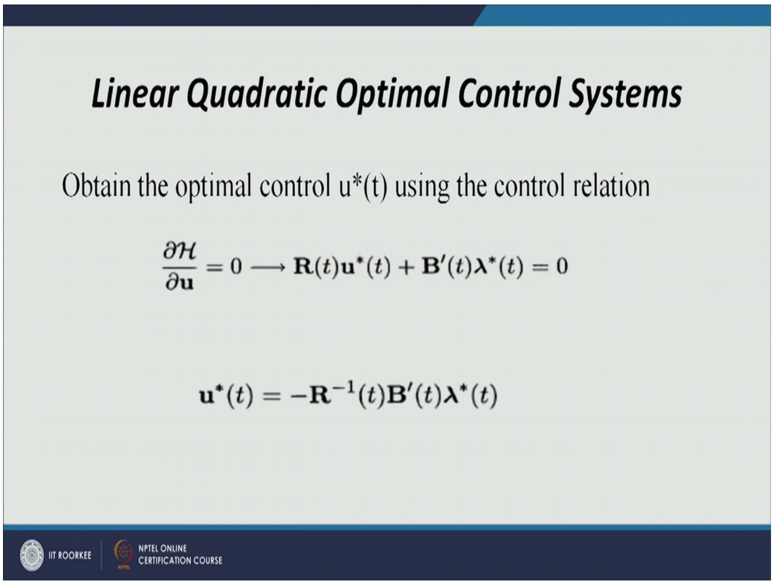

Then we define our Hamiltonian system x dot equal to A t minus E t minus Q t minus A t x lambda. So, this means x dot and lambda dots are represented in terms of the x and lambda.

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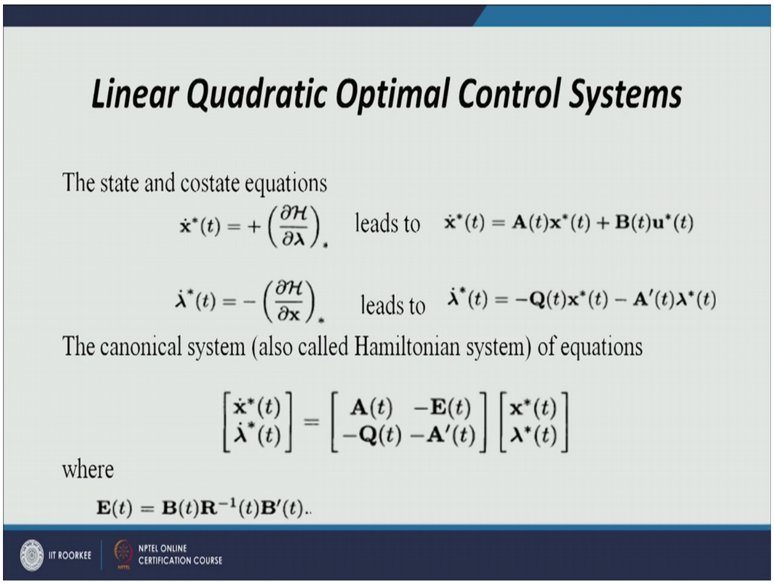

So, we are the n by 1 state vector n by 1 costate vector. So, total dimension of the system is 2 n by 2 n.

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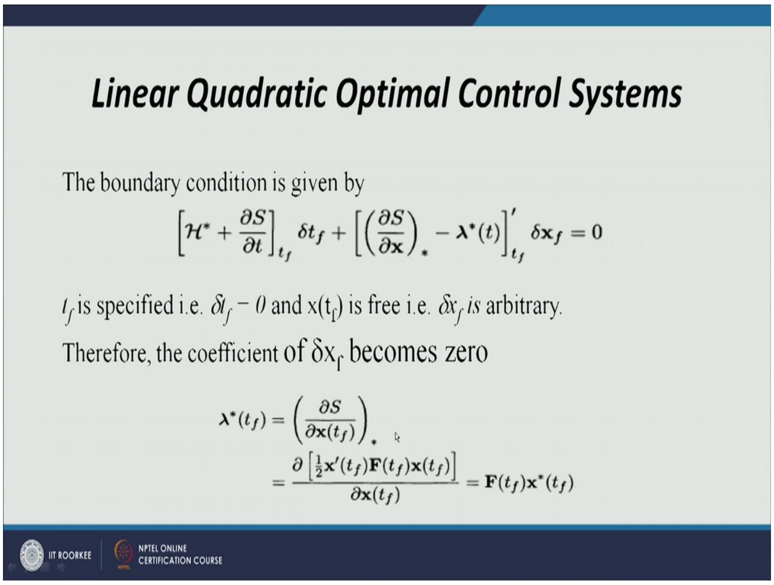

This Hamiltonian system along with the terminal condition we have taken as a lambda t f as F of t f, x of t f. So, we have to develop a state feedback system and if you will recall my u is nothing but R inverse B prime lambda t lambda is unknown quantity. So, this means we cannot directly feedback my system. To get the feedback I have to relate my u with x, but to develop the relation between lambda and x sorry; lambda t and x t we can take the intuition from the lambda t f, F of t f, x of t f.

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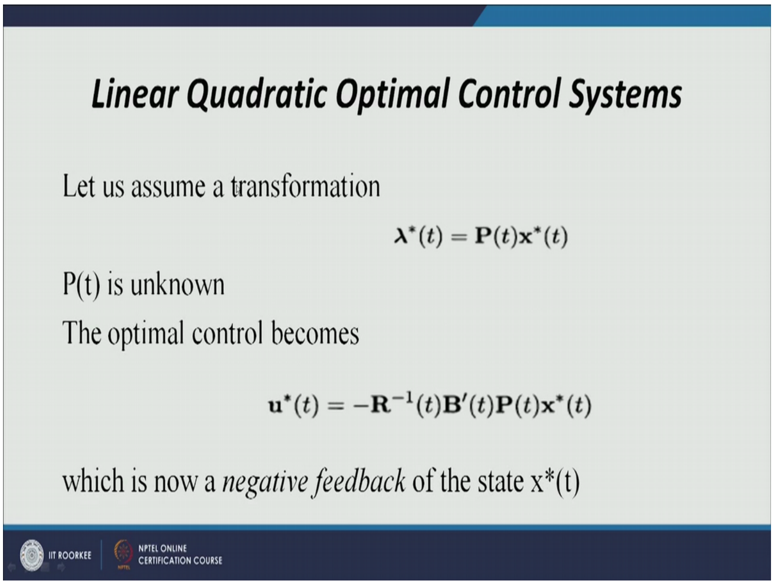

So, we assume a transformation which will give me lambda t as P t x t where P t is a unknown quantity. We have to find out the value of the P. So, if lambda is P t x t then u t I can simply write as minus R inverse t B prime P t x t.

So, this means if I know the x I can determine my R inverse B prime P. So, I can feedback. So, if you will recall in the previous lecture in a linear quadratic problem my control law u equal to minus k x. So, this R inverse t B t P t will be nothing, but my k. So, by this transformation I can transfer my system in a simple linear feedback system. So, this negative feedback of the state x t I can give to use it as my control feedback.

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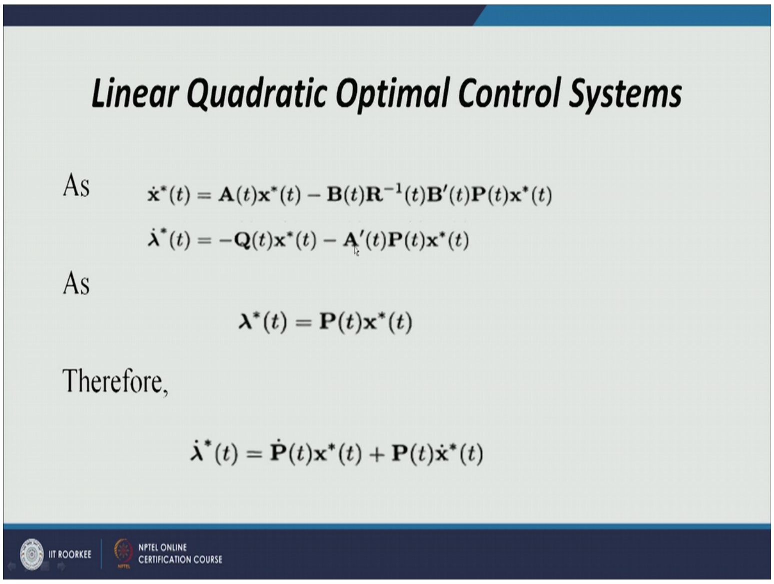

Now, from my Hamiltonian system my here x t as A x minus B R inverse B prime P t x t and lambda t as minus Q t x t, A transpose P t x t. Transformation we are considered as lambda t as P t x t, we have taken as lambda t - P t x t.

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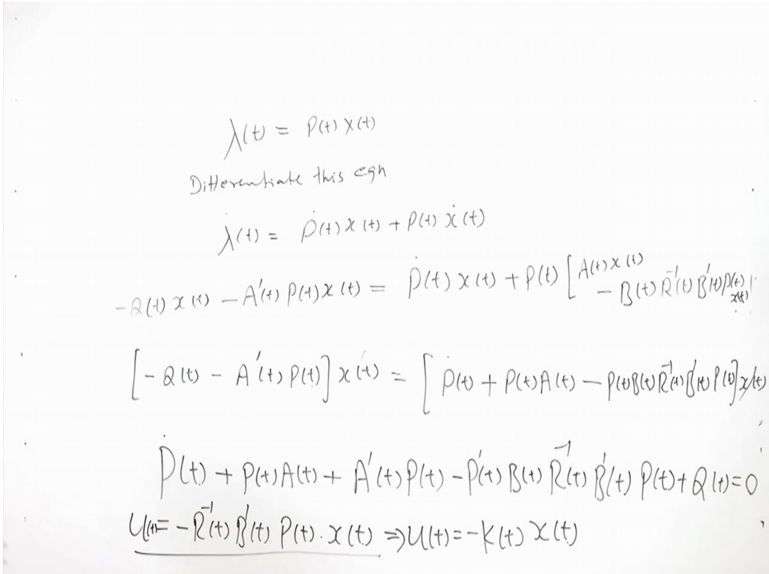

So, if I will differentiate this, differentiate this equation. So, I will get lambda dot t as P dot t x t plus P t x dot of t.

So, I have equation lambda dot t as P t x t plus P x dot t, and x dot and lambda dot value I can write in terms of my Hamiltonian equation. Say in this Hamiltonian equation if you will see I have the x dot in terms of the x t and if you will see P t x t is nothing, but my lambda t. So, from this relation I have replaced a lambda t by P t x t. With this I am writing this 2 relation as A x minus B R inverse B prime P x and lambda dot as Q x A prime P x. So, this means I am representing x dot and lambda dot both in terms of the x.

Now, in lambda dot t equal to P dot x plus P x dot. I am substituting the value of the lambda t and x t which will be; so what is my lambda dot t? Lambda dot t is minus Q t x t minus A transpose P t x t minus A transpose P t x t this is my lambda t, this I will keep as such P dot t x of t plus P t and x dot of t is A x, A t x t minus sorry, I have B, R inverse, B transpose t x t.

So, I have substituted x dot and lambda dot in place of this lambda dot and x dot. This is minus Q t, A transpose t x t and this side what we have? P dot plus P A minus P B R inverse B transpose sorry; this will be B transpose. Sorry here I have directly written the x t, but this is P transpose P t x t this is B transpose P t and x of t and this x of t we can cancel out and write the whole equation on one side, we have P dot t plus P t, A t. Second term I am writing here a transpose t P t minus P, B, R inverse, B transpose t, P t plus Q t equal to 0.

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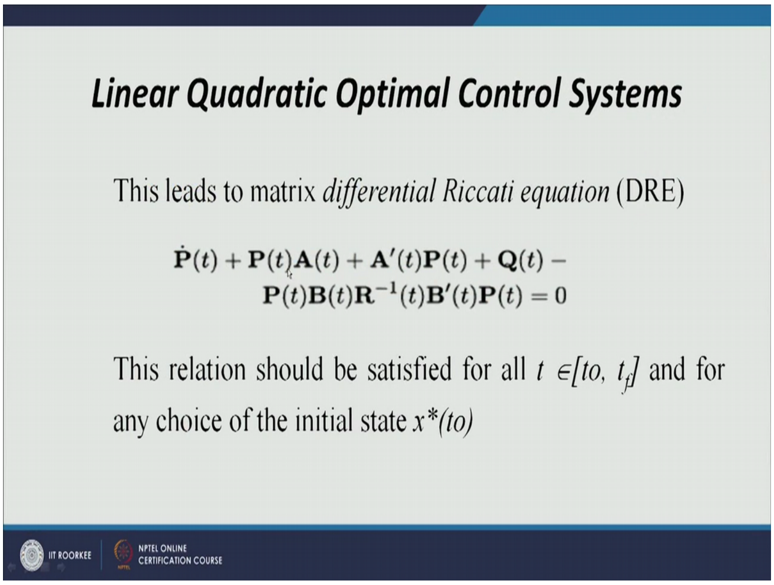

So, by this we got this equation P dot t plus P A plus A transpose P plus Q t minus P B R inverse B prime P t equal to 0 and this relations should satisfied for all for my specified time t 0 to t f, and for any choice of my initial condition x t 0. So, this means I have to solve this equation to get the value of the P and as we can see this is a differential equation which can be solved using my terminal condition which is lambda t f as P t of x t f equal to F of t f, x of t f; this means P t f is F of t f.

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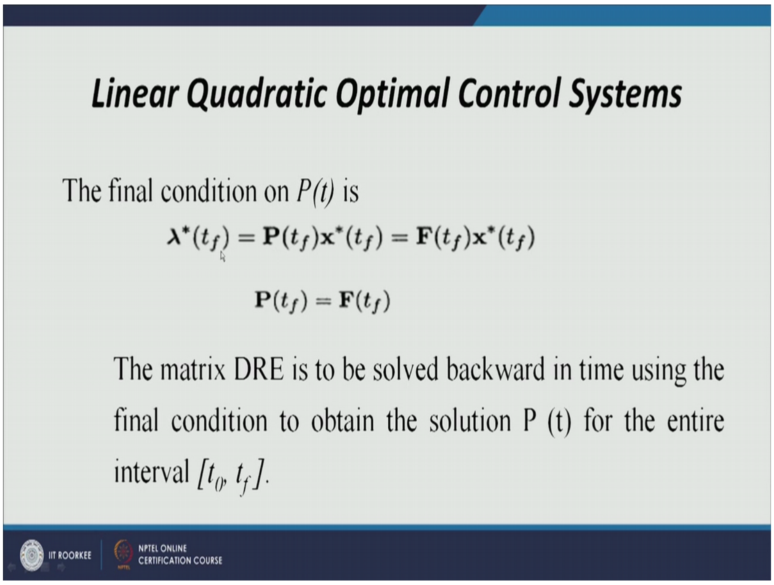

So, this is the matrix Differential Riccati Equation, this has to be solve backward in time and using the final condition to obtain the solution pt for the entire interval t 0 to t f. Once I know the P t them directly I can write my u which is nothing but minus R inverse B transpose now it is P t into x of t. So, this is my u, ut; which I am directly writing as this implies u t is minus K t x of t.

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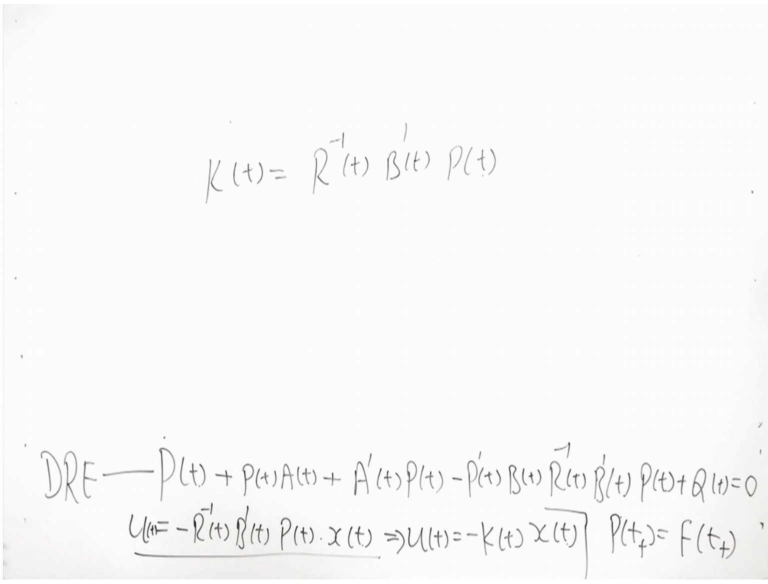

So, K is nothing, but R inverse, B transpose t, P t. So, for a linear quadratic regulator system I have to find out the value of the K t which is R inverse B transpose P t and P t is nothing, but the solution of my differential Riccati equation DRE and this DRE can be solved using the final condition as my P t f equal to F of t f. So, in backward we can solve this Riccati equation to get my K t. So, directly I can apply the u t as my complete state feedback.

So, I can measure the x t multiplied with the K t and give it as my u. So, this will, solution of this matrix DRE theory, it is a non-linear differential Riccati equation. So, the numerical technique can be adapted to solve this and for a simple system we can directly have the solution of this Riccati equation.

So, we will see it in form an example how this can be or how we can determine the optimal control law for a linear system.

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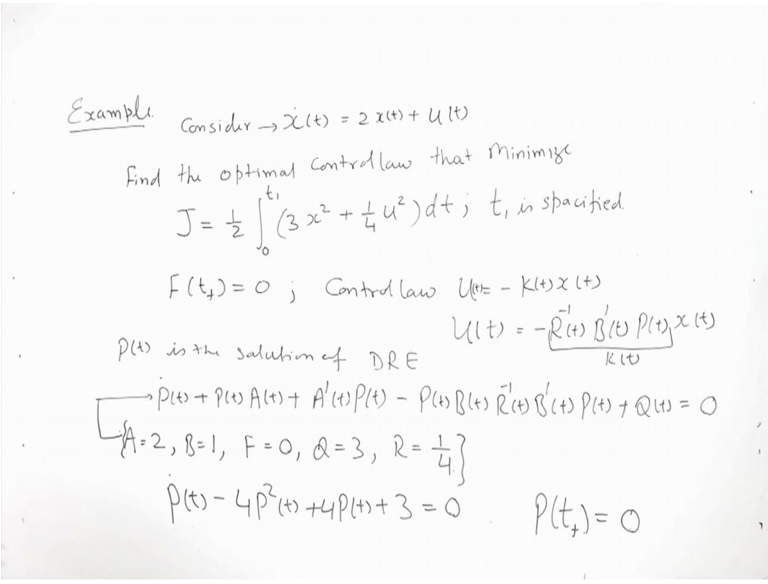

So, we take an example let me change this pen. So, let us consider a first order system as x dot t - 2 x t plus u t, consider a system; my objective is to find the optimal control law that minimize my J equal to and in this case my t 1 is specified.

So, my problem is for a first order system we have taken as x dot as 2 x plus u we have to obtain the optimal control law which is minimizing my performance index given here in a specified time t 1. So, t 1 is specified in this case. So, now, if we will see what is my terminal condition, my terminal condition here is because no terminal cost is given. So, this means F of t f equal to 0, my control law is u equal to minus sorry; u t equal to minus K t x t and K t is minus R inverse B transpose P t x t.

So, now with this if I will write what is the different values. So, before that if we will see to find out the sorry this is my u of t and this is my and this is my K t. So, u t can be determined and P t is the solution of my differential Riccati equation which is given as P dot t plus P A plus A transpose t minus P B R inverse, B transpose, P plus Q t equal to 0.

Now if you will see what is my A? 2 x t, so A is 2, B is 1, F is 0, Q is 3 and R is 1 by 4. So, if I will substitute these values into this all this values in this equation, so what I will get? I will get a final equation I am writing here as P dot t minus 4, P square t plus 4, P of t plus 3 equal to 0. So, this differential equation I have to solve with and what is my condition P t f is nothing, but 0 because F of t f is 0 with this condition I have to solve this equation.

So, it is a simple first order differential equation which can be solved some writing this as P dot d P by 4.

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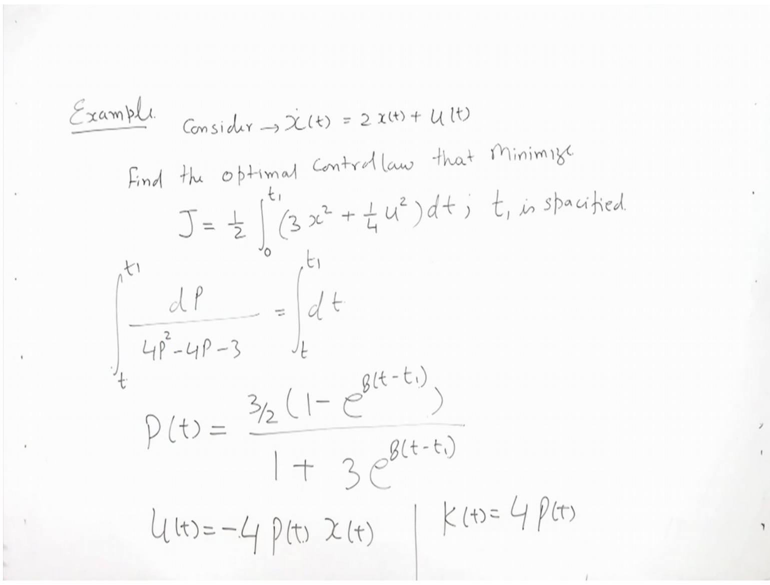

Just for simplicity I am dropping the t terms from here 4 P square minus 4 P minus 3 if I will take it there and this is nothing, but my d t this I will integrate from t 2 to t 1 t 2 as t 1 is final time. So, I can write this P sorry P t 1 as 0 because F we have taken as a t 1.

So, there are many approach to solve this I can make the factors of this, then directly take the integration I will integrate this. So, this you can perform the calculation yourself and its final solution I am giving you with this if I will calculate my P t comes out to be 3 by 2, 1 minus e to the power 8 t minus t 1 the whole divided by 1 plus 3 e to the power 8 t minus t 1.

So, this will be my P t and u t is nothing but minus R inverse B prime P which is coming out to be the 4 P t x t. So, this means my K t is nothing but 4 and P t and this P t we have already have been determined here. So, this will give me my final control law. My u t is minus four P t x t and my K t is 4 P t. So, in this way we can find out the solution of the matrix differential Riccati equation. For a first order system I have only the single equation, but as the order of the system will increase the number of equations will also increase.

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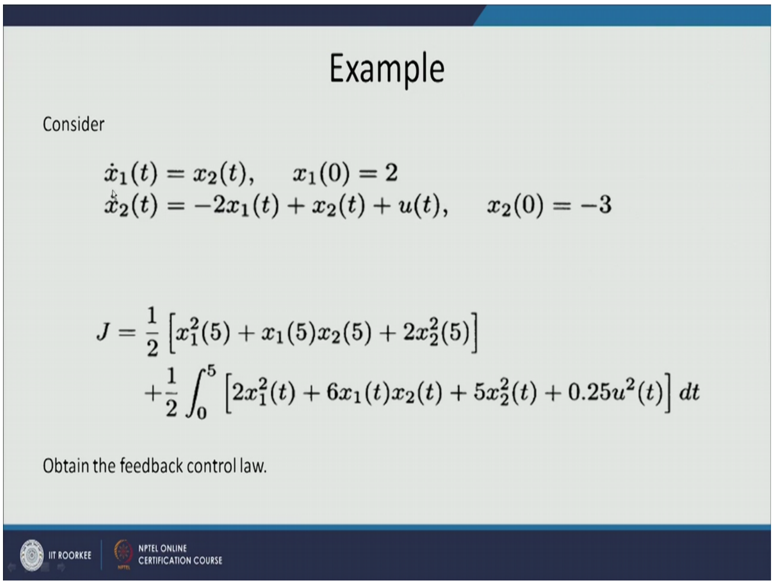

So, we take the another example as x 1 dot t x 2. So, we consider a second order system with initial condition as x 1 0 as to x 2 0 as minus 3. Our objective is to determine the optimal control law u t which will minimize the performance index J given here. So, we can see in this example we will have the terminal cost as well as my integral cost.

So, to solve this problem if you will see, so what will be my a b c d: my A is 0 1 minus 2 1, B t is 0 1, F t which is given here Q t 2 3 3 5, R t is 1 by 4 and t 0 and t f are specified as 0 and 5.

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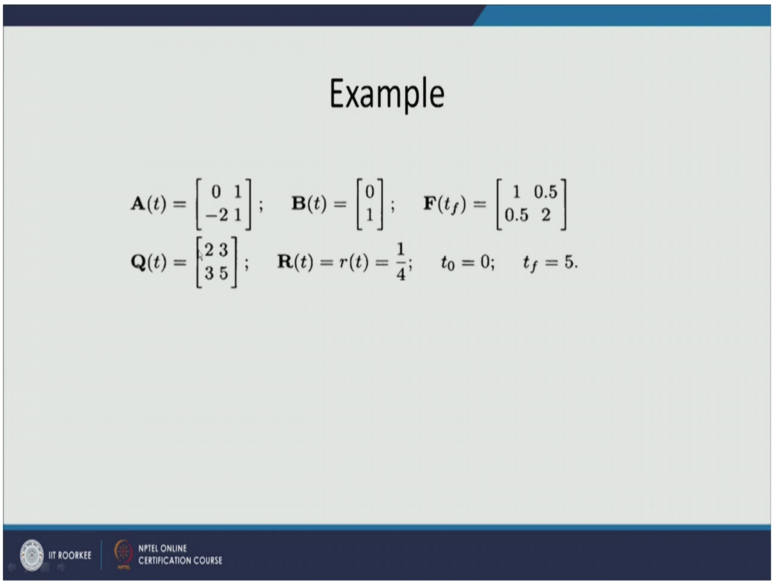

So, t f we have taken as the 5 and starting point is t 0 initial point is specified. To determine the u we have to write our Riccati equation.

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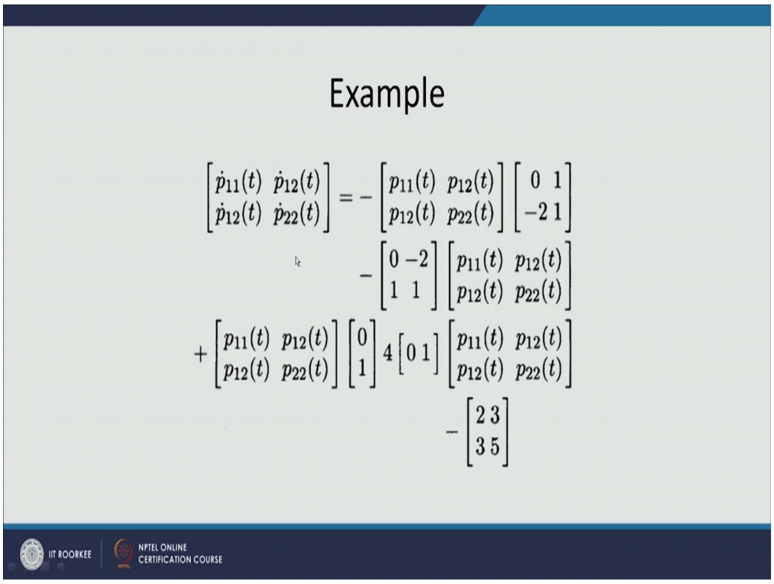

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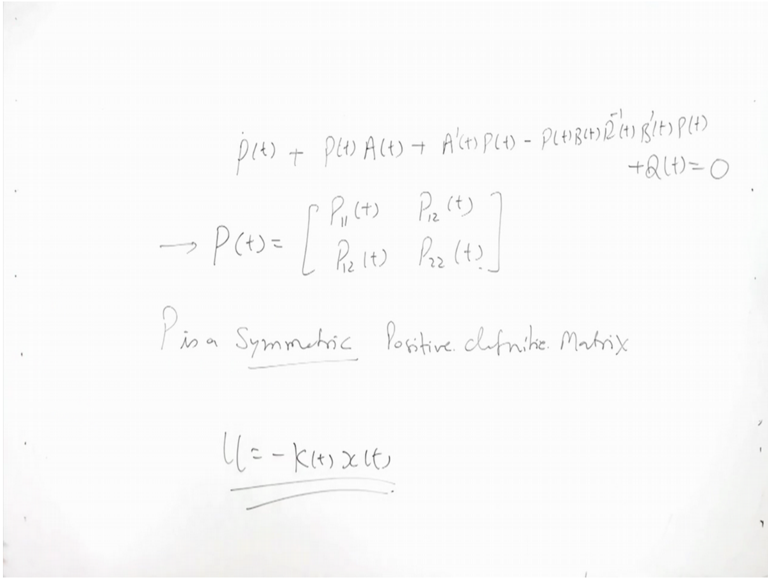

P dot t equal to P A plus A transpose P minus P B R inverse B transpose P plus Q t equal to 0. So, in this if you will put the value of A B F, A B Q R because A B Q R is required here. So, my equation will be, so now, what actually will be my P? For a second order system I consider my P t as P 11 of t, P 12 of t, P 12 of t and P 22 of t. So, we have considered this matrix as a symmetric matrix. So, P is a symmetric positive definite matrix.

So, in the next class we will discuss why P is a symmetric matrix. So, just for our consideration we take this P as my symmetric matrix here. So, with this value of the P and the value of the A B Q and R this, I can write my equation as P 11 dot P 12 dot P 12 dot P 22 dot as this whole equation.

So, now if you will see because this is a symmetric matrix, so in place of the 4 I will get only the 3 questions and these 3 questions so now, if I will simplify this matrix I will get these three equations.

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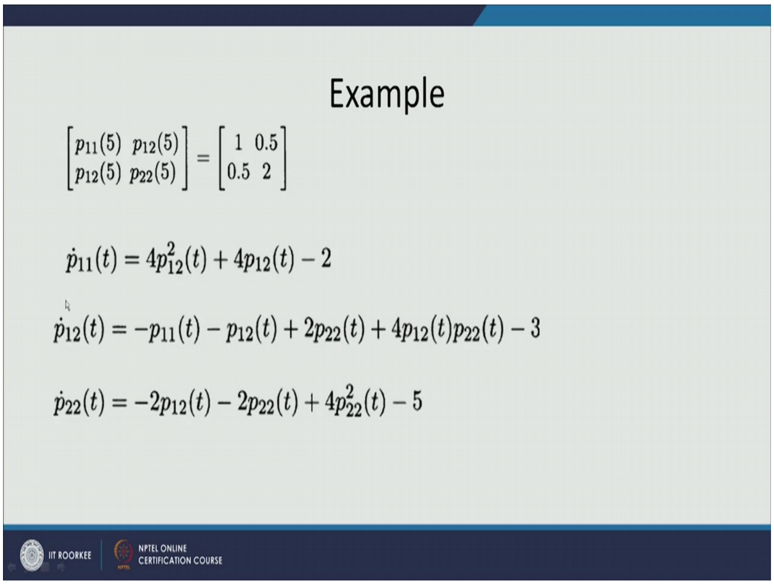

P 11 dot in terms of the P 12, so this is a interrelated equations, but I have to solve this equation for the P 11 dot P 12 dot and P 22 dot to get the value of my P matrix as this.

So, these equations with my final condition as P 11 5, P 22 5 which I am getting nothing, but from my matrix f, so this F matrix means my P t f equal to F of t f, so all this parameter will represent here as the final value of my P 11, P 12, P 12, P 22 as 1, 0.5 and 2. So, these three equation I have to solve with this three. Any numerical technique can be adopt to solve this equation and by this way we can determine the P matrix, if P is known to us then we can find out the K, if K is known to us we can find out the u which will be u equal to minus K t and x of t.

So, in this way we can solve our linear quadratic regulator problem for a finite time. If t f is finite then matrix differential Riccati equation we have to solve, this matrix differential Riccati equation we have to solve to find the value of the P and once the P is known we can find out the u.

So, this class I stop it here and in the next class we will discuss about the infinite horizon, time varying and the time invariant case.

Thank you very much.